On multicolor Ramsey number of paths versus cycles

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Abstract

Let $G_1, G_2, \ldots, G_t$ be graphs. The multicolor Ramsey number $R(G_1, G_2, \ldots, G_t)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. In this paper, we provide the exact value of $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$ for certain values of n_i and k. In addition, the exact values of $R(P_5, C_4, P_k)$, $R(P_4, C_4, P_k)$, $R(P_5, P_5, P_k)$ and $R(P_5, P_6, P_k)$ are given. Finally, we give a lower bound for $R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t})$ and we conjecture that this lower bound is the exact value of this number. Moreover, some evidence is given for this conjecture.

1 Introduction

In this paper, we are only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. The complement graph of a graph G is denoted by $\overline{G}$. As usual, the complete graph of order p is denoted by K_p and a complete bipartite graph with partite set (X, Y) such that $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. Throughout this paper, we denote a cycle and a path on m vertices by C_m and P_m, respectively. Also for a 3-edge coloring (say green, blue and red) of a graph G, we denote by G^g (resp. G^b and G^r) the subgraph induced by the edges of color green (resp. blue and red).

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Let $G_1, G_2, \ldots, G_t$ be graphs. The *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_t)$, is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [12]. Since their time, particulary since the 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic.

For $t \geq 3$, there is a few results about multicolor Ramsey number $R(G_1, G_2, \ldots, G_t)$. A survey including some results on Ramsey number of graphs, can be found in [11]. The multicolor Ramsey numbers $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$ and $R(P_{n_1}, P_{n_2}, \ldots, C_{n_t})$ are not known for $t \geq 3$. In the case $t = 2$, a well-known theorem of Gerencsér and Gyárfás [9] states that $R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$, where $n \geq m \geq 2$. Faudree and Schelp in [7] determined $R(P_{n_1}, P_{2n_2+\delta}, \ldots, P_{2n_t})$ where $\delta \in \{0, 1\}$ and n_1 is sufficiently large. In addition, they determined $R(P_{n_1}, P_{n_2}, P_{n_3})$ for the case $n_1 \geq 6(n_2 + n_3)^2$ and they conjectured that

$$R(P_n, P_n, P_n) = \begin{cases} 2n - 1 & \text{if } n \text{ is odd,} \\ \\ 2n - 2 & \text{if } n \text{ is even.} \end{cases}$$

This conjecture was established by Gyárfás et al. [10] for sufficiently large n. In asymptotic form, this was proved by Figaj and Luczak in [8] as a corollary of more general results about the asymptotic results of the Ramsey number for three long even cycles.

Recently, determination of some exact values of Ramsey numbers of type $R(P_i, P_j, C_k)$ such as $R(P_4, P_4, C_k)$, $R(P_4, P_6, C_k)$ and $R(P_3, P_5, C_k)$ have been investigated. For more details related to three-color Ramsey numbers for paths versus a cycle, see [3, 4, 5, 13]. In this paper, we provide the exact value of the Ramsey numbers $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$ for certain values of n_i and k and then we determine the exact values of some three-color Ramsey numbers of type $R(P_i, P_j, C_k)$ as corollaries of our result. Moreover, we determine the exact value of the multicolor Ramsey number $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$, if at most one n_i is odd and k is sufficiently large. Consequently, we obtain an improvement of the result of Faudree and Schelp [7] on multicolor Ramsey number $R(P_{n_1}, P_{2n_2+\delta}, \ldots, P_{2n_t})$. In addition, we determine the exact values of some three-color Ramsey numbers such as $R(P_5, C_4, P_k)$, $R(P_4, C_4, P_k)$, $R(P_5, P_5, P_k)$ and $R(P_5, P_6, P_k)$. Finally, we give a lower bound for $R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t})$ and we conjecture that, with giving some evidences, this lower bound is the exact value of this number.

2 Multicolor Ramsey number $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$

In this section, we determine the exact value of $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$ when at most one of n_i is odd and k is sufficiently large. Also, the exact values of some known three-color Ramsey numbers of type $R(P_i, P_j, C_k)$ are given as some corollaries. For this purpose, we need some definitions and notations. A graph G is called *H-free* if it does not contain H as a subgraph. The notation $ex(p,H)$ is defined the maximum number of edges in a H-free graph on p vertices. It is well known that [6] $ex(p,P_n) \leq \frac{(n-2)}{2}p$, for every n. Moreover, $ex(p,C_k)$ is known for some values of p and k. The following theorem can be found in the appendix IV of [1].

Theorem 2.1 ([1]) *Assume that $k \geq \frac{1}{2}(p+3)$. Then*

$$ex(p,C_k) = \binom{p-k+2}{2} + \binom{k-1}{2}.$$

Now, we are ready to establish the main result of this section.

Theorem 2.2 *Let $k \geq n_1 \geq n_2 \geq \cdots \geq n_t \geq 3$ and $l \geq 1$ be a positive integer that can be written as $l = \sum_{i=1}^{t} x_i$ for some x_i such that $2x_i + 1 < n_i$. Then in the following cases, we have $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k) = k + l$.*

(i) *If $k \geq 2l^2 + 5l + 5$ and $\sum_{i=1}^{t} n_i = 2l + 2t + 1$,*

(ii) *If $k \geq l^2 + 2l + 3$ and $\sum_{i=1}^{t} n_i = 2l + 2t$.*

Proof. Let R denote the multicolor Ramsey number $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}, C_k)$. By Theorem 2.1, we obtain that $ex(k+l, C_k) = \frac{1}{2}(k^2 + l^2 - 3k + 3l + 4)$ where $k \geq l + 3$. Clearly $R \leq k + l$ if the following inequality holds.

$$\sum_{i=1}^{t} ex(k+l, P_{n_i}) + ex(k+l, C_k) < \binom{k+l}{2}.$$

In the other words, $R \leq k + l$ if

$$\frac{k+l}{2}\Big(\sum_{i=1}^{t} n_i - 2t\Big) + \frac{1}{2}(k^2 + l^2 - 3k + 3l + 4) < \binom{k+l}{2},$$

or simply

$$\sum_{i=1}^{t} n_i < (2t + 2l + 2) - \frac{2l^2 + 6l + 4}{k+l}. \tag{1}$$

In each case of the theorem, inequality (1) holds and so $R \leq k+l$. Now consider the graph $K_{k-1} \cup K_l$ and partition the vertices of K_l into t classes $V_1, V_2, \ldots, V_t$ such that $|V_i| = x_i$, $1 \leq i \leq t$. Color the edges of K_{k-1} and K_l by color α_{t+1} and also color the edges having an end vertex in V_i, $1 \leq i \leq t$, and one in K_{k-1} by color α_i. Since for $i = 1, 2, \ldots, t$, the inequality $2|V_i| + 1 < n_i$ holds, this coloring of K_{k+l-1} contains no P_{n_i} in color α_i, $1 \leq i \leq t$, and no C_k in color α_{t+1}. This means that $R \geq k + l$, which completes the proof. $\square$

In the following theorem, we determine the exact value of $R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t}, C_k)$ for sufficiently large k.

Theorem 2.3 *Assume that $\delta \in \{0,1\}$ and Σ denotes $\sum_{i=1}^{t}(n_i - 1)$. Then*

$$R(P_{2n_1+\delta}, P_{2n_2}, \ldots, P_{2n_t}, C_k) = k + \Sigma,$$

where $k \geq \Sigma^2 + 2\Sigma + 3$ if $\delta = 0$ and $k \geq 2\Sigma^2 + 5\Sigma + 5$, otherwise.

Proof. The assertion holds from Theorem 2.2 where $x_i = n_i - 1$ for $1 \leq i \leq t$. $\square$

As an application of Theorem 2.3, we have the following corollary which determine some known three-color Ramsey numbers of small paths versus a cycle.

Corollary 2.4 *Let k be a positive integer. Then*

(i) ([3]) $R(P_4, P_4, C_k) = k + 2$ *for* $k \geq 11$,

(ii) ([4]) $R(P_3, P_4, C_k) = k + 1$ *for* $k \geq 12$,

(iii) ([13]) $R(P_4, P_5, C_k) = k + 2$ *for* $k \geq 23$,

(iv) ([13]) $R(P_4, P_6, C_k) = k + 3$ *for* $k \geq 18$.

We end this section by giving the following consequent of Theorem 2.3.

Corollary 2.5 *Let k be a positive integer. Then*

(i) $R(P_3, P_6, C_k) = k + 2$ *for* $k \geq 23$,

(ii) $R(P_6, P_6, C_k) = R(P_4, P_8, C_k) = k + 4$ *for* $k \geq 27$,

(iii) $R(P_6, P_7, C_k) = k + 4$ *for* $k \geq 57$.

3 Some three-color Ramsey numbers

In this section, we provide the exact values of some three-color Ramsey numbers such as $R(P_5, C_4, P_m)$, $R(P_4, C_4, P_m)$, $R(P_5, P_5, P_m)$ and $R(P_5, P_6, P_m)$. First, we recall a result of Faudree and Schelp.

Theorem 3.1 ([7]) *If G is a graph with $|V(G)| = nt + r$ where $0 \leq r < n$ and G contains no path on $n+1$ vertices, then $|E(G)| \leq t\binom{n}{2} + \binom{r}{2}$ with equality if and only if either $G \cong tK_n \cup K_r$ or if n is odd, $t > 0$ and $r = (n \pm 1)/2$*

$$G \cong lK_n \cup \left(K_{(n-1)/2} + \overline{K}_{((n+1)/2+(t-l-1)n+r)}\right),$$

for some $0 \leq l < t$.

By Theorem 3.1, it is easy to obtain the following corollary.

Corollary 3.2 *For all integer $n \geq 3$,*

$$ex(n, P_4) = \begin{cases} n & \textit{if } n = 0 \ (\textit{mod } 3), \\ n-1 & \textit{if } n = 1, 2 \ (\textit{mod } 3). \end{cases}$$

$$ex(n, P_5) = \begin{cases} 3n/2 & \textit{if } n = 0 \ (\textit{mod } 4), \\ 3n/2 - 2 & \textit{if } n = 2 \ (\textit{mod } 4), \\ (3n-3)/2 & \textit{if } n = 1, 3 \ \textit{mod } 4. \end{cases}$$

$$ex(n, P_6) = \begin{cases} 2n & \textit{if } n = 0 \ (\textit{mod } 5), \\ 2n-2 & \textit{if } n = 1, 4 \ (\textit{mod } 5), \\ 2n-3 & \textit{if } n = 2, 3 \ \textit{mod } 5. \end{cases}$$

In order to prove the main results of this section, we need some lemmas.

Lemma 3.3 ([13]) *Let G be a complete bipartite graph $K_{3,4}$ with two partite sets X and Y where $|X| = 3$ and $|Y| = 4$. If each edge of G is colored green or blue, then G contains either a green P_5 or a blue C_4.*

Lemma 3.4 ([13]) *Let G be a graph obtained by removing two edges from K_6. If each edge of G is colored green or blue, then G contains either a green P_5 or a blue C_4.*

Using Lemma 3.3, we have the following lemma.

Lemma 3.5 *Let G be a complete bipartite graph $K_{3,5}$ with two partite sets X and Y where $|X| = 3$ and $|Y| = 5$. If each edge of G is colored green or blue, then G contains a monochromatic graph P_5.*

Proof. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. By Lemma 3.3, G must contain a green P_5 or a blue C_4. If a green P_5 occur, we are done. So let G contains a blue C_4 on vertices x_1, y_1, x_2, y_2, in this order. If one of the edges x_iy_j, $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, is blue we obtain a blue P_5. Otherwise, we may assume that these edges are all in green color. Clearly this gives a green $P_5 = y_5x_2y_4x_1y_3$, which completes the proof. □

Now, we use previous results to prove the following lemma, which help us to calculate the three-color Ramsey number $R(P_5, C_4, P_m)$.

Lemma 3.6 *Let $m \geq 5$ and the edges of K_{m+2} be colored with colors green, blue and red such that G^r contains a copy of P_{m-1} as a subgraph. Then K_{m+2} contains either a green P_5, a blue C_4 or a red P_m.*

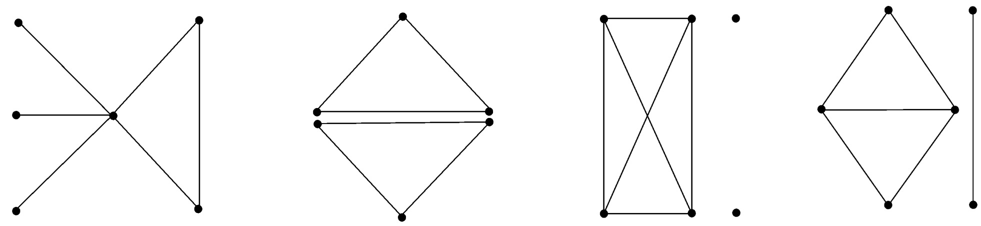

Fig. 1: P_5-free graphs on 6 vertices and 6 edges

Proof. Assume that $V(K_{m+2}) = \{v_1, v_2, \ldots, v_{m+2}\}$ and $P = v_1v_2 \ldots v_{m-1}$ is the desired copy of P_{m-1} in G^r. We suppose that G^r contains no copy of P_m, then we prove that K_{m+2} contains either a green P_5 or a blue C_4. First assume that $v_1v_{m-1} \in E(G^r)$. If one of the vertices v_m, v_{m+1} or v_{m+2} is adjacent to P in G^r then we obtain a red P_m, a contradiction. So each edge between $\{v_m, v_{m+1}, v_{m+2}\}$ and P is colored green or blue. Since $m \geq 5$, we obtain the complete bipartite graph $K_{3,4}$ on two partite set $X = \{v_m, v_{m+1}, v_{m+2}\}$ and $Y = \{v_1, v_2, v_{m-2}, v_{m-1}\}$ with all edges are colored green or blue. Using Lemma 3.3, we obtain a green P_5 or a blue C_4. Hence we may assume that $v_1v_{m-1} \notin E(G^r)$. Also all edges between $\{v_1, v_{m-1}\}$ and $\{v_m, v_{m+1}, v_{m+2}\}$ are colored by green or blue, otherwise we have a red P_m. Let H be a subgraph of G^r induced by the edges of color red on vertices $\{v_m, v_{m+1}, v_{m+2}\}$. We have the following cases.

Case 1. $|E(H)| = 0$.

Since $|E(H)| = 0$, all edges between vertices $T = \{v_1, v_{m-1}, v_m, v_{m+1}, v_{m+2}\}$ are colored by green or blue. We find a vertex $v \in P$ such that $T \cup \{v\}$ are the vertices of a complete graph on six vertices with at most two red edges and then we use Lemma 3.4, which guaranties the existence of a green P_5 or a blue C_4. If there is a vertex $v \in P - \{v_1, v_{m-1}\}$ such that for each $i \in \{m, m+1, m+2\}$, $vv_i \notin E(G^r)$, then this vertex is the desired vertex. Also note that two consecutive vertices of P are not adjacent in G^r to a vertex in $\{v_m, v_{m+1}, v_{m+2}\}$, otherwise we have a red copy of P_m, a contradiction. So, without loss of generality, let $v_2v_m, v_3v_{m+1} \in E(G^r)$. If $v_3v_1 \in E(G^r)$, then $P_m = v_mv_2v_1v_3v_4 \ldots v_{m-1}$ is a red P_m and so $v_3v_1 \notin E(G^r)$. By the same argument, $v_2v_{m-1} \notin G^r$. Now let $v = v_3$ if $v_3v_{m+2} \notin E(G^r)$ and $v = v_2$ otherwise. In any case, $T \cup \{v\}$ form a complete graph on six vertices with at most two red edges.

Case 2. $|E(H)| = 1$.

Let $E(H) = \{v_mv_{m+1}\}$. Since $P_m \nsubseteq G^r$, v_2 (also v_{m-2}) is not adjacent to v_m or v_{m+1} in G^r. If $v_2v_{m-1}, v_1v_3 \in E(G^r)$, then G^r contains $C_{m-1} = v_2v_1v_3 \ldots v_{m-1}v_2$ and so each edge between $X = \{v_m, v_{m+1}, v_{m+2}\}$ and $Y = \{v_1, v_2, v_{m-2}, v_{m-1}\}$ is colored green or blue, since $P_m \nsubseteq G^r$. Using Lemma 3.3, we obtain either a green P_5 or a blue C_4. Therefore if $v_2v_{m-1} \in E(G^r)$, then $v_1v_3 \notin E(G^r)$. Now, assume that $v_2v_{m+2} \notin E(G^r)$. If $v_2v_{m-1} \notin E(G^r)$, then $\{v_1, v_2, v_{m-1}, v_m, v_{m+1}, v_{m+2}\}$ are the vertices of a complete graph on six vertices with at most two red edges. Also if $v_2v_{m-1} \in E(G^r)$, then for each $i \in \{m, m+1, m+2\}$, $v_3v_i \notin E(G^r)$, otherwise we have a red P_m. In this case $\{v_1, v_3, v_{m-1}, v_m, v_{m+1}, v_{m+2}\}$ are the vertices of a complete graph on six vertices with at most two red edges. Using Lemma 3.4, we obtain a green P_5 or blue C_4, as desired. So we may assume that v_2v_{m+2} is an edge of G^r. If $m = 5$, then $\{v_1, v_3, v_{m-1}, v_m, v_{m+1}, v_{m+2}\}$ are the vertices of a complete graph on six vertices such that each edge is colored green or blue except at most two edges. Now let $m \geq 6$. By the same argument, we may assume that $v_{m-2}v_{m+2} \in E(G^r)$. If for some $i \in \{m, m+1, m+2\}$, $v_3v_i \in E(G^r)$, then we obtain $P_m = v_1v_2v_{m+2}v_{m-2} \dots v_3v_i$ in G^r. Also if $v_1v_3 \in E(G^r)$, then we obtain a copy of $P_m = v_{m+2}v_2v_1v_3 \dots v_{m-1}$ in G^r, a contradiction. Hence $\{v_1, v_3, v_{m-1}, v_m, v_{m+1}, v_{m+2}\}$ are the vertices of a complete graph on six vertices such that each edge is colored green or blue except at most two edges. Lemma 3.4, guaranties the existence of a green P_5 or a blue C_4.

Case 3. $|E(H)| \geq 2$.

Let $X = \{v_m, v_{m+1}, v_{m+2}\}$ and $Y = \{v_1, v_2, v_{m-2}, v_{m-1}\}$. All edges having one end in X and one in Y, are colored by green or blue, otherwise we obtain a red P_m. So we obtain the complete bipartite graph $K_{3,4}$ on two partite set X and Y with all edges are colored green or blue. Again using Lemma 3.3, we obtain a green P_5 or a blue C_4, which completes the proof of theorem. □

Corollary 3.7 $R(P_5, C_4, P_5) = 7$.

Proof. By a result in [13], $R(P_5, C_4, P_4) = 7$ and clearly $R(P_5, C_4, P_5) \geq R(P_5, C_4, P_4)$. So it is sufficient to prove that $R(P_5, C_4, P_5) \leq 7$. Assume the edges of K_7 are arbitrary colored by green, blue and red. Since $R(P_5, C_4, P_4) = 7$, we may assume that G^r contains a copy of P_4 as a subgraph. By Lemma 3.6, K_7 must contains either a green P_5, a blue C_4 or a red P_5, which completes the proof. □

Using Lemma 3.6 and Corollary 3.7, we have the following theorem.

Theorem 3.8 *For all integers $m \geq 5$, $R(P_5, C_4, P_m) = m + 2$.*

Proof. Color all edges crossing a vertex of K_m by green and other edges by red. Adjoin a new vertex to all vertices of colored graph K_m and color all new edges by blue. This yields a 3-colored graph K_{m+1} with no a green P_5, a blue C_4 and a red P_m and so $R(P_5, C_4, P_m) > m+1$. Now assume that the edges of K_{m+2} are colored with colors green, blue and red. We prove that K_{m+2} contains either a green P_5, a blue C_4 or a red P_m. We prove the claim by induction on m. By Corollary 3.7, this claim is true when $m = 5$. Assume that $R(P_4, C_4, P_{m-1}) = m+1$ for $m \geq 6$. By the induction assumption, we obtain that K_{m+2} contains a red P_{m-1}. Using Lemma 3.6, we obtain that K_{m+2} contains a green P_5, a blue C_4 or a red P_m, which completes the proof. □

Corollary 3.9 *For all integers $m \geq 5$, $R(P_4, C_4, P_m) = m + 2$.*

Proof. Using Theorem 3.8, we have $R(P_4, C_4, P_m) \leq m + 2$. On the other hand, the 3-colored graph K_{m+1} in the proof of Theorem 3.8, implies that $R(P_4, C_4, P_m) > m + 1$. $\square$

Before establishing the other results of this section, we give the following lemmas which help us to calculate the Ramsey number $R(P_5, P_5, P_m)$.

Lemma 3.10 *Let G be a graph obtained by removing two edges from K_6. If each edge of G is colored green or blue, then G contains a monochromatic graph P_5.*

Proof. By Corollary 3.2, $ex(6, P_5) = 7$. Since $|E(G)| = 13$, so without loss of generality, we may assume that $|E(G^b)| = 6$ and $|E(G^g)| = 7$. Since $|E(G^b)| = 6$, G^b is isomorphic to one of the graphs shown in Fig. 1. So G^g is isomorphic to a graph obtained by removing any two edges of $\overline{G^b}$. One can easily check that $\overline{G^b}$ is isomorphic to $K_5 - e$, $K_{3,3}$ or $K_{2,4}$ with one additional edge and any graph obtained by removing two edges from these graphs, still contains a P_5, which completes the proof. $\square$

Lemma 3.11 *Let G be a graph obtained by removing an edge from the complete bipartite graph $K_{4,5}$ with partite sets X and Y. If each edge of G is colored green or blue, then G contains either a green P_5 or a blue P_6.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Also without loss of generality, let $e = x_4y_5$ be the edge of $K_{4,5}$ such that $G = K_{4,5} - e$. By Lemma 3.5, $G - x_4$ (particulary G) contains a monochromatic P_5. If G contains a green P_5, we are done. So we may assume that G contains a blue P_5 such as P. Suppose t and z are the end vertices of P. First let $t, z \in X$ and $Y \cap V(P) = \{y_1, y_2\}$. If one of the edges ty_i or zy_i, $i \in \{3, 4, 5\}$, is blue we have a blue P_6. Otherwise the path $y_3ty_5zy_4$ is a green P_5. So let $t, z \in Y$ and $X \cap V(P) = \{x_1, x_2\}$.

Let $Y \cap V(P) = \{y_1, y_2, y_3\}$ such that $t = y_1$ and $z = y_3$. If one of the edges y_1x_i or y_3x_i, $i \in \{3, 4\}$, is blue we have a blue P_6. So we may assume that these edges are colored green. Now if one of the edges x_3y_i, $i \in \{2, 4, 5\}$, is green we have a green P_5. Otherwise the path $y_5x_3y_2x_1y_3x_2$ is a blue P_6. If $y_5 \in Y \cap V(P)$, by the same argument, one can easily find either a green P_5 or a blue P_6 in G, which completes the proof. $\square$

In the following theorem, the values of $R(P_5, P_5, P_5)$ and $R(P_5, P_5, P_6)$ are given.

Theorem 3.12 *Let $n \in \{5, 6\}$. Then $R(P_5, P_5, P_n) = 9$.*

Proof. First we prove that $R(P_5, P_5, P_n) \geq 9$. To see this, let $v_1, v_2, \ldots, v_8$ be the vertices of K_8 in the clockwise order. Let G^1 be the union of two K_4 on vertices $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$, G^2 be the union of two C_4 on vertices $\{v_1, v_5, v_2, v_6\}$ and $\{v_3, v_7, v_4, v_8\}$ and G^3 be the union of two C_4 on $\{v_1, v_7, v_2, v_8\}$ and $\{v_3, v_6, v_4, v_5\}$ in this order. Color the edges of G^i by color i. This gives a 3-edge coloring of K_8 which contains no P_5 in color 1, no P_5 in color 2 and no P_n in color 3. So $R(P_5, P_5, P_n) \geq 9$. Now we prove that $R(P_5, P_5, P_n) \leq 9$. Let $c : E(K_9) \longrightarrow \{1, 2, 3\}$ be an arbitrary 3-edge coloring of K_9. Also assume that G^i denotes the spanning subgraph of K_9 induced by the edges of color i.

Case 1. $n = 5$.

Using Corollary 3.2, we have $ex(9, P_5) = 12$. Since $E(K_9) = 36$, we may assume that $|E(G^1)| = |E(G^2)| = |E(G^3)| = 12$. By Theorem 3.1, $G^1 \cong 2K_4 \cup K_1$. This implies that $K_{4,5} \subseteq \overline{G^1}$. Now using Lemma 3.5, we obtain a monochromatic P_5.

Case 2. $n = 6$.

Again by Corollary 3.2, $ex(9, P_5) = 12$ and $ex(9, P_6) = 16$. If $|E(G^1)| = 12$, by the same argument as in case 1, we obtain that $K_{4,5} \subseteq \overline{G^1}$. Using Lemma 3.11, we obtain either a P_5 in color 2 or a P_6 in color 3. Also if $|E(G^2)| = 12$, by a similar argument, one can obtain the desired result. If $|E(G^3)| = 16$, then Theorem 3.1 implies that $G^3 \cong K_5 \cup K_4$. Again $K_{4,5} \subseteq \overline{G^3}$, and hence $\overline{G^3}$ contains a copy of P_5 in color 1 or 2, by Lemma 3.5. Without loss of generality, we may assume that $|E(G^1)| = 11$. Since $|E(G^1)| = 11$, G^1 is not connected, otherwise we obtain a copy of P_5 in color 1. Since $|E(G^1)| = 11$, so there exists a component of G^1 such as H such that $|H| = 4$ and hence $K_{4,5} \subseteq \overline{G^1}$. Using Lemma 3.11, we obtain a copy of P_5 in color 2 or a copy of P_6 in color 3, which completes the proof. □

In order to determine the exact value of the Ramsey number $R(P_5, P_5, P_7)$, we need the following lemma which can be obtained by an argument similar to the proof of Lemma 3.6 and using Lemma 3.5 and Lemma 3.10.

Lemma 3.13 *Let $m \geq 7$ and the edges of K_{m+2} are colored by colors green, blue and red such that G^r contains a copy of P_{m-1} as a subgraph. Then K_{m+2} contains either a green P_5, a blue P_5 or a red P_m.*

As an easy consequent of Lemma 3.13, we have the following corollary.

Corollary 3.14 *$R(P_5, P_5, P_7) = 9$.*

Proof. By Theorem 3.12, $R(P_5, P_5, P_6) = 9$ and clearly $R(P_5, P_5, P_7) \geq R(P_5, P_5, P_6)$, so it is sufficient to prove that $R(P_5, P_5, P_7) \leq 9$. Assume that the edges of K_9 are arbitrary colored green, blue and red. Since $R(P_5, P_5, P_6) = 9$, we may assume that G^r contains a copy of P_6 as a subgraph. By Lemma 3.13, K_9 must contains either a monochromatic P_5 in color green or blue or a red P_6, which completes the proof. □

Now, we are ready to calculate the exact value of $R(P_5, P_5, P_m)$ for $m \geq 7$.

Theorem 3.15 *For all integers $m \geq 7$, $R(P_5, P_5, P_m) = m + 2$.*

Proof. Consider the graph $K_{m-1} \cup K_2$ and color the complete graphs K_{m-1} and K_2 by color red. Consider a vertex of K_2, say v, and color the edges which are incident with v and having another end in K_{m-1} by blue and finally, color the remaining edges by green. This coloring contains neither a green P_5, a blue P_5, nor a red P_m, which means that $R(P_5, P_5, P_m) \geq m + 2$. Now assume that the graph K_{m+2} is 3-edge colored by colors green, blue and red. We prove that K_{m+2} contains either a green P_5, a blue P_5 or a red P_m. We use induction on m. By Corollary 3.14, the claim is true when $m = 7$. Let us assume that $R(P_5, P_5, P_{m-1}) \leq m + 1$ for $m \geq 8$. By the induction assumption, we obtain that K_{m+2} contains a red copy of P_{m-1}. Using Lemma 3.13, we obtain that K_{m+2} contains a green P_5, a blue P_5 or a red P_m, which completes the proof. $\square$

We need the following lemma to determine the exact value of $R(P_5, P_6, P_m)$.

Lemma 3.16 *Let G be a graph obtained by removing three edges from K_7. If each edge of G is colored green or blue, then G contains either a green P_5 or a blue P_6.*

Proof. By Corollary 3.2, $ex(7, P_5) = 9$ and $ex(7, P_6) = 11$. Since $|E(G)| = 18$, we may assume that $|E(G^g)| \in \{7, 8, 9\}$. If $|E(G^g)| = 9$, then by Theorem 3.1, $G^g \cong K_4 \cup K_3$ which implies that $K_{3,4} \subseteq \overline{G^g}$. But removing any three edges from $K_{3,4}$, retains a copy of P_6. If $|E(G^g)| = 7$, then $|E(G^b)| = 11$, since $|E(G)| = 18$. Now by Theorem 3.1, $G^b \cong K_5 \cup K_2$ or $G^b \cong K_2 + \overline{K}_5$ which implies that $K_{2,5} \subseteq \overline{G^b}$ or $K_5 \subseteq \overline{G^b}$. But removing any three edges from $K_{2,5}$ or K_5, retains a copy of P_5. So we may assume that $|E(G^g)| = 8$. We have the following cases.

Case 1. G^g is connected.

Clearly G^g contains no C_4, otherwise the connectivity of G^g implies a copy of P_5. So G^g contains a triangle C. The induced subgraph of G^g on $V(K_7) - V(C)$ is an independent set, since otherwise we have a copy of P_5 in G^g. Since $|E(G^g)| = 8$, two vertices of C must contain a common neighbor outside C, which gives a copy of C_4 and hence a copy of P_5 in G.

Case 2. G^g is disconnected.

Since $ex(6, P_5) = 7$, $ex(5, P_5) = 6$ by Corollary 3.2, and $|E(G^g)| = 8$, so G^g can not have two components H_1 and H_2 such that $|V(H_1)| \leq 2$. Hence one can easily find $K_{3,4} \subseteq \overline{G^g}$ and clearly removing any three edges from $K_{3,4}$, retains a copy of P_6, which completes the proof. $\square$

Using Lemma 3.11 and Lemma 3.16, we have the following lemma.

Lemma 3.17 *Let $m \geq 6$ and K_{m+3} is 3-edge colored with colors green, blue and red such that G^r contains a copy of P_{m-1} as a subgraph. Then K_{m+3} contains either a green P_5, a blue P_6 or a red P_m.*

Proof. Assume that $v_1, v_2, \ldots, v_{m+3}$ are vertices of K_{m+3} and $P = v_1 v_2 \ldots v_{m-1}$ is the desired copy of P_{m-1} in G^r. Also let $P_m \nsubseteq G^r$. We prove that K_{m+3} contains either a green P_5 or a blue P_6. First assume that $v_1 v_{m-1} \in E(G^r)$. If one of the vertices v_m, v_{m+1}, v_{m+2} or v_{m+3} is adjacent to P by a red edge, then we obtain a red P_m. So we may assume that each edge between $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and P is colored by green or blue. Since $m \geq 6$, we obtain a bipartite graph $K_{4,5}$ with two partite sets $X = \{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $Y = \{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ such that all edges colored green or blue and so by Lemma 3.11, we obtain a green P_5 or a blue P_6. Hence we may assume that $v_1 v_{m-1} \notin E(G^r)$. Since $P_m \nsubseteq G^r$, all edges having ends in both $\{v_1, v_{m-1}\}$ and $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ are colored by green or blue. Now let H be the subgraph induced by edges of color red between vertices $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$. We have the following cases.

Case 1. $|E(H)| = 0$.

Since $|E(H)| = 0$, then all edges among vertices $T = \{v_1, v_{m-1}, v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ are colored by green or blue. We find a vertex v such that $T \cup \{v\}$ are the vertices of a complete graph on seven vertices and each edge is colored green and blue except at most three edges. If there exists a vertex $v \in P - \{v_1, v_{m-1}\}$ such that $vv_i \in E(G^r)$ for at most one $i \in \{m, m+1, m+2, m+3\}$, then this vertex is the desired vertex. Note that since $P_m \nsubseteq G^r$, then two consecutive vertices of P are not adjacent in G^r to a vertex in $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$. So let $v_2 v_i \in E(G^r)$ for $i \in \{m, m+1\}$ and $v_3 v_i \in E(G^r)$ for $i \in \{m+2, m+3\}$. Now, if $v_3 v_1 \in E(G^r)$, then $P_m = v_m v_2 v_1 v_3 \ldots v_{m-1}$ is a red P_m, a contradiction. So $v_3 v_1 \notin E(G^r)$ and hence the induced subgraph on $\{v_1, v_3, v_{m-1}\}$ has at most one edge in G^r. Therefore $T \cup \{v_3\}$ are the vertices of a complete graph on seven vertices with at most three red edges. Using Lemma 3.16, we have either a green P_5 or a blue P_6.

Case 2. $|E(H)| = 1$.

Let $v_m v_{m+1} \in E(G^r)$ be the edge of H and $T = \{v_1, v_{m-1}, v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$. We find a vertex v such that $T \cup \{v\}$ are the vertices of a complete graph on seven vertices and each edge is colored green and blue except at most three edges. If there exists a vertex $v \in P - \{v_1, v_{m-1}\}$ such that $vv_i \notin E(G^r)$, for each $i \in \{m, m+1, m+2, m+3\}$, then this vertex is the desired vertex. So we assume that for some $i \in \{m, m+1, m+2, m+3\}$, $vv_i \in E(G^r)$. In G^r the vertex v_2 (also v_{m-2}) is not adjacent to any of v_m or v_{m+1}, otherwise we obtain a red P_m. So without loss of generality, let $v_2 v_{m+2} \in E(G^r)$. If $v_{m-2} v_{m+2} \in E(G^r)$, then $v_3 v_i \notin G^r$ for each $i \in \{m, m+1, m+2, m+3\}$, otherwise we obtain a red $P_m = v_1 v_2 v_{m+2} v_{m-2} \ldots v_3 v_i$. So v_{m+3} is the only vertex outside P such that $v_{m-2} v_{m+3} \in E(G^r)$. Finally, let $v = v_{m-3}$ if $v_1 v_{m-2} \in E(G^r)$ and $v = v_{m-2}$ otherwise. In any case, v is the vertex such that $T \cup \{v\}$ are the vertices of a complete graph on seven vertices at most three red edges. Using Lemma 3.16, we obtain a green P_5 or a blue P_6.

Case 3. $|E(H)| = 2$.

First let $H = 2K_2$, where $E(H) = \{v_m v_{m+1}, v_{m+2} v_{m+3}\}$. Since $P_m \nsubseteq G^r$, for each $i \in \{m, m+1, m+2, m+3\}$ we have $v_2 v_i, v_{m-2} v_i \notin E(G^r)$. If $v_3 v_i \notin E(G^r)$ for each $i \in \{m, m+1, m+2, m+3\}$, then we obtain the complete bipartite $K_{4,5}$ with partite set $X = \{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $Y = \{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ with all edges colored green or blue. Using Lemma 3.11, we obtain either a green P_5 or a blue P_6. So without loss of generality, we may assume that $v_3 v_m \in E(G^r)$. Also $v_2 v_{m-1} \notin E(G^r)$, otherwise we obtain a red copy of P_m. Now, $\{v_1, v_2, v_{m-1}, v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ are the vertices of a complete graph on seven vertices with at most three red edges. Using Lemma 3.16, we obtain either a green P_5 or a blue P_6.

Now let $H = P_3 = v_m v_{m+1} v_{m+2}$. By the same argument, one can easily obtain either a complete graph on seven vertices with at most three red edges or a complete bipartite graph $K_{4,5}$ with all edges colored green or blue. Using Lemmas 3.11 and 3.16, we obtain either a green P_5 or a blue P_6.

Case 4. $|E(H)| \geq 3$.

If either $H \cong P_4$ or $|E(H)| \geq 4$, then all edges between $\{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ and $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ are colored by green or blue, otherwise we obtain a red copy of P_m. Since $m \geq 6$, we obtain the complete bipartite graph $K_{4,5}$ with partite set $X = \{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $Y = \{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ with all edges colored green or blue. Using Lemma 3.11, we obtain either a green P_5 or a blue P_6. So it is sufficient to consider the cases that H is either a star with center v_m or the graph $K_3 \cup K_1$ with isolated vertex v_m.

In the first case, all edges having end vertices in both $\{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $\{v_1, v_2, v_{m-2}, v_{m-1}\}$ are colored green or blue, otherwise we obtain a red copy of P_m. If $v_3 v_i \notin E(G^r)$, $i \in \{m, m+1, m+2, m+3\}$, then we obtain the complete bipartite graph $K_{4,5}$ with partite set $X = \{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $Y = \{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ with all edges colored green or blue. Using Lemma 3.11, we obtain either a green P_5 or a blue P_6. So we may assume that $v_3 v_m \in E(G^r)$. Now $v_1 v_{m-2} \notin E(G^r)$, otherwise the path $P_m = v_2 v_1 v_{m-1} \ldots v_3 v_m v_{m+1}$ is a copy of P_m in G^r, a contradiction. Also $v_2 v_{m-1} \notin E(G^r)$. Hence $\{v_1, v_2, v_{m-1}, v_{m-2}, v_{m+1}, v_{m+2}, v_{m+3}\}$ are the vertices of a complete graph on seven vertices with at most three red edges. Again using Lemma 3.16, we obtain either a green P_5 or a blue P_6.

Now let $H = K_3 \cup K_1$ with isolated vertex v_m. It is clear that there is no any red edge having ends in both $\{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ and $\{v_{m+1}, v_{m+2}, v_{m+3}\}$. If either $v_2 v_m, v_{m-2} v_m \notin G^r$ or $v_2 v_m \in G^r$ and $v_{m-2} v_m \notin G^r$ then $X = \{v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ and $Y = \{v_1, v_2, v_3, v_{m-2}, v_{m-1}\}$ form a complete bipartite graph $K_{4,5}$ with at most one red edge. Using Lemma 3.11, we obtain either a green P_5 or a blue P_6. So let both edges v_2v_m and $v_{m-2}v_m$ be red. In this case, $v_3v_1 \notin G^r$ otherwise $v_mv_2v_1v_3\dots v_{m-1}$ is a copy of P_m in G^r. Also $v_3v_{m-1} \notin G^r$, otherwise $P_m = v_1v_2v_mv_{m-2}v_{m-1}v_3\dots v_{m-3}$ is a copy of P_m in G^r. So $\{v_1, v_3, v_{m-1}, v_m, v_{m+1}, v_{m+2}, v_{m+3}\}$ form a K_7 with at most three red edges. Using Lemma 3.16, we obtain either a green P_5 or a blue P_6. □

Corollary 3.18 *$R(P_5, P_6, P_6) = 9$.*

Proof. By Theorem 3.12, $R(P_5, P_6, P_5) = 9$ and clearly $R(P_5, P_6, P_6) \geq R(P_5, P_6, P_5)$. So it is sufficient to prove $R(P_5, P_6, P_6) \leq 9$. Assume that the graph K_9 is 3-edge colored by colors green, blue and red. We prove that K_9 contains either a green P_5, a blue P_6 or a red P_6. Since $R(P_5, P_6, P_5) = 9$, so we may assume that G^r contains a copy of P_5. Using Lemma 3.17, we obtain that K_9 contains either a green P_5, a blue P_6 or a red P_6, which completes the proof. □

Finally we end this section by the following theorem.

Theorem 3.19 *For all integers $m \geq 6$, $R(P_5, P_6, P_m) = m + 3$.*

Proof. Consider the graph $K_{m-1} \cup K_3$ and color the complete graphs K_{m-1} and K_3 by color red. Consider two vertices of K_3, say u, v, and color the edges which are incident with u and v and having another end in K_{m-1} by blue and finally, color the remaining edges by green. This coloring contains neither a green P_5, a blue P_6, nor a red P_m, so $R(P_5, P_6, P_m) \geq m + 3$. The upper bound follows by induction on m. By Corollary 3.18, theorem is true when $m = 6$. Let us assume that $R(P_5, P_6, P_{m-1}) \leq m + 2$ for $m \geq 7$. By the induction assumption, we obtain that K_{m+3} contains a red P_{m-1}. Using Lemma 3.17, we obtain that K_{m+3} contains either a green P_5, a blue P_6 or a red P_m, which completes the proof. □

Corollary 3.20 *For all integers $m \geq 6$, $R(P_4, P_6, P_m) = m + 3$.*

4 Multicolor Ramsey number of paths

In this section, we give an improvement of a result of Faudree and Schelp [7] on multicolor Ramsey number $R(P_{n_1}, P_{2n_2+\delta}, \dots, P_{2n_t})$. In addition,, we use a simple lemma to give a lower bound for the multicolor Ramsey number $R(P_{n_1}, P_{n_2}, \dots, P_{n_t})$ and we conjecture that this lower bound is the exact value of this Ramsey number if all n_i's are even integers greater than three. Moreover, we give some evidences for this conjecture. Before that we need a definition. By a *stripe* mK_2 we mean that a graph on $2m$ vertices and m independent edges. In [2], the exact value of the multicolor Ramsey number of stripes is given as follows.

Theorem 4.1 ([2]) *Let $n_1 \geq n_2 \geq \cdots \geq n_t$ and Σ denote $\Sigma_{i=1}^{t}(n_i - 1)$. Then*

$$R(n_1K_2, n_2K_2, \ldots, n_tK_2) = n_1 + \Sigma + 1.$$

In the following lemma, we give a lower bound for the multicolor Ramsey number $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$.

Lemma 4.2 *Assume that $G_1, G_2, \ldots, G_t$ are arbitrary graphs and for $i = 1, 2, \ldots, t$, $H_i \subseteq G_i$. Also let $n_1 \geq n_2 \geq \cdots \geq n_t \geq 3$ and Σ denote $\Sigma_{i=1}^{t}(\lfloor \frac{n_i}{2} \rfloor - 1)$. Then*

(i) $R(H_1, H_2, \ldots, H_t) \leq R(G_1, G_2, \ldots, G_t)$,

(ii) $\lfloor \frac{n_1}{2} \rfloor + \Sigma + 1 \leq R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$,

(iii) *If $n_1 > \Sigma_{i=2}^{t}(\lfloor \frac{n_i}{2} \rfloor - 1)$, then $n_1 + \Sigma_{i=2}^{t}(\lfloor \frac{n_i}{2} \rfloor - 1) \leq R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$,*

(iv) *If $2n_1 > \Sigma_{i=2}^{t}(n_i - 1)$, then $n_1 + \Sigma_{i=1}^{t}(n_i - 1) + 1 \leq R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t})$.*

Proof. Part (i) is clear. Part (ii) is a direct consequent of part (i) and Theorem 4.1. To see (iii), let $m = \Sigma_{i=2}^{t}(\lfloor \frac{n_i}{2} \rfloor - 1)$ and consider the graph $K_{n_1-1} \cup K_m$. Partition K_m into subsets $V_2, V_3, \ldots, V_t$ of size $\lfloor \frac{n_2}{2} \rfloor - 1, \lfloor \frac{n_3}{2} \rfloor - 1, \ldots, \lfloor \frac{n_t}{2} \rfloor - 1$, respectively. For $i = 2, 3, \ldots, t$, color the edges of $K_{n_1-1} \cup K_m$ having one end in V_i and another end in K_{n_1-1} by the i-th color and the remaining edges by color 1. Clearly this coloring of K_{n_1+m-1} contains no P_i in color i, which means that part (iii) holds. Part (iv) is a direct consequent of part (iii). □

The following theorem, gives an improvement of a result in [7], which follows from Theorem 2.3 and Lemma 4.2.

Theorem 4.3 *Assume that $\delta \in \{0, 1\}$ and Σ denotes $\Sigma_{i=1}^{t}(n_i - 1)$. Then*

$$R(P_{2n_1+\delta}, P_{2n_2}, \ldots, P_{2n_t}, P_k) = k + \Sigma,$$

where $k \geq \Sigma^2 + 2\Sigma + 3$ if $\delta = 0$ and $k \geq 2\Sigma^2 + 5\Sigma + 5$ otherwise.

In the following theorem, we give the exact value of some multicolor Ramsey number of paths with even number of vertices.

Theorem 4.4 *Let $n_1 \geq n_2 \geq \cdots \geq n_t \geq 2$ and m be positive integers. Also let Σ denote $\Sigma_{i=1}^{t}(n_i - 1)$. Then*

(i) $R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t}) = n_1 + \Sigma + 1$ *for* $2n_1 \geq (\Sigma - n_1 + 2)^2 + 2$,

(ii) $R(P_4, P_4, P_{2m}) = 2m + 2$ *for* $m \geq 2$,

(iii) $R(P_4, P_6, P_{2m}) = 2m + 3$ *for* $m \geq 3$,

(iv) $R(P_6, P_6, P_{2m}) = R(P_4, P_8, P_{2m}) = 2m + 4$ *for* $m \geq 14$.

Proof. (i) This part is a consequent of Theorem 4.3.

(ii) First we prove that $R(P_4, P_4, P_4) = 6$. By part (iv) of Lemma 4.2, $R(P_4, P_4, P_4) \geq 6$. For the upper bound, let the edges of K_6 be colored by green, blue and red colors and also let G^g be the graph induced by the green edges. Since $ex(6, P_4) = 6$, so we may assume that $|E(G^g)| \leq 6$. This implies that $\overline{G^g}$ contains either $K_{3,3}$ or K_5 as a subgraph. If $\overline{G^g}$ contains a copy of K_5, we can find a copy of P_4 in blue or red, since $R(P_4, P_4) = 5$. If $\overline{G^g}$ contains a copy of $K_{3,3}$, then it is easy to check that any two coloring of $K_{3,3}$ with colors blue and red contains a monochromatic copy of P_4. This means that $R(P_4, P_4, P_4) \leq 6$. For $m \geq 3$, the result follows from Corollary 3.9 and Lemma 4.2.

(iii) This part is a direct consequent of Corollary 3.20.

(iv) This part is an easy consequent of Corollary 2.5 and Lemma 4.2.

□

As mentioned before, it is proved that [7], $R(P_{n_1}, P_{n_2}, P_{n_3}) = n_1 + \lfloor \frac{n_2}{2} \rfloor + \lfloor \frac{n_3}{2} \rfloor - 2$ if $n_1 \geq 6(n_2 + n_3)^2$ and both n_2, n_3 are not odd numbers. This result can be obtained by Theorem 4.3. Theorem 4.3 shows that the lower bound in part (iii) of Lemma 4.2 is the exact value of the multicolor ramsey number $R(P_{n_1}, P_{n_2}, \ldots, P_{n_t})$ if at most one of $n_2, n_3, \ldots, n_t$ is odd and n_1 is sufficiently large. For the case $t = 4$, it seems that $R(P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}) \in \{r, r+1, r+2\}$, where $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 3$ and $r = n_1 + \lfloor \frac{n_2}{2} \rfloor + \lfloor \frac{n_3}{2} \rfloor + \lfloor \frac{n_4}{2} \rfloor - 3$. Anyway we end this paper by proposing the following conjecture, which gives the exact value of the multicolor Ramsey number of paths with even number of vertices.

Conjecture 1 *For positive integers* $n_1 \geq n_2 \geq \cdots \geq n_t \geq 2$, *we have*

$$R(P_{2n_1}, P_{2n_2}, \ldots, P_{2n_t}) = n_1 + \sum_{i=1}^{t} (n_i - 1) + 1.$$

Theorem 4.4, gives some evidences for this conjecture. We think the following conjecture is also true, which is a generalization of the previous conjecture.

Conjecture 2 *Let* $n_1 \geq n_2 \geq \cdots \geq n_t \geq 4$ *be positive integers such that at most one of* $n_2, n_3, \ldots, n_t$ *is odd. Then*

$$R(P_{n_1}, P_{n_2}, \ldots, P_{n_t}) = n_1 + \sum_{i=2}^{t} (\lfloor \frac{n_i}{2} \rfloor - 1).$$

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