Proof of the $(n/2 - n/2 - n/2)$ Conjecture for large n

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Abstract

A conjecture of Loebl, also known as the $(n/2 - n/2 - n/2)$ Conjecture, states that if G is an n-vertex graph in which at least $n/2$ of the vertices have degree at least $n/2$, then G contains all trees with at most $n/2$ edges as subgraphs. Applying the Regularity Lemma, Ajtai, Komlós and Szemerédi proved an approximate version of this conjecture. We prove it exactly for sufficiently large n. This immediately gives a tight upper bound for the Ramsey number of trees, and partially confirms a conjecture of Burr and Erdős.

1 Introduction

For a graph G, let $V(G)$ (or simply V) and $E(G)$ denote its vertex set and edge set, respectively. The *order* of G is $v(G) = |V(G)|$ or $|G|$, and the *size* of G is $e(G) = |E(G)|$ or $||G||$. For $v \in V$ and a set $X \subseteq V$, $N(v, X)$[1] represents the set of the neighbors of v in X, and $\deg(v, X) = |N(v, X)|$ is the degree of v in X. In particular $N(v) = N(v, V)$ and $\deg(v) = \deg(v, V)$.

Let G be a graph and T be a tree with $v(T) \le v(G)$. Under what condition must G contain T as a subgraph? Applying the greedy algorithm, one can easily derive the following fact.

Fact 1.1. *Every graph G with $\delta(G) = \min \deg(v) \ge k$ contains all trees T on k edges as subgraphs.*

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[1]We prefer $N(v, X)$ to the widely used notation $N_X(v)$ because we want to save the subscript for the underlying graph.

Extending Fact 1.1, Erdős and Sós [7] conjectured that the same holds when $\delta(G) \geq k$ is weakened to $a(G) > k-1$, where $a(G)$ is the average degree of G.

Conjecture 1.2 (Erdős-Sós). *Every graph on n vertices and with more than $(k-1)n/2$ edges contains, as subgraphs, all trees with k edges.*

This celebrated conjecture was open till the early 90's, when Ajtai, Komlós and Szemerédi [1] proved an approximate version by using the celebrated Regularity Lemma of Szemerédi [17].

Another way to strengthen Fact 1.1 is replacing $\delta(G)$ by the median degree of G. The $k = n/2$ case of this direction was conjectured by Loebl [8] and became known as the $(n/2 - n/2 - n/2)$ Conjecture (see [9] page 44).

Conjecture 1.3 (Loebl). *If G is a graph on n vertices, and at least $n/2$ vertices have degree at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

The general case was conjectured by Komlós and Sós [8].

Conjecture 1.4 (Komlós-Sós). *If G is a graph on n vertices, and at least $n/2$ vertices have degree at least k, then G contains, as subgraphs, all trees with at most k edges.*

Conjecture 1.4 is trivial for stars and was verified by Bazgan, Li and Woźniak [3] for paths. Applying the Regularity Lemma, Ajtai, Komlós and Szemerédi proved [2] an approximate version of Conjecture 1.3.

Theorem 1.5 (Ajtai-Komlós-Szemerédi). *For every $\rho > 0$ there is a threshold $n_0 = n_0(\rho)$ such that the following statement holds for all $n \geq n_0$: If G is a graph on n vertices, and at least $(1+\rho)n/2$ vertices have degree at least $(1+\rho)n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

The main goal of this paper is to prove Conjecture 1.3 *exactly* for sufficiently large n. Below we add floor and ceiling functions around $n/2$ to make the case when n is odd more explicit.

Theorem 1.6 (Main Theorem). *There is a threshold n_0 such that Conjecture 1.3 holds for all $n \geq n_0$. In other words, if G is a graph of order $n \geq n_0$, and at least $\lceil n/2 \rceil$ vertices have degree at least $\lceil n/2 \rceil$, then G contains, as subgraphs, all trees with at most $\lfloor n/2 \rfloor$ edges.*

It was shown in [2] that Conjecture 1.4 is best possible when $k+1$ divides n. But the sharpness of Conjecture 1.3 appears not to have been studied before. Clearly the $n/2$ as the degree condition cannot be weakened because T could be a star with $n/2$ edges. Is the other $n/2$, the number of large degree vertices, best possible? The following construction shows that this is essentially the case, more exactly, this $n/2$ cannot be replaced by $n/2 - \sqrt{n} - 2$.

Construction 1.7. *Let T be a tree with $n/2+1$ vertices distributed in 3 levels: the root has $n/4$ children, each of which has exactly one leaf. Let G be a graph such that $V(G) = V_1 + V_2$, $|V_1| = |V_2| = n/2$ and each $V_i = A_i + B_i$ with $|A_i| = n/4 - \sqrt{n}/2 - 1$. Each vertex $v \in A_i$ is adjacent to all other vertices in V_i and exactly one vertex in B_j for $j \neq i$. The $n/4 - \sqrt{n}/2 - 1$ edges between A_i and B_j make up $\sqrt{n}/2$ vertex-disjoint stars centered at B_j of size either $\sqrt{n}/2 - 1$ or $\sqrt{n}/2 - 2$.*

Clearly the $n/2 - \sqrt{n} - 2$ vertices in $A_1 \cup A_2$ have degree $n/2$. We claim that G does not contain T. In fact, by symmetry in G, we only consider two possible locations for the root r of T: A_1 or B_1. Suppose that r is mapped to some $u \in B_1$. Since $\deg(u) \leq |A_1| + \sqrt{n}/2 - 1 = n/4 - 2$, there is no room for the $n/4$ children of r. Suppose that r is mapped to some $u \in A_1$. Let m be the size of a largest family of paths of length 2 sharing only u (*u-2-paths*). There are two kinds of u-2-paths containing *no* vertices from $A_1 \setminus \{u\}$: u to B_1 to A_2, and u to B_2 to A_2. Since the size of a maximal matching between B_1 and A_2 is $\sqrt{n}/2$ and $\deg(u, B_2) = 1$, we conclude that $m \leq |A_1| - 1 + \sqrt{n}/2 + 1 = n/4 - 1$. Hence there is no room for the $n/4$ 2-paths in T.

Define $\ell(G) = |\{u \in V(G) : \deg(u) \geq v(G)/2\}|$. Denote by $\mathcal{T}_k$ the set of trees on k edges. We write $G \supset \mathcal{T}_k$ when the graph G contains *all* members of $\mathcal{T}_k$ as subgraphs. Conjecture 1.4 leads us to the following extremal problem. Let $m(n,k)$ be the smallest m such that every n-vertex graph G with $\ell(G) \geq m$ contains all trees on k edges, *i.e.*, $G \supset \mathcal{T}_k$. Conjecture 1.4 says that $m(n,k) \leq n/2$ for all $k < n$, in particular, Conjecture 1.3 says that $m(n, n/2) \leq n/2$. Theorem 1.6 confirms that $m(n, n/2) \leq n/2$ for $n \geq n_0$ while Construction 1.7 shows that $m(n, n/2) > n/2 - \sqrt{n} - 2$. At present, we do not know the exact value of $m(n, n/2)$ or $m(n,k)$ for most values of k.

When studying an extremal problem on graphs, researchers are also interested in the structure of graphs whose size is close to the extreme value. Let $\mathrm{ex}(n, F)$ be the usual Turán number of a graph F. The stability theorem of Erdős-Simonovits [16] from 1966 proved that n-vertex graphs without a fixed subgraph F with close to $\mathrm{ex}(n, F)$ edges have similar structures: they all look like the extremal graph. In this paper, though we can not determine $m(n, n/2)$ exactly, we are able to describe the structure of n-vertex graphs G with $\ell(G)$ about $n/2$ and $G \not\supset \mathcal{T}_{n/2}$.

Definition 1.8. *The half-complete graph H_n is a graph on n vertices with $V = V_1 + V_2$ such that $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$. The edges of H_n are all the pairs inside V_1 and between V_1 and V_2. In other words, $H_n = K_n - E(K_{\lceil n/2 \rceil})$.*

For a graph G and $k \in \mathbf{N}$, we denote by kG the graph that consists of k *disjoint* copies of G, in other words, $V(kG)$ has a partition $\cup_{i=1}^{k} V_i$ such that its induced subgraph on each V_i is isomorphic to G.

Theorem 1.9 (Stability Theorem)**.** *For every $\beta > 0$ there exist $\zeta > 0$ and $n_0 \in \mathbf{N}$ such that the following statement holds for all $n \geq n_0$: if a $2n$-vertex graph G with $\ell(G) \geq (1-\zeta)n$ does not contain some $T \in \mathcal{T}_n$, then $G = 2H_n \pm \beta n^2$,* i.e., *G can be transformed to two vertex-disjoint copies of H_n by changing at most βn^2 edges.*

The structure of the paper is as follows. In the next section we discuss the application of Theorem 1.6 on graph Ramsey theory. In Section 3 we outline the proof of Theorem 1.6, comparing it with the proof of Theorem 1.5, and define two extremal cases. Section 4 contains the Regularity Lemma and some properties of regular pairs. Section 5 contains a few embedding lemmas for tress and forests; an involved proof (of Lemma 5.4 Part 3) is left to the appendix. In Section 6 we extend the ideas in [2] to prove the non-extremal case, where Subsection 6.5 contains most of our new ideas and many technical details. The extremal cases are covered in Section 7, in which we also give the proof of Theorem 1.9. The last section contains some concluding remarks.

Notation: Let $[n] = \{1, 2, \ldots, n\}$. For two disjoint sets A and B we sometimes write $A + B$ for $A \cup B$. Let $G = (V, E)$ be a graph. If $U \subset V$ is a vertex subset, we write $G - U$ for $G[V \setminus U]$, the induced subgraph on $V \setminus U$. When $U = \{v\}$ is a singleton, we often write $G - v$ instead of $G - \{v\}$. For a subgraph H of G, we write $G - H$ for the subgraph of G obtained by removing all edges in H and all vertices $v \in V(H)$ that are only incident to edges of H.[2] Given two *not necessarily disjoint* subsets A and B of V, $e(A, B)$ denotes the number of *ordered* pairs (a, b) such that $a \in A$, $b \in B$ and $\{a, b\} \in E$. The *density* $d(A, B)$ between A and B and the minimum degree $\delta(A, B)$ from A to B are defined as follows:

$$d(A, B) = \frac{e(A, B)}{|A||B|}, \quad \delta(A, B) = \min_{a \in A} \deg(a, B).$$

Trees in this paper are always rooted (though we may change roots if necessary). Let T be a tree with root r. Then T is associated a partial order $<$ with r as the maximum element. In other words, for two distinct vertices x, y on T, we write $x < y$ if and only if y lies on the unique connecting r and x. For any vertex $x \neq r$, the *parent* $p(x)$ is the unique neighbor of x such that $x < p(x)$, the set of *children* is $C(x) = N(x) \setminus p(x)$. Furthermore, let $T(x)$ denote the subtree induced by $\{y : y \leq x\}$.

A forest F is a disjoint union of trees. We write $T \in F$ if the tree T is a component of F. The number of the components of F is denoted by $c(F)$. Hence $v(F) = e(F) + c(F)$. We partition the vertices of F by levels, namely, their distances to the roots such that $Level_i(F)$ denotes the set of vertices whose distance to the roots is i. In particular, we write $Rt(F) = Level_0(F)$, and $Rt(F)$ denotes the root (instead of the set of the root) if F is a tree. We also write $Level_{\geq i}(F) = \bigcup_{j \geq i} Level_j(F)$, $F_{even} = \bigcup Level_i(F)$ for all even i, and $F_{odd} = \bigcup Level_i(F)$ for all odd i. For a tree T, $T_{even} \cup T_{odd}$ is the unique bipartition of $V(T)$. A forest with c components has 2^{c-1} non-isomorphic bipartitions, which are determined by the location of its roots. Finally we define $Ratio(F) = |F_{odd}|/v(F)$.

For two graphs G and H, we write $H \to G$ if H can be embedded into G, *i.e.*, there is an injection $\phi : V(H) \to V(G)$ such that $\{\phi(u), \phi(v)\} \in E(G)$ whenever $\{u, v\} \in E(H)$. For $X \in V(H)$ and $A \subseteq V(G)$, $\phi(X)$ stands for the union of $\phi(x)$, $x \in X$. When $\phi : H \to G$ and $\phi(X) \subseteq A$, we write $X \to A$.

[2]This is not a standard notation: many researchers instead define $G - H := G - V(H)$.

2 Ramsey number of trees

An immediate consequence of Theorem 1.6 is a tight upper bound for the *Ramsey number* of trees. The Ramsey number $R(H)$ of a graph H is the minimum integer k such that every 2-edge-coloring of K_k yields a monochromatic copy of H. Let T be a tree on n vertices. What can we say about upper bounds for $R(T)$?

It is easy to see that $R(T) \leq 4n-3$. In fact, every 2-edge-coloring of K_{4n-3} yields a monochromatic graph G on $4n-3$ vertices with at least $\frac{1}{2}\binom{4n-3}{2}$ edges. Since every graph with average degree d contains a subgraph whose minimal degree is at least $d/2$, G contains a subgraph G' with minimal degree at least $(4n-4)/4 = n-1$. By Fact 1.1, G' thus contains a copy of T.

Burr and Erdős [5] made the following conjecture.[3]

Conjecture 2.1 (Burr-Erdős). *For every tree T on n vertices, $R(T) \leq 2n-2$ when n is even and $R(T) \leq 2n-3$ when n is odd.*

Note that [9] page 18 says that Burr and Erdős conjectured that $R(T) \leq 2n-2$, and [14] says that Loebl conjectured $R(T) \leq 2n$.

The bounds in Conjecture 2.1 are tight when T is a star on n vertices. For example, when n is even, there exists an $(n-2)$-regular graph G_1 on $2n-3$ vertices. Consequently the 2-edge-coloring K_{2n-3} with G_1 as the red graph contains no monochromatic star on n vertices.

It is easy to check that the Erdős-Sós Conjecture implies Conjecture 2.1. On the other hand, Conjecture 1.3 implies that $R(T) \leq 2n-2$. To see this, suppose a 2-edge-coloring partitions K_{2n-2} into two subgraphs G_1 and G_2. Then either G_1 contains at least $n-1$ vertices of degree at least $n-1$ or G_2 contains at least n vertices of degree at least $n-1$. Conjecture 1.3 thus implies that either G_1 or G_2 contains all trees of order n. Our main theorem (Theorem 1.6) therefore confirms Conjecture 2.1 for large even integers n.

Corollary 2.2. *If n is sufficiently large and T is a tree on n vertices, then $R(T) \leq 2n-2$.*

Given two graphs H_1, H_2, the asymmetric Ramsey number $R(H_1, H_2)$ is the minimum integer k such that every 2-edge-coloring of K_k by red and blue yields either a red H_1 or a blue H_2. Theorem 1.6 actually implies that for any two trees T', T'' on n vertices and sufficiently large n, $R(T', T'') \leq 2n-2$. Furthermore, the Komlós-Sós Conjecture implies that $R(T', T'') \leq m+n-2$, where T', T'' are arbitrary trees on n, m vertices, respectively.

Finally, when the bipartition of T is known, Burr conjectured [4] a upper bound for $R(T)$ which implies Conjecture 2.1, in terms of $|T_{even}|$ and $|T_{odd}|$. See [4, 10, 11] for progress on this conjecture.

3 Structure of our proofs

In this section we sketch the proofs of the main theorem and Theorem 1.9.

[3]This is a different conjecture from their well-known conjecture on Ramsey numbers for graphs with degree constraints.

Let us first recall the proof of Theorem 1.5. Given T and G as in Theorem 1.5, the authors of [2] first prepared T and G as follows: T is folded such that it looks like a *bi-polar* tree, namely, a tree having two vertices (called *poles*) under which all subtrees are small, and G is treated with the Regularity Lemma which yields a reduced graph G_r whose vertices represents the clusters of G. Then they applied the Gallai–Edmonds decomposition to G_r and found two clusters A, B of large degree and a matching M covering the neighbors of A and B. Finally they embedded the bi-polar version of T into $\{A, B\} \cup M$ and showed how to convert this embedding to an embedding of T in G.

The two ρ's in Theorem 1.5 are to compensate the following losses. Assume that ε, d, γ are some small positive numbers determined by ρ. After applying the Regularity Lemma with parameters ε, d, the degrees of the vertices of L are reduced by $(d+\varepsilon)n$. In addition, the regularity of a regular pair (A, B) only guarantees (by a corollary of Lemma 5.1) an embedding of a forest (consisting of small-size trees) of order $(1-\gamma)(|A|+|B|)$, instead of $|A|+|B|$. Clearly the above losses are unavoidable as long as the Regularity Lemma is applied. In other words, without these two ρ's, we can only expect to embed trees of size smaller than $v(G)/2$ by copying the proof of Theorem 1.5.

In order to prove Theorem 1.6 which contains no error terms, we have to study the structure of G more carefully and also consider the structure of T in order to find a series of sufficient conditions for embedding T in G. If none of these conditions holds, then G can be split into two equal parts such that between them, there exist either almost no edges or almost all possible edges. In such extremal cases, we show that all trees with n edges can be found in the original graph G without using the Regularity Lemma.

Without loss of generality, we may assume that the order of the host graph G is even. In fact, when $v(G) = 2k-1$, the assumption of Theorem 1.6 says that there are at least k vertices of degree at least k in G. After adding one isolated vertex to G, the new graph $\tilde{G}$ still has at least k vertices of degree at least k. If a tree (on k edges) can be found in $\tilde{G}$, then it must be a subgraph of G. **From now on we assume that G is a graph of order $2n$.**

Given $0 \le \alpha \le 1$, we define two *extremal cases*[4] *with parameter* α. We say that G is in Extremal Case 1 with parameter α if

EC1: *$V(G)$ can be evenly partitioned into two subsets V_1 and V_2 such that $d(V_1, V_2) \ge 1-\alpha$.*

We say that G is in Extremal Case 2 with parameter α if

EC2: *$V(G)$ can be evenly partitioned into two subsets V_1 and V_2 with $d(V_1, V_2) \le \alpha$.*

Note that if G is in **EC1** (or **EC2**) with parameter α, then G is in **EC1** (or **EC2**) with parameter x for any positive $x < \alpha$.

Our next two results show that $G \supset \mathcal{T}_n$, *i.e.*, G containing all trees on n edges if $\ell(G) \ge n$ and G is in either of the extremal cases.

[4]As noted by a referee, we may only define one extremal case since G is in **EC1** if and only if its complement $\bar{G}$ is in **EC2**.

Proposition 3.1. *For any $0 < \sigma < 1$, there exist $n_1 \in \mathbf{N}$ and $0 < c < 1$ such that the following holds for all $n \geq n_1$. Let G be a $2n$-vertex graph with $\ell(G) \geq 2\sigma n$. If G is in **EC1** with parameter c, then $G \supset \mathcal{T}_n$.*

Theorem 3.2. *There exist $\alpha_2 > 0$ and $n_2 \in \mathbf{N}$ such that the following holds for all $0 < \alpha \leq \alpha_2$ and $n \geq n_0$. Let G be a $2n$-vertex graph with $\ell(G) \geq n$. If G is in **EC2** with parameter α, then $G \supset \mathcal{T}_n$.*

To prove Theorem 1.6, we only need the $\sigma = 1/2$ case of Proposition 3.1. But Theorem 1.9 need the $\sigma < 1/2$ case. The core step in our proof is the following theorem, which describes the structure of hypothetical G with $\ell(G) \geq (1-\varepsilon)n$ and $G \not\supset \mathcal{T}_n$.

Theorem 3.3. *For every $\alpha > 0$ there exist $\varepsilon > 0$ and $n_3 = n_3(\alpha) \in \mathbf{N}$ such that the following statement holds for all $n \geq n_0$: if a $2n$-vertex graph G with $\ell(G) \geq (1-\varepsilon)n$ does not contain some $T \in \mathcal{T}_n$, then G is in either of the two extremal cases with parameter α.*

Similarly, to prove Theorem 1.6, we only need to prove Theorem 3.3 under the stronger assumption $\ell(G) \geq n$. This general Theorem 3.3 is necessary for the proof of Theorem 1.9 and becomes useful if one wants to show that $G \supset \mathcal{T}_n$ under a (slightly) smaller value of $\ell(G)$.

Proof of Theorem 1.6. Let n_1, c be given by Proposition 3.1 with $\sigma = 1/2$. Let α_2, n_2 be given by Theorem 3.2. We let $\alpha := \min\{c, \alpha_2\}$, and let $n_3 = n_3(\alpha)$ be given by Theorem 3.3. Finally set $n_0 := \max\{n_1, n_2, n_3\}$.

Now let G be a graph of order $2n$ with $\ell(G) \geq n$ for some $n \geq n_0$. By Theorem 3.3, either $G \supset \mathcal{T}_n$ or G is in either of the two extremal cases with parameter α. If G is in **EC1** with parameter $\alpha \leq c$, then Proposition 3.1 (with $\sigma = 1/2$) implies that $G \supset \mathcal{T}_n$. If G is in **EC2** with parameter $\alpha \leq \alpha_2$, then Theorem 3.2 implies that $G \supset \mathcal{T}_n$. We thus have $G \supset \mathcal{T}_n$ in all cases. □

We will prove our stability result (Theorem 1.9) in Section 7.2. It easily follows from Proposition 3.1, Theorem 3.3, and Lemma 7.4, where Lemma 7.4 is also the main step in the proof of Theorem 3.2.

4 Regular pairs and the Regularity Lemma

In this section we state the Regularity Lemma along with some properties of regular pairs. Recall for two vertex sets A, B in a graph, $d(A,B) = e(A,B)/(|A||B|)$.

Definition 4.1. *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-*regular *(*regular *if ε is clear from the context) if for every $X \subseteq A$ and $Y \subseteq B$, satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X,Y) - d(A,B)| < \varepsilon$.*

We use the following version of the Regularity Lemma from [13].

Lemma 4.2 (Regularity Lemma - Degree Form)**.** *For every $\varepsilon > 0$ there is an $M(\varepsilon)$ such that if $G = (V,E)$ is any graph and $d \in [0,1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ partition sets $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- *$\ell \le M(\varepsilon)$,*
- *$|V_0| \le \varepsilon|V|$; all clusters V_i, $i \ge 1$, are of the same size $N \le \varepsilon|V|$,*
- *$\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)|V|$ for all $v \in V$,*
- *V_i, $i \ge 1$, is an independent set in G',*
- *all pairs (V_i, V_j), $1 \le i < j \le \ell$, are ε-regular in G', each with density either 0 or greater than d.*

Like in many other problems to which the Regularity Lemma is applied, it suffices to consider the subgraph $G'' = G' - V_0$ as the underlying graph except for the extremal case. We therefore skip the subscript G'' unless we consider G'' and G at the same time. Let $V' = V \setminus V_0$ denote the vertex set of $V(G'')$.

Given two vertex sets X and Y, recall that $\delta(X,Y) = \min_{v\in X} \deg(v,Y)$ denotes the minimum degree from X to Y. We now define the average degree from X to Y as

$$\overline{\deg}(X,Y) = \frac{1}{|X|}e(X,Y) = d(X,Y)\,|Y|.$$

Note the *asymmetry* of $\delta(X,Y)$ and $\overline{\deg}(X,Y)$. When $X = \{v\}$, we have $\overline{\deg}(v,Y) = \deg(v,Y)$. Finally we let $\overline{\deg}(X) = \overline{\deg}(X,V')$.

We call $V_1,\dots,V_\ell$ *clusters*. Denote by $\mathcal{V}$ the family of all the clusters and use capital letters X, Y, A, B for elements of $\mathcal{V}$. For $X, Y \in \mathcal{V}$, if $d(X,Y) \ne 0$, *i.e.*, $d(X,Y) > d$, then we write $X \sim Y$ and call $\{X,Y\}$ a *non-trivial* regular pair.

Definition 4.3. *After applying the Regularity Lemma to G, we define the reduced graph G_r as follows: the vertices are $1 \le i \le \ell$, which correspond to clusters V_i, $1 \le i \le \ell$, and for $1 \le i < j \le \ell$ there is an edge between i and j if $V_i \sim V_j$.*

For a cluster $X = V_i \in \mathcal{V}$, we may abuse our notation by writing $\deg_{G_r}(X)$ or $N(X)$ instead of $\deg_{G_r}(i)$ or $N_{G_r}(i)$. The degree of X, $\overline{\deg}(X)$ and $\deg_{G_r}(X)$ have the following relationship

$$\overline{\deg}(X) = \frac{1}{|X|}e(X,V) = \sum_{Y\in\mathcal{V}, Y\sim X} d(X,Y)N \le \sum_{Y\in\mathcal{V}, Y\sim X} N = \deg_{G_r}(X)\,N. \tag{4.1}$$

Definition 4.4.
- *Given an ε-regular pair (A,B), a vertex $u \in A$ is called ε-*typical *(*typical *if ε is clear from the context) to a set $Y \subseteq B$ if $\deg(u,Y) > (d(A,B)-\varepsilon)|Y|$.*
- *Given a cluster $A \in \mathcal{V}$ and a family of clusters $\mathcal{S} \subseteq \mathcal{V}$, a vertex $u \in A$ is called* typical *to a family $\mathcal{Y} = \{Y \subseteq B : B \in \mathcal{S}\}$ if u is typical to all but at most $\sqrt{\varepsilon}|\mathcal{Y}|$ sets of $\mathcal{Y}$.*
- *In earlier cases we say u is* atypical *to Y or $\mathcal{Y}$ otherwise.*

One immediate consequence of (A, B) being regular is that all but at most $\varepsilon|A|$ vertices $u \in A$ are typical to any subset Y of B with $|Y| > \varepsilon|B|$. In the following proposition, Part 1 says that for any $A \in \mathcal{V}$ and family $\mathcal{Y} = \{Y \subseteq V_i : V_i \in \mathcal{V}, |Y| > \varepsilon N\}$, most vertices in A are typical to $\mathcal{Y}$. As a corollary of Part 1, Part 2 says that the degree of a cluster is about the same as the degree of most vertices in the cluster.

Proposition 4.5. *Suppose that $V_1, V_2, \dots, V_\ell$ are obtained from Lemma 4.2 and $n' = |V'|$. Let $i_0 \in [\ell]$, $I \subseteq [\ell] \setminus \{i_0\}$ and $Y_I = \cup_{i\in I} Y_i$, where each Y_i is a subset of V_i containing at least εN vertices. For every $u \in V_{i_0}$ we define*

$$I_u = \{i \in I : \deg(u, Y_i) \le (d(V_{i_0}, V_i) - \varepsilon)|Y_i|\}.$$

Then the following statements hold:

1. *All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy $|I_u| \le \sqrt{\varepsilon}|I|$.*
2. *All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy*

$$\deg(u, Y_I) > \overline{\deg}(V_{i_0}, Y_I) - (2\varepsilon + \sqrt{\varepsilon})N|I| \ge \overline{\deg}(V_{i_0}, Y_I) - 2\sqrt{\varepsilon}n'.$$

All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy $\deg(u, Y_I) < \overline{\deg}(V_{i_0}, Y_I) + 2\sqrt{\varepsilon}n'$.

Proof. *Part 1.* Suppose instead, that $|\{u \in V_{i_0} : |I_u| > \sqrt{\varepsilon}|I|\} > \sqrt{\varepsilon}N$. Then

$$\sum_{i\in I} |\{u \in V_{i_0} : i \in I_u\}| = \sum_{u\in V_{i_0}} |I_u| > \sqrt{\varepsilon}N\sqrt{\varepsilon}|I| = \varepsilon N|I|.$$

Therefore we can find $i_1 \in I$ such that $|S| > \varepsilon N$ for $S = \{u \in V_{i_0} : i_1 \in I_u\}$. By the definition of I_u, we have

$$d(S, Y_{i_1}) = \sum_{u\in S} \frac{\deg(u, Y_{i_1})}{|S||Y_{i_1}|} \le d(V_{i_0}, V_{i_1}) - \varepsilon,$$

which contradicts the regularity between V_{i_0} and V_{i_1}.

Part 2. For every $u \in V_{i_0}$,

$$\begin{aligned}
\deg(u, Y_I) &\ge \sum_{i\notin I_u} \deg(u, Y_i) > \sum_{i\notin I_u} (d(V_{i_0}, V_i) - \varepsilon)|Y_i| > \sum_{i\notin I_u} (d(V_{i_0}, Y_i) - 2\varepsilon)|Y_i| \\
&= \sum_{i\in I} d(V_{i_0}, Y_i)|Y_i| - \sum_{i\in I_u} d(V_{i_0}, Y_i)|Y_i| - 2\varepsilon \sum_{i\notin I_u} |Y_i| \\
&\ge \overline{\deg}(V_{i_0}, Y_I) - \sum_{i\in I_u} |V_i| - 2\varepsilon N|I|.
\end{aligned}$$

According to Part I, all but $\sqrt{\varepsilon}N$ vertices of V_{i_0} further satisfy

$$\deg(u, Y_I) > \overline{\deg}(V_{i_0}, Y_I) - \sqrt{\varepsilon}N|I| - 2\varepsilon N|I| > \overline{\deg}(V_{i_0}, Y_I) - 2\sqrt{\varepsilon}n'.$$

The second claim can be proved similarly. □

5 Lemmas on embedding (small) trees and forests

In this section we give a few technical lemmas that embed trees or forests into G'', the resulting subgraph of G after we apply the Regularity Lemma. Some of these lemmas (or their variations) appeared in [2] with very brief proofs. The reason why we state and (re)prove them is to make them applicable under new assumptions (the readers who are familiar with [2] may want to skip this section first).

Throughout this section, we assume that $0 < \varepsilon \ll \gamma \ll d < 1$. Let N be an integer such that $\varepsilon N \geq 1$. Let $\mathcal{V}$ be a family of clusters of size N such that any two clusters of $\mathcal{V}$ form a regular pair with density either 0 or greater than d.

One advantage of a regular pair is that regardless of its density, it behaves like a complete bipartite graph when we embed many small trees in it. This follows from repeatedly applying the following fundamental lemma, which gives an *online* embedding algorithm (embedding vertices one by one, without having the entire input available from the start). Let us first introduce a notation to represent the flexibility of such an embedding. Suppose that an algorithm embeds the vertices of a graph H_1 one by one into another graph H_2. For a vertex $x \in V(H_1)$, a real number $p \neq 0$ and a set $A \subseteq V(H_2)$, we write $x \xrightarrow{p} A$ to indicate the flexibility of the embedding. When $p > 0$, it means that (at the moment when we consider x), our algorithm allows at least p vertices of A to be the image of x. When $p = -q < 0$, it means that all but at most q vertices of A can be chosen as the image of x. Note that no matter which of these vertices we finally select as the image of x, we can always embed the remaining vertices of H_1 (with corresponding flexibility). Such a flexibility is needed in Lemma 6.3 when we connect several forests into a tree. For a set $S \subseteq V(H_1)$, we write $S \xrightarrow{p} A$ if $S \rightarrow A$ and $x \xrightarrow{p} A$ for every $x \in S$.

Lemma 5.1. *Let $X, Y \in \mathcal{V}$ be two clusters such that $X \sim Y$, namely, (X, Y) is regular with $d(X, Y) \geq d$. Suppose that $X_0, X_1 \subset X$, $Y_1 \subset Y$ satisfy $|X_0| \geq 3\varepsilon N$, $|X_1| \geq \gamma N$, $|Y_1| \geq \gamma N$. Then for any tree T of order εN with root r, there exists an online algorithm embedding $V(T)$ into $X_0 \cup X_1 \cup Y_1$ such that $r \xrightarrow{2\varepsilon N} X_0$, $T_{even} \setminus \{r\} \xrightarrow{2\varepsilon N} X_1$, and $T_{odd} \xrightarrow{2\varepsilon N} Y_1$.*

Proof. First we embed r to a typical vertex $u \in X_0$ such that $\deg(u, Y_1) \geq (d(X, Y) - \varepsilon)|Y_1|$. Since at most εN vertices of X are atypical to Y_1 and $|X_0| \geq 3\varepsilon N$, at least $2\varepsilon N$ vertices of X_0 can be chosen as u.

We now embed $D_i := Level_i(T), i \geq 1$ into $X_1 \cup Y_1$. Suppose that $D_1, \ldots, D_{i-1}$ have been embedded to X_1 and Y_1 by a function ϕ with the following property. When $j < i$ is even, D_j is embedded to X_1 such that $\deg(\phi(x), Y_1) > (d - \varepsilon)|Y_1|$ for every $x \in D_j$; when $j < i$ is odd, D_j is embedded to Y_1 such that $\deg(\phi(y), X_1) > (d - \varepsilon)|X_1|$ for every $y \in D_j$. Below we assume that D_{i-1} is embedded into X_1. Consider the vertices in D_i in any order. Let $y \in D_i$ and assume that $x = p(y) \in D_{i-1}$. We want to embed y to an unoccupied vertex $u \in N(\phi(x), Y_1)$ which is typical to X_1, *i.e.*, $\deg(u, X_1) > (d - \varepsilon)|X_1|$. If this is possible, this process may continue for all levels. By the regularity between X and Y, at most εN vertices in Y_1 are atypical to X_1 (note that $|X_1| \geq \gamma N > \varepsilon N$). On the other hand, at most $(\sum_{j \leq i} |D_i|) - 1$ vertices of Y_1 may already be occupied. The following inequality thus guarantees that at least $2\varepsilon N$ vertices can be chosen as u:

$$(d-\varepsilon)|Y_1| - \varepsilon N - \left(\sum_{j\le i} |D_i|\right) + 1 \ge 2\varepsilon N.$$

It suffices to have $(d-\varepsilon)|Y_1| \ge v(T) + 3\varepsilon N$. This holds because $|Y_1| \ge \gamma N$, $v(T) \le \varepsilon N$ and $\varepsilon \ll \gamma \ll d$.[5] □

The following variant of Lemma 5.1 is needed for the proof of Lemma 5.9.

Lemma 5.2. *Let X, Y, Z be three clusters such that $X \sim Y$ and $X \sim Z$. Suppose $X_0, X_1 \subseteq X$, $Y_1 \subseteq Y$, and $Z_1 \subseteq Z$ are subsets of sizes $|X_0| \ge 5\varepsilon N$, $|X_1|, |Y_1|, |Z_1| \ge \gamma N$. Then any forest F of order at most εN can be embedded into $X_0 \cup X_1 \cup Y_1 \cup Z_1$ such that $Rt(F) \xrightarrow{2\varepsilon N} X_0$, $F_{even} \setminus Rt(F) \xrightarrow{2\varepsilon N} X_1$, and each $y \in F_{odd}$ can be mapped to either Y_1 or Z_1, each with flexibility $2\varepsilon N$.*

Proof. We follow the proof of Lemma 5.1 and only elaborate on what is different here. We embed each $r \in Rt(F)$ to an unoccupied vertex $u \in X_0$ that is typical to Y_1 and Z_1. Since at most $2\varepsilon N$ vertices of X are atypical to either Y_1 or Z_1, $v(F) \le \varepsilon N$, and $|X_0| \ge 5\varepsilon N$, at least $2\varepsilon N$ vertices of X_0 can be chosen as u. Suppose $D_0, \dots, D_{i-1}$ have been embedded for some $i \ge 1$ and we need to embed D_i. When i is even, we map every $x \in D_i$ to an unoccupied vertex in X_1 that is typical to both Y_1 and Z_1. As long as $(d-\varepsilon)|X_1| \ge v(T) + 4\varepsilon N$, at least $2\varepsilon N$ vertices of X_1 may be chosen as the image of x. When i is odd, for each $y \in D_i$, since its parent $p(y) \in D_{i-1}$ has been mapped to a vertex that is typical to Y_1 and Z_1, we can map y to either Y_1 or Z_1, up to our choice. Since $(d-\varepsilon)\gamma N \ge v(T) + 3\varepsilon N$, at least $2\varepsilon N$ vertices of Y_1 and at least $2\varepsilon N$ vertices of Z_1 can be chosen as the image of y. □

Recall that $T(x)$ denotes the maximal subtree in a rooted tree T containing a vertex x but not its parent $p(x)$.

Definition 5.3. *Let $m > 0$ be a real number.*

- *A tree T with root r is called an m-tree if $v(T(x)) \le m$ for every $x \ne r$.*
- *A forest F is called an m-forest if all the components of F are m-trees. An ordered m-forest is an m-forest with an ordered $Rt(F)$, in other words, it is a sequence of m-trees.*

Let C, X, Y be three distinct clusters in $\mathcal{V}$ with $X \sim Y$. Let F be an ordered εN-forest. We write $F \to (C, \{X, Y\})$ if there exists an online algorithm embedding the trees of F in order such that $Rt(F) \overset{-3\varepsilon N}{\to} C$ and $F - Rt(F) \overset{2\varepsilon N}{\to} \{X, Y\}$, which means that $v \overset{2\varepsilon N}{\to} X$ or $v \overset{2\varepsilon N}{\to} Y$ for every $v \in V(F) \setminus Rt(F)$.

Given an εN-forest F, our first lemma gives three sufficient conditions for $F \to (C, \{X, Y\})$. The flexibility of the embedding will allow us to connect F into a tree

[5] For example, assuming $8\varepsilon < \gamma^2 < \gamma < d$ we have $(d-\varepsilon)\gamma > \frac{d}{2}\gamma > \frac{\gamma^2}{2} > 4\varepsilon$.

later. The most general case, Part 1, was proved in [2] and sufficed for their purpose. Recall that $||F||$ is the number of edges in a forest F, which equals to the number of vertices in $F - Rt(F)$. The ratio of a tree T is $|T_{odd}|/|T|$.

Lemma 5.4. *Let C, X, Y be three distinct clusters in $\mathcal{V}$ with $X \sim Y$. Write $d_x = d(C, X)$, $d_y = d(C, Y)$. Let F be an ordered εN-forest with $s \leq \varepsilon N$ components. Then $F \to (C, \{X, Y\})$ if either of the following cases holds. Furthermore, the first root in F can be embedded into* any *vertex $u \in C$ that is typical to both X and Y.*

1. *$||F|| \leq (d_x + d_y - 2\gamma - 2\varepsilon)N$.*
2. *Every tree in $F - Rt(F)$ has ratio between c and $1-c$ (inclusively) for some $0 \leq c \leq \frac{1}{2}$ and $||F|| \leq (d_x + d_y - 2\gamma - 3\varepsilon)N + \frac{c}{1-c}|d_y - d_x|N$.*
3. *Every tree in $F - Rt(F)$ contains at least two vertices, and there exists $0 \leq \lambda \leq \frac{1}{2}$ such that $\lambda \leq \{d_x, d_y\} \leq 1 - \lambda$, and $||F|| \leq (d_x + d_y + \lambda - 2\gamma - 13\varepsilon)N$.*

Proof. We present proofs of Part 1 and Part 2 here, and leave the proof of Part 3 to the appendix due to its complexity.

Without loss of generality, assume that $d_x \leq d_y$. We also assume that $d_y > 0$ otherwise there is nothing to prove. We will embed trees in F in order. For the ith tree in F, we map its root r_i to an unoccupied vertex $u_i \in C$ that is typical to both[6] X and Y. In other words, $\deg(u_i, X) > (d_x - \varepsilon)N$ and $\deg(u_i, Y) > (d_y - \varepsilon)N$. By the regularity of (C, X) and (C, Y), all but at most $2\varepsilon N + s \leq 3\varepsilon N$ can be chosen as u_i.

Let $F^o = F - Rt(F)$. Then $v(F^o) = v(F) - |Rt(F)| = ||F||$. Following the order of $Rt(F)$, we may regard F^o as a sequence $\{T_1, \ldots, T_t\}$ such that $T_1, \ldots, T_{i_1}$ are under the first root, $T_{i_1+1}, \ldots, T_{i_2}$ are under the second root of F, etc. Since F is an εn-forest, each T_i has at most εN vertices. We claim that it suffices to show that F^o has a bipartition[7] (A, B) satisfying the following properties.

(I). $|A|, |B| \leq (d_y - \gamma)N$.

There exists $0 \leq i_0 \leq t$ such that

(II). $|A_i|, |B_i| \leq (d_x - \gamma)N$ for $i \leq i_0$, where $A_i = A \cap (V(T_1) \cup \cdots \cup V(T_i))$ and $B_i = B \cap (V(T_1) \cup \cdots \cup V(T_i))$.

(III). $Rt(T_i) \in B$ for $i > i_0$.

Note that (II) forces $i_0 = 0$ whenever $d_x = 0$. If such a bipartition (A, B) exists, we can sequentially embed $T_1, \ldots, T_t$ such that A is mapped to X and B is mapped to Y as follows. Let $i \geq 1$. Suppose that $T_1, \ldots, T_{i-1}$ have been embedded, and the root $r \in Rt(F)$ that is adjacent to $Rt(T_i)$ has been embedded to a typical vertex $u \in C$. Let X^*, Y^* denote the set of unoccupied vertices in X, Y, respectively, and P the set of available vertices in $N(u, X)$ (in $N(u, Y)$) if $Rt(T_i) \in A$ ($Rt(T_i) \in B$). In order to embed T_i by Lemma 5.1, we need to verify that $|X^*|, |Y^*| \geq \gamma N$ and $|P| \geq 3\varepsilon N$. From (I),

[6] If $d_x = 0$, then all vertices $u \in C$ are typical to X because $\deg(u, X) \geq 0 > -\varepsilon N$.

[7] This means that there is a partition $V(F^o) = A \cup B$ such that A, B are independent.

$|A|,|B| \le (d_y-\gamma)N \le (1-\gamma)N$, thus we immediately obtain that $|X^*|,|Y^*| \ge \gamma N$. When $i \le i_0$ (then $d_x > 0$), since u is typical to X and Y, by (II), we have

$$|P| \ge \begin{cases} \deg(u,X) - |A_i| > (d_x-\varepsilon)N - (d_x-\gamma)N > 3\varepsilon N & \text{if } P \subseteq X; \\ \deg(u,Y) - |B_i| > (d_y-\varepsilon)N - (d_x-\gamma)N > 3\varepsilon N & \text{if } P \subseteq Y. \end{cases}$$

When $i > i_0$, by (III), we have $|P| \ge \deg(u,Y) - |B| > (d_y-\varepsilon)N - (d_y-\gamma)N > 3\varepsilon N$. Finally, the embedding provided by Lemma 5.1 guarantees that $v \overset{2\varepsilon N}{\to} X$ or $v \overset{2\varepsilon N}{\to} Y$ for every $v \in V(T_i)$.

We now show that a bipartition satisfying (I)-(III) always exists under the hypothesis of Parts 1 and 2.

Part 1. Starting with $A'_0 = B'_0 = \emptyset$, we inductively obtain a bipartition (A'_i, B'_i) of $T_1 \cup \cdots \cup T_i$ for $i = 1, \ldots, t$ such that $||A'_i| - |B'_i|| < \varepsilon N$ and $|A'_i| \ge |B'_i|$. Suppose that such a bipartition exists for some $i \ge 0$, and assume that $|(T_{i+1})_{even}| \ge |(T_{i+1})_{odd}|$ (the other case is analogous). Let A'_{i+1} be the larger of the two sets $A'_i \cup (T_{i+1})_{odd}$ and $B'_i \cup (T_{i+1})_{even}$, and let B'_{i+1} be the smaller one. Then

$$0 \le |A'_{i+1}| - |B'_{i+1}| = \Big| |A'_i| - |B'_i| - \big(|(T_{i+1})_{even}| - |(T_{i+1})_{odd}|\big) \Big|.$$

Since both $|A'_i| - |B'_i|$ and $|(T_{i+1})_{even}| - |(T_{i+1})_{odd}|$ are non-negative and less than εN, we have $||A'_{i+1}| - |B'_{i+1}|| < \varepsilon N$.

Let i_0 be the largest index such that $|A'_i| \le (d_x-\gamma)N$. We let

$$A := A'_{i_0} \cup \bigcup_{i>i_0} (T_i)_{odd} \quad \text{and} \quad B := B'_{i_0} \cup \bigcup_{i>i_0} (T_i)_{even}.$$

Clearly (III) holds. Since $|B'_{i_0}| \le |A'_{i_0}| \le (d_x-\gamma)N$ and $\{A_i, B_i\} = \{A'_i, B'_i\}$ for $i \le i_0$, (II) also holds. It remains to verify (I): $|A|,|B| \le (d_y-\gamma)N$. If $i_0 = t$, then $|B| \le |A| < (d_x-\gamma)N \le (d_y-\gamma)N$, as desired. Otherwise assume $i_0 < t$. We first show that

$$|A'_{i_0}| > (d_x-\gamma-\varepsilon)N, \quad \text{and} \quad |B'_{i_0}| > (d_x-\gamma-2\varepsilon)N. \tag{5.1}$$

For instead, that $|A'_{i_0}| \le (d_x-\gamma-\varepsilon)N$ (then $|B'_{i_0}| \le (d_x-\gamma-\varepsilon)N$ as well). The definition of A'_{i_0+1} implies that $|A'_{i_0+1}| \le (d_x-\gamma-\varepsilon)N + \varepsilon N \le (d_x-\gamma)N$, contradicting the maximality of i_0. Assuming $|A'_{i_0}| > (d_x-\gamma-\varepsilon)N$, we obtain $|B'_{i_0}| \ge (d_x-\gamma-2\varepsilon)N$ from $|A'_{i_0}| - |B'_{i_0}| < \varepsilon N$.

By (5.1), we have $|A| \ge |A'_{i_0}| \ge (d_x-\gamma-\varepsilon)N$. By assumption, we have $|A|+|B| = v(F^o) = ||F|| \le (d_x+d_y-2\gamma-2\varepsilon)N$. Consequently $|B| \le (d_y-\gamma-\varepsilon)N$. On the other hand, using $|B'_{i_0}| \ge (d_x-\gamma-2\varepsilon)N$, we obtain that $|A| \le (d_y-\gamma)N$.

Part 2. Let us first rewrite the assumption on $||F||$ as

$$||F|| \le (2d_x-2\gamma-3\varepsilon)N + \frac{1}{1-c}(d_y-d_x)N. \tag{5.2}$$

We follow the same bipartition of F as in Part 1. Again it suffices to show that $|A|,|B| \le (d_y-\gamma)N$. First consider the $i_0 = t$ case. We have $0 \le |A|-|B| < \varepsilon N$ in this case. Since

$|A| + |B| = v(F^o) = ||F||$, it follows that $|A| \le (||F|| + \varepsilon N)/2$. Using (5.2) and $c \le 1/2$, we derive that

$$||F|| \le (2d_x - 2\gamma - 3\varepsilon)N + 2(d_y - d_x)N = (2d_y - 2\gamma - 3\varepsilon)N,$$

which implies that $|A| \le (d_y - \gamma - \varepsilon)N$.

When $i_0 < t$, (5.1) holds. Let $A' = A - A'_{i_0}$ and $B' = B - B'_{i_0}$. By (5.1) and (5.2), we have $|A'| + |B'| \le \frac{1}{1-c}(d_y - d_x)N$. Since (A', B') is a bipartition of a forest of trees of ratio between c and $1 - c$, it follows that

$$\max\{|A'|, |B'|\} \le (1 - c)(|A'| + |B'|) \le (d_y - d_x)N.$$

Together with $|B'_{i_0}| \le |A'_{i_0}| \le (d_x - \gamma)N$, we have $\max\{|A|, |B|\} \le (d_x - \gamma + d_y - d_x)N = (d_y - \gamma)N$, as desired. □

Definition 5.5. *1. A* cluster-matching *is a family $\mathcal{M}$ of disjoint regular pairs in $\mathcal{V}$. The set of the clusters covered by $\mathcal{M}$ is denoted by $V(\mathcal{M})$ (hence the* size $|\mathcal{M}|$ *of $\mathcal{M}$ is the half of $|V(\mathcal{M})|$).*

2. For a cluster $A \in \mathcal{V}$, we define $\overline{\deg}(A, \mathcal{M}) = \sum_{X \in V(\mathcal{M})} \overline{\deg}(A, X)$ to be the (average) degree of A to $\mathcal{M}$.

3. For $e = \{X, Y\} \in \mathcal{M}$, a cluster A and a vertex u, we simply write $\overline{\deg}(A, e)$ as $\overline{\deg}(A, X) + \overline{\deg}(A, Y)$, $d(A, e)$ as $d(A, X) + d(A, Y)$, and $\deg(u, e)$ as $\deg(u, X) + \deg(u, Y)$.

Let $\mathcal{M}$ be a cluster-matching, A be a cluster not in $V(\mathcal{M})$, F be an ordered εN-forest. We write $F \stackrel{p}{\to} (A, \mathcal{M})$ if there is an online algorithm embedding the trees in F to $A \cup \bigcup_{C \in V(\mathcal{M})} C$ in order such that $Rt(F) \stackrel{p}{\to} A$ and $F - Rt(F) \xrightarrow{2\varepsilon N} \mathcal{M}$, which means that for each tree T in $F - Rt(F)$ there exists $\{X, Y\} \in \mathcal{M}$ such that for each vertex $v \in V(T)$, either $v \stackrel{2\varepsilon N}{\to} X$, or $v \stackrel{2\varepsilon N}{\to} Y$. We simply write $F \to (A, \mathcal{M})$ if $p = -2\sqrt{\varepsilon}N$.

Definition 5.6. *1. A subtree of a tree T is called a* root-subtree *if it is obtained from T by removing $\{T(x) : x \in C\}$ for some subset $C \subseteq Level_1(T)$. We call the root-subtree with only one vertex (the root)* trivial.

2. A root-subforest F' of a forest F consists of root-subtrees of some trees in F. Formally, if $F = \{T_1, \ldots, T_s\}$, then $F' = \{T'_i : i \in I\}$, where T'_i is a root-subtree of T_i and I is a subset of $[s]$.

3. In a forest F, two root-subforests F' and F'' form a root-partition *of F if $E(F') \cup E(F'')$ is a partition of $E(F)$ (this implies that $V(F') \cap V(F'') \subseteq Rt(F)$).*

The following proposition says that if an εN-forest F has a root-partition $F_1 \cup F_2$ such that F_1 and F_2 can be embedded into A and two disjoint matchings[8] $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, then F can be embedded into $(A, \mathcal{M}_1 \cup \mathcal{M}_2)$ under a slightly weaker flexibility.

[8]Two matchings are *disjoint* if they have no vertex in common.

Proposition 5.7. *Let F be an ordered εN-forest with $c(F) \le \varepsilon N$. Let $\mathcal{M}_0$, $\mathcal{M}_1$ be two disjoint cluster-matchings and A be a cluster not in $V(\mathcal{M}_0 \cup \mathcal{M}_1)$. If there is a root-partition $F_0 \cup F_1$ of F such that $F_0 \to (A, \mathcal{M}_0)$, $F_1 \to (A, \mathcal{M}_1)$, then $F \overset{-4\sqrt{\varepsilon}N}{\longrightarrow} (A, \mathcal{M}_0 \cup \mathcal{M}_1)$.*

Proof. For $j = 0, 1$, let ϕ_j be the function which embeds $Rt(F_j) \overset{-2\sqrt{\varepsilon}N}{\longrightarrow} A$ and $F_j - Rt(F_j) \overset{2\varepsilon N}{\longrightarrow} \mathcal{M}_j$. We sequentially embed the trees in F by following ϕ_0 and ϕ_1. Consider the ith tree T in F. Let T_0, T_1 be the restriction of F_0, F_1 on $V(T)$, respectively. If say, T_0 is the empty graph, then we embed T by ϕ_1 but need to avoid the images of $Rt(F_0)$ when embedding $Rt(T)$. Since $|Rt(F_0)| \le \varepsilon N$ and $Rt(F_1) \overset{-2\sqrt{\varepsilon}N}{\longrightarrow} A$, all but at most $\varepsilon N + 2\sqrt{\varepsilon}N < 4\sqrt{\varepsilon}N$ vertices of A can be chosen as the image of $Rt(T)$. Otherwise both T_0 and T_1 contain $Rt(T)$. Since $Rt(F_0) \overset{-2\sqrt{\varepsilon}N}{\longrightarrow} A$ and $Rt(F_1) \overset{-2\sqrt{\varepsilon}N}{\longrightarrow} A$, all but at most $4\sqrt{\varepsilon}N$ vertices of A can be chosen as the image of $Rt(T)$. Since $\mathcal{M}_0$ and $\mathcal{M}_1$ are disjoint, the rest of T can be embedded by simply following ϕ_0 or ϕ_1. $\square$

The following lemma is the most important one in this section; in particular, Part 1 will be frequently used in Section 6. Its three parts follow from the three parts in Lemma 5.4.

Lemma 5.8. *Suppose that $\mathcal{M}$ is a cluster-matching of size m and A is a cluster not in $V(\mathcal{M})$. Let F be an ordered εN-forest with at most εN components. Then $F \to (A, \mathcal{M})$ if any of the following holds:*

1. *$||F|| \le \overline{\deg}(A, \mathcal{M}) - 3\gamma n$.*
2. *There exist constants $0 \le c \le 1/2$ and $\lambda \ge 0$ such that $|d(A, X) - d(A, Y)| \ge \lambda$ for all $(X, Y) \in \mathcal{M}$, all trees in F have ratio between c and $1 - c$ (inclusively), and $||F|| \le \overline{\deg}(A, \mathcal{M}) + \frac{c}{1-c}\lambda N m - 3\gamma n$.*
3. *There exists $0 \le \lambda \le \frac{1}{2}$ such that $\lambda \le d(A, X) \le 1 - \lambda$ for all $X \in V(\mathcal{M})$, every tree in F has at least two vertices, and $||F|| \le \overline{\deg}(A, \mathcal{M}) + \lambda N m - 3\gamma n$.*

Proof. Following the corresponding part of Lemma 5.4, we define the capacity of an edge $e = \{X, Y\} \in \mathcal{M}$ hosting εN-forests (with respect to A)

$$w(e) := \begin{cases} \overline{\deg}(A, e) - 2(\gamma + \varepsilon)N & \text{for Part 1} \\ \overline{\deg}(A, e) + \frac{c}{1-c}\lambda N - (2\gamma + 3\varepsilon)N & \text{for Part 2} \\ \overline{\deg}(A, e) + (\lambda - 2\gamma - 13\varepsilon)N & \text{for Part 3.} \end{cases} \tag{5.3}$$

It is easy to see that $w(e) < 2N$ in all cases. For example, for Part 2, since $0 \le c \le 1/2$, we have $\frac{c}{1-c} \le 1$. Together with $|d(A, X) - d(A, Y)| \ge \lambda$, this implies that

$$w(e) \le \overline{\deg}(A, e) + \lambda N - (2\gamma + 3\varepsilon)N \le 2\max\{d(A, X), d(A, Y)\}N - (2\gamma + 3\varepsilon)N < 2N.$$

Since $\varepsilon < \sqrt{\varepsilon} \ll \gamma$ and $mN \le n$, for the three parts of the lemma, it suffices to prove that $F \to (A, \mathcal{M})$ under the uniform assumption

$$||F|| \le \left(\sum_{e \in \mathcal{M}} w(e)\right) - (4\sqrt{\varepsilon} + \varepsilon)Nm. \tag{5.4}$$

Suppose that $F = \{T_1, \ldots, T_s\}$ with $r_i = Rt(T_i)$. Define $F_i = \{T_1, \ldots, T_i\}$ for $1 \le i \le s$ and $F_0 = \emptyset$. Our goal is to prove the following claim.

Claim: *For every $0 \le i \le s$, there exists a sub-forest F'_i of F_i such that the following holds.*

(i) If $F'_i \neq \emptyset$, then there exists $i_0 \le i$ such that $F'_i = \{T'_{i_0}, T_{i_0+1}, \ldots, T_i\}$, where T'_{i_0} is a non-trivial root-subtree of T_{i_0}.

(ii) If $F'_i \neq \emptyset$, then there exists $e_i = \{X_i, Y_i\} \in \mathcal{M}$ such that $0 < ||F'_i|| \le w(e_i) - \varepsilon N$; otherwise $e_i = \emptyset$.

(iii) $F_i - F'_i \to (A, \mathcal{M} \setminus \{e_i\})$.[9] *Furthermore, for every $e \in \mathcal{M}$, denote by $F_i(e)$ the portion of F_i embedded in e. Let $\mathcal{M}_i$ be the set of $e \in \mathcal{M} \setminus \{e_i\}$ such that $|F_i(e)| > 0$. Then for every $e \in \mathcal{M}_i$,*

$$w(e) - \varepsilon N < |F_i(e)| \le w(e). \tag{5.5}$$

Finally, if $F'_i \neq \emptyset$ and $T'_{i_0} \neq T_{i_0}$ (thus $r_{i_0} \in V(F_i - F'_i)$), then r_{i_0} is mapped to a vertex $a_{i_0} \in A$ that is typical to X_i and Y_i.

If the claim holds for $i = s$, then we can derive $F \to (A, \mathcal{M})$ as follows. If $F'_s = \emptyset$, then the embedding follows from (iii) immediately. When $F'_s \neq \emptyset$, by (i), there exists $s_0 \le s$ such that $F'_s = \{T'_{s_0}, \ldots, T_s\}$. By (ii), there exists $e_s = \{X_s, Y_s\} \in \mathcal{M}$ such that $||F'_s|| \le w(e_s)$. Since F'_s is an εN-forest with at most εN components, we can apply Lemma 5.4 to embed $F'_s \to (A, e_s)$, *i.e.*, $Rt(F'_s) \xrightarrow{-3\varepsilon N} A$ and $F'_s - Rt(F'_s) \xrightarrow{2\varepsilon N} \{X_s, Y_s\}$. Furthermore, if r_{s_0} has been mapped to a vertex $a_{s_0} \in A$ that is typical to X_s and Y_s by (iii), then Lemma 5.4 allows us to map r_{s_0} to a_{s_0}. Together with $F_s - F'_s \to (A, \mathcal{M} \setminus \{e_s\})$ from (iii), this gives the desired embedding $F \to (A, \mathcal{M})$. Note that for each root $r \in Rt(F'_s)$, we have $r \xrightarrow{-4\varepsilon N} A$ because at most εN vertices may have been embedded into A before r. As $2\sqrt{\varepsilon}N > 4\varepsilon N$, this proves Lemma 5.8.

We now prove the claim by induction on i. Since $F_0 = \emptyset$, the claim trivially holds for $i = 0$. Suppose that it holds for some $0 \le i < s$. We consider the following cases.

Case 1. $||T_{i+1}|| + ||F'_i|| \le w(e_i) - \varepsilon N$.

In this case we do not need to embed anything. Simply let $F'_{i+1} = F'_i \cup T_{i+1}$ and $e_{i+1} = e_i$. Then the claim holds for $i + 1$.

Case 2. $||T_{i+1}|| + ||F'_i|| > w(e_i) - \varepsilon N$.

Let $\mathcal{M}'_{i+1} = \mathcal{M}_i \cup \{e_i\}$, $\mathcal{M}' = \mathcal{M} \setminus \mathcal{M}'_{i+1}$, and $m' = |\mathcal{M}'|$. Since T_{i+1} is an εN-tree, we can partition it into two εN-*root-subtrees* T'_{i+1} and T''_{i+1} such that

$$w(e_i) - \varepsilon N < ||T'_{i+1}|| + ||F'_i|| \le w(e_i). \tag{5.6}$$

Then $F'_i \cup T'_{i+1}$ is an εN-forest with at most εN components and with at most $w(e_i)$ edges. Applying Lemma 5.4, we can embed $F'_i \cup T'_{i+1} \to (A, e_i)$ such that $r_{i_0} \to a_{i_0}$ if r_{i_0} was mapped to a_{i_0} when we embedded $F_i - F'_i$. By Lemma 5.4, all but at most $3\varepsilon N$ vertices of A can be the image of r_{i+1}. We, in particular, map r_{i+1} to an unoccupied vertex $a_{i+1} \in A$ that is typical to the cluster-set $V(\mathcal{M}')$, that is, typical to at least $(1 - \sqrt{\varepsilon})|V(\mathcal{M}')|$ clusters in $V(\mathcal{M}')$. By Proposition 4.5, all but at most $\sqrt{\varepsilon}N$ vertices in

[9] Recall that if G_2 is a subgraph of G_1, we let $G_1 - G_2$ be the subgraph of G_1 obtained by removing all edges of G_2 and all vertices that are only incident to edges of G_2.

A are typical to $V(\mathcal{M}')$. Since $i \leq s-1$ roots of F have been mapped to A, all but at most $(s-1)+3\varepsilon N+\sqrt{\varepsilon}N < 2\sqrt{\varepsilon}N$ can be chosen as a_{i+1}. Let $\mathcal{M}^* \subseteq \mathcal{M}'$ denote the set of all $e \in \mathcal{M}'$ such that a_{i+1} is typical to both ends of e. Then

$$|\mathcal{M}' \setminus \mathcal{M}^*| \leq \sqrt{\varepsilon}|V(\mathcal{M}')| = 2\sqrt{\varepsilon}m'. \tag{5.7}$$

By (5.5) and (5.6), we have $||F_i|| + ||T'_{i+1}|| \geq \sum_{e\in\mathcal{M}'_{i+1}}(w(e)-\varepsilon N)$. It follows that

$$\begin{aligned}
||T''_{i+1}|| &\leq ||F|| - (||F_i|| + ||T'_{i+1}||) \\
&\leq \left(\sum_{e\in\mathcal{M}} w(e)\right) - (4\sqrt{\varepsilon}+\varepsilon)Nm - \sum_{e\in\mathcal{M}'_{i+1}}(w(e)-\varepsilon N) \quad \text{by (5.4)} \\
&\leq \left(\sum_{e\in\mathcal{M}'} w(e)\right) - (4\sqrt{\varepsilon}+\varepsilon)Nm', \\
&\leq \left(\sum_{e\in\mathcal{M}'} (w(e)-\varepsilon N)\right) - 2N|\mathcal{M}' \setminus \mathcal{M}^*| \quad \text{by (5.7)} \\
&\leq \sum_{e\in\mathcal{M}^*} (w(e)-\varepsilon N)
\end{aligned}$$

We may therefore partition T''_{i+1} into root-subtrees $\{T_{i+1}(e) : e \in \mathcal{M}^*\}$ such that

$$w(e) - \varepsilon N < ||T_{i+1}(e)|| \leq w(e) \tag{5.8}$$

for all but at most one nonempty $T_{i+1}(e)$. Denote by e_{i+1} this exceptional edge of $\mathcal{M}^*$ if it exists. We have $0 < |T_{i+1}(e_{i+1})| \leq w(e_{i+1}) - \varepsilon N$. Let $\mathcal{M}''_{i+1}$ be the set of $e \in \mathcal{M}^*$ satisfying (5.8). For each $e = \{X, Y\} \in \mathcal{M}''_{i+1}$, since a_{i+1} is typical to X and Y, we can apply Lemma 5.4 embedding $T_{i+1}(e) \to (A, (X, Y))$ such that $r_{i+1} \to a_{i+1}$. Now it is easy to see that the claim holds for $i+1$. In fact, (i) and (ii) hold by letting $F'_{i+1} = T_{i+1}(e_{i+1})$ if e_{i+1} exists, otherwise $F'_{i+1} = \emptyset$. Let $\mathcal{M}_{i+1} = \mathcal{M}'_{i+1} \cup \mathcal{M}''_{i+1}$. Then (5.5) holds for every $e \in \mathcal{M}_{i+1}$ because of the definition of T'_{i+1} and $T_{i+1}(e)$. By the definition of $\mathcal{M}^*$, the image of r_{i+1} is typical to both ends of e_{i+1}. Thus (iii) holds. □

We need the next Lemma for Section 6.5.3. Its proof is similar to those of Lemma 5.4 and Lemma 5.8. The difference is that a forest F is embedded into three layers (A, $\mathcal{C}$ and $\mathcal{M}$) in Lemma 5.9 Part 2, instead of two layers as in Lemma 5.8.

Let F by an ordered εN-forest, A be a cluster, $\mathcal{C}$ be a family of clusters not containing A, and $\mathcal{M}$ be a cluster-matching such that $V(\mathcal{M}) \cap (\{A\} \cup \mathcal{C}) = \emptyset$. We write $F \to (A, \mathcal{C}, \mathcal{M})$ if there is an online algorithm embedding $V(F)$ to $A \cup \bigcup_{X\in\mathcal{C}\cup V(\mathcal{M})} X$ such that for any set $S \subseteq F_{odd}$ of size $|S| \leq \varepsilon N$,

$$Rt(F) \xrightarrow{-2\sqrt{\varepsilon}N} A, \quad Level_1(F) \cup S \xrightarrow{2\varepsilon N} \mathcal{C}', \quad Level_{\geq 2}(F) - S \xrightarrow{2\varepsilon N} \mathcal{M}, \tag{5.9}$$

where $\mathcal{C}' = \{C \in \mathcal{C} : A \sim C\}$. The purpose of introducing S can be seen from the proof of Lemma 6.3, in which we need to embed at most εN vertices from $Level_{\geq 3}(F)$ to $\mathcal{C}'$.

Lemma 5.9. *1. Let C be a cluster with a subset $P \subseteq C$. Suppose that $\mathcal{M}$ is a cluster-matching not containing C such that $d(C,e) > 0$ for all $e \in \mathcal{M}$. Let $O \subseteq \bigcup_{X \in V(\mathcal{M})} X$ be a vertex set. Suppose that $F = \{T_1, T_2, \ldots, T_t\}$ and each T_i is a trees of order εN. Let S be a subset of F_{even} of size $|S| \leq \varepsilon N$. If $t \leq |P| - (\varepsilon + \gamma)N$ and $|O| + ||F|| \leq (1-\gamma)|\mathcal{M}|N$, then F can embedded into $(P, \mathcal{M})$ such that $Rt(F) \cup S \stackrel{2\varepsilon N}{\longrightarrow} P$ and $F - Rt(F) - S \stackrel{2\varepsilon N}{\longrightarrow} \bigcup_{X \in V(\mathcal{M})} X \setminus O$.*

2. Let A be a cluster, $\mathcal{C}$ be a family of clusters that are adjacent to A, and $\mathcal{M}$ be a cluster-matching such that $V(\mathcal{M}) \cap (\{A\} \cup \mathcal{C}) = \emptyset$. Let $m = \min_{C \in \mathcal{C}} |\{e \in \mathcal{M} : d(C,e) > 0\}|$. If $F = \{T_1, T_2, \ldots, T_t\}$ is an ordered εN-forest such that

$$t \leq \varepsilon N, \quad |Level_1(F)| \leq \overline{\deg}(A, \mathcal{C}) - 2\gamma|\mathcal{C}|N, \quad \textit{and} \quad |Level_{\geq 2}(F)| \leq (1-\gamma)mN,$$

then $F \to (A, \mathcal{C}, \mathcal{M})$.

Proof. For both parts, we will embed $T_1, \ldots, T_t$ inductively. Suppose $i \geq 1$ and $T_1, \ldots, T_{i-1}$ has been embedded via a function $\phi = \phi(i)$.

Part 1. For each pair $\{X, Y\} \in \mathcal{M}$, let X^* and Y^* denote the sets of unoccupied vertices in $X \setminus O$ and $Y \setminus O$, respectively. If either $|X^*| < \gamma N$ or $|Y^*| < \gamma N$, then $|(X \cup Y) \cap (\phi(F) \cup O)| > (1-\gamma)N$. If this is the case for all $\{X, Y\} \in \mathcal{M}$, then $||F|| + |O| > (1-\gamma)|\mathcal{M}|N$ (because only vertices in $F - Rt(F)$ are embedded to $\mathcal{M}$), a contradiction. Hence there exists $\{X, Y\} \in \mathcal{M}$ such that both $|X^*|, |Y^*| \geq \gamma N$. By assumption, $d(C, \{X, Y\}) > 0$. Without loss of generality, suppose that $d(C, X) > 0$. Let us first embed $Rt(T_i)$ into an unoccupied vertex $u_i \in P$ typical to X^*, namely, $|N(u_i, X^*)| > (d(C,X) - \varepsilon)|X^*| > 4\varepsilon N$. Since only vertices from $Rt(F) \cup S$ have been embedded to P and $|S| \leq \varepsilon N$, by the assumption on $|P|$, at least $|P| - t - |S| - \varepsilon N > 2\varepsilon N$ vertices of P can be chosen as u_i. Let P^* be the set of unoccupied vertices in P after selecting u_i. We know that $|P^*| \geq |P| - t - |S| \geq \gamma N$. We now apply Lemma 5.2 with $X_0 = N(u_i, X^*)$, $X_1 = X^*$, $Y_1 = Y^*$, and $Z_1 = P^*$ to embed the forest $T_i - Rt(T_i)$ into $P^* \cup X^* \cup Y^*$ such that $S \stackrel{2\varepsilon N}{\to} P^*$ and $T_i - Rt(T_i) - S \stackrel{2\varepsilon N}{\to} \{X^*, Y^*\}$.

Part 2. Without loss of generality, assume that every $C \in \mathcal{C}$ is adjacent to A (otherwise remove such C from $\mathcal{C}$ and $\overline{\deg}(A, \mathcal{C})$ does not change). Let $S \subseteq F_{odd}$ be a set of at most εN vertices that we will embed to $\mathcal{C}$.

We first embed $Rt(T_i)$ into an unoccupied vertex $a_i \in A$ that is typical to $\mathcal{C}$, namely, there exists a subfamily $\mathcal{C}_i \subseteq \mathcal{C}$ of size at least $(1-\sqrt{\varepsilon})|\mathcal{C}|$ such that $\deg(a_i, C) > (d(A,C) - \varepsilon)N$ for every $C \in \mathcal{C}_i$. By Proposition 4.5, all but $\sqrt{\varepsilon}N + (i-1) < 2\sqrt{\varepsilon}N$ vertices of A can be chosen as a_i. For each cluster $C \in \mathcal{C}_i$ let P_C denote the set of unoccupied vertices in $N(a_i, C)$. Define $F_j = T_j - Rt(T_j)$ for all $j \leq i$. Since $\{Rt(F_j) \cup (S \cap V(F_j)), j < i\}$

has been embedded to $\mathcal{C}$, we have

$$\begin{aligned}\sum_{C\in\mathcal{C}_i}|P_C| &\geq \sum_{C\in\mathcal{C}_i}|N(a_i,C)| - \sum_{j<i}|Rt(F_j)| - |S| \\ &\geq \overline{\deg}(A,\mathcal{C}) - \varepsilon|\mathcal{C}_i|N - \sqrt{\varepsilon}|\mathcal{C}|N - \sum_{j<i}|Rt(F_j)| - \varepsilon N \\ &\geq \overline{\deg}(A,\mathcal{C}) - 2\sqrt{\varepsilon}|\mathcal{C}|N - \sum_{j<i}|Rt(F_j)|.\end{aligned}$$

Together with the assumption

$$|Rt(F_i)| + \sum_{j<i}|Rt(F_j)| \leq |Level_1(F)| \leq \overline{\deg}(A,\mathcal{C}) - 2\gamma|\mathcal{C}|N,$$

this implies that $|Rt(F_i)| \leq \sum_{C\in\mathcal{C}_i}(|P_C| - (\varepsilon+\gamma)N)$. We then partition F_i into forests $\bigcup_{C\in\mathcal{C}_i} F_C$ such that $|Rt(F_C)| \leq |P_C| - (\varepsilon+\gamma)N$ for all $C \in \mathcal{C}_i$.

We will apply Part 1 to embed each F_C to $P_C \cup \bigcup_{X\in V(\mathcal{M})} X$. Consider a cluster $C \in \mathcal{C}_i$. Let $\mathcal{M}_C$ denote the set of those $e \in \mathcal{M}$ such that $d(C,e) > 0$. By assumption, $|\mathcal{M}_C| \geq m$. Let O denote the set of the vertices in $\bigcup_{X\in V(\mathcal{M})} X$ occupied by $T_1,\dots,T_{i-1}$ and the trees in F_i embedded before F_C. In order to embed F_C by Part 1, it suffices to have $||F_C|| + |O| \leq (1-\gamma)|\mathcal{M}_C|N$. Since only the vertices in $Level_{\geq 2}(F)$ are embedded to the clusters in $V(\mathcal{M})$, this is guaranteed by the assumption $|Level_{\geq 2}(F)| \leq (1-\gamma)mN$. □

6 The non-extremal case

The purpose of this section is to prove Theorem 3.3. We use the following parameters:

$$0 < \varepsilon \ll \gamma \ll d \ll \eta \ll \rho \ll \alpha \ll 1, \tag{6.1}$$

where $a \ll b$ can be specified as, for example, $10^5 a \leq b^{12}$.

We assume that n is sufficiently large, in particular,

$$n \geq \left(\frac{M(\varepsilon)}{\varepsilon}\right)^2, \tag{6.2}$$

where $M(\varepsilon)$ is given by the Regularity Lemma.

Let $G = (V,E)$ be a $2n$-vertex graph with $\ell(G) \geq (1-\varepsilon)n$, *i.e.*, at least $(1-\varepsilon)n$ vertices of degree at least n. We assume that G is *not* in **EC1** or **EC2** with parameter α.

We apply the Regularity Lemma (Lemma 4.2) to G, and obtain the subgraph G'' and the reduced graph G_r. Then G'' contains ℓ clusters $V_1,\dots,V_\ell$, each of which is of size N. We first observe that both εN and $\sqrt{d}\ell$ are large. By Lemma 4.2, we have $\ell \leq M(\varepsilon)$ and $|V_0| \leq \varepsilon(2n)$. Thus $\ell N \geq (1-\varepsilon)2n$, which gives $N \geq (1-\varepsilon)2n/M(\varepsilon)$. By (6.2), we have

$$\varepsilon N \geq 2(1-\varepsilon)\left(\frac{M(\varepsilon)}{\varepsilon}\right)^2 \frac{\varepsilon}{M(\varepsilon)} \geq \frac{M(\varepsilon)}{\varepsilon}. \tag{6.3}$$

On the other hand, since $N \leq \varepsilon(2n)$, we have $2n \leq (\ell+1)\varepsilon(2n)$ or $\ell \geq \frac{1}{\varepsilon} - 1$. Since $\varepsilon \ll d \ll 1$, both εN and $\sqrt{d}\ell$ are large.

Now let $k = \lfloor \ell/2 \rfloor$. We have

$$k \geq \frac{\ell - 1}{2} \geq \frac{1}{2\varepsilon} - 1. \tag{6.4}$$

If ℓ is odd, then we eliminate one cluster by moving all the vertices in this cluster to V_0. As a result, $V' = V(G'')$ contains $2k$ clusters and $|V_0| \leq 2\varepsilon|V| = 4\varepsilon n$. Hence $|V'| = 2Nk \geq 2n - 4\varepsilon n$, which implies that

$$n - 2\varepsilon n \leq Nk \leq n \tag{6.5}$$

Throughout Section 6, *we assume omit floors and ceilings unless they are crucial*. For example, we assume that error terms, such as εN, $\sqrt{d}N$, are integers. In fact, if εN is not an integer, then we can replace ε by ε' such that $\varepsilon - \frac{1}{N} < \varepsilon' \leq \varepsilon$ and $\varepsilon' N$ is an integer. As $\frac{1}{N}$ is very small, the new parameter ε' still satisfies (6.1).

The rest of the proof is divided into five subsections. In Section 6.1 we prove G'' and G_r have similar properties to G. In Section 6.2 we partition a tree T into a forest F such that $F - Rt(F)$ consists of small trees. In Section 6.3 we give several sufficient conditions for embedding F and correspondingly T into G''. In Section 6.4 we prove a Tutte-type one-factor theorem, which provides a large matching in G_r. Since **EC1** does not hold in G, this immediately provides an embedding of trees of size near n into G''. In Section 6.5 we carefully check case by case when we can embed a tree of size n and conclude that **EC2** is the only exception.

6.1 Preparation of G

The goal of this subsection is to prove Claim 6.1, which gives the properties of G'' and G_r. Before stating the Lemma, we need the following preliminaries. Let L be the set of vertices in G of degree at least n. We call these *large* vertices, and call vertices in $V \setminus L$ *small* vertices. Since deleting edges between small vertices does not change our assumption, we assume that there is no edge between any two small vertices.

We call a cluster **large** if it contains $2\sqrt{d}N$ large vertices (though the reason we set the threshold as $2\sqrt{d}N$ can only be seen in the proof of Claim 6.17). The set of large clusters is denoted by $\mathcal{L}$. We delete all the edges of G between two small clusters and thus assume every (non-trivial) regular-pair (of clusters) contains at least one large cluster.

Claim 6.1. *1. For every $X \in \mathcal{L}$, we have $\overline{\deg}(X) > n - 4dn$ and $\deg_{G_r}(X) \geq (1-4d)k$. Furthermore, all but at most $\sqrt{\varepsilon}N$ vertices in X have degree in G'' greater than $n - 5dn$.*

2. $|\mathcal{L}| \geq (1 - 4\sqrt{d})k$.

3. $\mathcal{L}$ is not independent.

Proof. *Part 1.* Applying Proposition 4.5 Part 2 to X and $Y_I = V' \setminus X$, we know that all but at most $\sqrt{\varepsilon}N$ vertices $u \in X$ satisfy

$$\deg(u, V' \setminus X) < \overline{\deg}(X, V' \setminus X) + 2\sqrt{\varepsilon}|V'|.$$

Note that the underlying graph is G''. Since $\deg_{G''}(u) = \deg(u, V' \setminus X)$ and $\overline{\deg}(X) = \overline{\deg}(X, V' \setminus X)$, it follows that

$$\deg_{G''}(u) < \overline{\deg}(X) + 4\sqrt{\varepsilon}n. \tag{6.6}$$

Since $|X \cap L| \geq 2\sqrt{d}N > \sqrt{\varepsilon}N$, we let u be a vertex of $X \cap L$. The definitions of G'' and L imply that

$$\deg_{G''}(u) \geq \deg_G(u) - (d+\varepsilon)2n - |V_0| \geq n - (d+3\varepsilon)2n > n - 3dn, \tag{6.7}$$

where the last inequality holds because $\varepsilon \ll d$ from (6.1). By putting (6.6) and (6.7) together, we conclude that $\overline{\deg}(X) > (1-3d)n - 4\sqrt{\varepsilon}n > n - 4dn$. Because of (4.1) and (6.5), we also have $\deg_{G_r}(X) \geq (1-4d)n/N \geq (1-4d)k$. Furthermore, by Proposition 4.5 Part 2, all but at most $\sqrt{\varepsilon}N$ vertices in X have degree in G'' at least $\overline{\deg}(X) - 4\sqrt{\varepsilon}n > n - 5dn$.

Part 2. From $|L| \geq (1-\varepsilon)n$ and the definition of $\mathcal{L}$, we have

$$n - 5\varepsilon n \leq |L| - |V_0| = |L \cap V'| \leq |\mathcal{L}|N + 2\sqrt{d}N\left(2k - |\mathcal{L}|\right),$$

or $(N - 2\sqrt{d}N)|\mathcal{L}| \geq n - 5\varepsilon n - 4\sqrt{d}Nk$, which implies that $|\mathcal{L}| \geq (1-4\sqrt{d})k$ because of (6.1) and (6.5).

Part 3. Suppose instead, that $\mathcal{L}$ is an independent set in G_r. Let U_1 be the set of the vertices of G contained in all the large clusters, and $U_2 := V \setminus U_1$. For all $v \in U_1$, we have $\deg_{G''}(v, U_1) = 0$, which implies that $\deg_{G''}(v, U_2) = \deg_{G''}(v)$. By Part 1, at least $(1-\sqrt{\varepsilon})N$ vertices v in a large cluster satisfy $\deg_{G''}(v) > n - 5dn$. By using $|\mathcal{L}| \geq (1-4\sqrt{d})k$ from Part 2, we have

$$\begin{aligned} e_{G''}(U_1, U_2) &> (n-5dn)(1-\sqrt{\varepsilon})N|\mathcal{L}| \\ &\geq (n-5dn)(1-\sqrt{\varepsilon})N(1-4\sqrt{d})k > (1-10\sqrt{d})n^2. \end{aligned}$$

Since $|U_1| = |\mathcal{L}|N \geq (1-4\sqrt{d})kN > (1-5\sqrt{d})n$, we can move at most $5\sqrt{d}n$ vertices from U_2 to U_1 such that $|U_1| = n$. The resulting sets U_1, U_2 satisfy

$$e_G(U_1, U_2) \geq e_{G''}(U_1, U_2) > (1-10\sqrt{d})n^2 - 5\sqrt{d}n^2 > (1-\alpha)n^2$$

since $d \ll \alpha$. This contradicts our assumption that G is not in **EC1** with parameter α. □

6.2 Partition a tree into a forest

In this subsection we associate every tree with an ordered εN-forest. Recall that F is an ordered m-forest if $Rt(F)$ is ordered, and any tree in $F - Rt(F)$ has at most m vertices.

Definition 6.2. *Fix a positive integer m and a rooted tree T. An ordered m-forest $F = \{T_1, T_2, \ldots, T_s\}$ is called an m-forest of T if it satisfies the following properties.*

- *F contains $s-1$ (not necessarily distinct) special vertices $p_2, \ldots, p_s$ (we call them* parent-vertices*). Suppose $r_i = Rt(T_i)$ for $1 \le i \le s$. Then F is obtained from T by removing the $s-1$ edges $r_2p_2, \ldots, r_sp_s$.*
- *Let $R_a = Rt(F) \cap T_{even}$ and $R_b = Rt(F) \cap T_{odd}$. Then $|R_a|, |R_b| \le \frac{v(T)+m}{m+1}$.*
- *For each $j \ge 2$, p_j is contained in T_i for some $i < j$. Furthermore, if $r_i \in R_a$ (resp. R_b), then either $p_j = r_i$ or $r_j \in R_a$ (resp. R_b).*

Following the definitions of R_a and R_b, we partition F into two ordered m-forests F_a and F_b, e.g., $F_a = \{T_i \in F : Rt(T_i) \in R_a\}$.

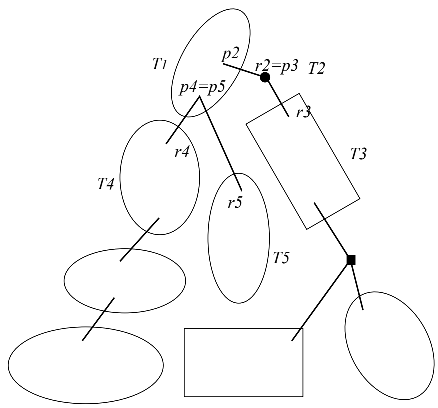

Figure 1: An m-forest of T (ovals = trees in F_a, rectangles = trees in F_b)

Note that F_a, F_b are interchangeable because T_{even} and T_{odd} are interchangeable (by pick $Rt(T)$ differently).

Given a tree T, we now describe an algorithm which returns an ordered m-forest of T. In a tree t, a vertex x is called an m-vertex of t if $|t(x)| > m$ and $|t(y)| \le m$ for every $y \in C(x)$. Let us start with $F = \emptyset$ and add subtrees of T to F as follows. We first remove subtrees $T(x)$ for each m-vertex x (note that these subtrees are disjoint in T), and then add them in an arbitrary order to F. Naturally each m-vertex x is the root of $T(x)$. Let T' denote the remaining part of T. We next remove subtrees $T'(x)$ for each m-vertex x of T', and add them (in an arbitrary order) to F. We repeat this procedure till at most m vertices remain.[10] We add the subtree on these remaining vertices to F with $Rt(T)$ as its root. Label the trees in F by $T_1, \ldots, T_t$ in the reversing order that they were added to F, *e.g.*, the tree added at last is T_1. Except for T_1, every tree in F has at least $m+1$ vertices, consequently $t \le \frac{v(T)-1}{m+1} + 1 = \frac{v(T)+m}{m+1}$. The roots of F form an ordered set $R_0 = \{v_1, \ldots, v_t\}$ with $v_i = Rt(T_i)$.

In order to obtain item 3 in Definition 6.2, we refine F as follows. We call a vertex in F even (or odd) if the distance from it to $Rt(T)$ in T is even (or odd), for example,

[10] It is easy to see that any tree with more than m vertices must contain an m-vertex.

$v_1 = Rt(T)$ is even. We call two roots $v_i, v_j \in R_0$, $i < j$, *linked* if the parent u_j of v_j is a vertex of T_i. we now cut the subtree $T_i(u_j)$ from T_i whenever two linked roots v_i, v_j have different parity and $u_j \neq v_i$. The new tree is inserted right before T_j in F; the new root u_j has the same parity as v_i. Let $R = \{r_1, \dots, r_s\}$ be the set of roots in the resulting F, with subsets R_a and R_b of the even roots and the odd roots, respectively. We have $|R_a|, |R_b| \leq |R_0|$ because, for example, each vertex of R_a is either an even vertex from R_0 or the parent of some odd vertex in R_0.

Let T be a rooted tree with n edges. Let ε be as in (6.1) and N be the size of clusters. Suppose that F is an ordered εN-forest of T. By item 2 in Definition 6.2 and $v(T) + \varepsilon N < 2n - 4\varepsilon n$, we have

$$|R_a|, |R_b| \leq \frac{v(T) + \varepsilon N}{\varepsilon N + 1} \leq \frac{2n - 4\varepsilon n}{\varepsilon N} \overset{(6.5)}{\leq} \frac{2Nk}{\varepsilon N} \leq \frac{M(\varepsilon)}{\varepsilon} \overset{(6.3)}{\leq} \varepsilon N. \tag{6.8}$$

6.3 Sufficient conditions for embedding large trees

In this subsection we prove several lemmas which give sufficient conditions for embedding large trees into G'' (and thus in G). Our first lemma gives two sufficient conditions for $T \subseteq G$ based on the embedding of F_a and F_b.

Lemma 6.3. *Let T be a tree of order n and $F = F_a \cup F_b$ be an ordered εN-forest of T. Let A, B be two adjacent clusters of size N in G with subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0|, |B_0| \geq \sqrt{d}N$. Then T can be embedded into G with $Rt(F) \to A_0 \cup B_0$ if any of the following holds.*

1. *There are two disjoint cluster-matchings $\mathcal{M}_a$ and $\mathcal{M}_b$ from $\mathcal{V} \setminus \{A, B\}$ such that $F_a \overset{-4\sqrt{\varepsilon}N}{\longrightarrow} (A, \mathcal{M}_a)$ and $F_b \overset{-4\sqrt{\varepsilon}N}{\longrightarrow} (B, \mathcal{M}_b)$.*
2. *There are two sub-forests F_0 and F_1 of F_a such that $E(F_0) \cup E(F_1)$ is a partition of $E(F_a)$ and $V(F_0) \cap V(F_1) \subseteq Rt(F_a)$. There are a cluster-set $\mathcal{C} \subset \mathcal{V} \setminus \{A, B\}$ and three disjoint cluster-matchings $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_b$ from $\mathcal{V} \setminus (\{A, B\} \cup \mathcal{C})$ such that $F_0 \to (A, \mathcal{C}, \mathcal{M}_0)$, $F_1 \to (A, \mathcal{M}_1)$, and $F_b \to (B, \mathcal{M}_b)$.*

Proof. Suppose that $F = \{T_1, \dots, T_s\}$ with roots $r_1, \dots, r_s$ and parent-vertices $p_2, \dots, p_s$. Let ϕ be the given embedding function of F_a and F_b (into $\mathcal{M}_a$, $\mathcal{M}_b$ or $\mathcal{M}_0$). The key point in our proof is to select $\phi(p_i), \phi(r_i)$ carefully such that $\phi(p_i)$ and $\phi(r_i)$ are adjacent for all $i \geq 2$. More precisely, we will sequentially embed $T_1, T_2, \dots$ such that

$$\textit{each } p_i \textit{ is mapped to a vertex typical to } A_0 \textit{ (resp. } B_0\textit{) if } T_i \in F_a \textit{ (resp. } T_i \in F_b\textit{).} \tag{6.9}$$

Given $i \geq 1$, suppose that $T_1, \dots, T_{i-1}$ have been embedded and (6.9) holds for all parent-vertices in $V(T_1 \cup \dots \cup T_{i-1})$. It suffices to show that T_i can be embedded such that (6.9) holds for all parent-vertices contained in T_i.

Part 1. Without loss of generality, assume that $T_i \in F_a$. Since $p_i \in V(T_1 \cup \dots \cup T_{i-1})$, by (6.9), p_i has been mapped to a vertex w_i typical to A_0. As $F_a \overset{-4\sqrt{\varepsilon}N}{\longrightarrow} (A, \mathcal{M}_a)$, all but at most $4\sqrt{\varepsilon}N$ vertices of A can be chosen as $\phi(r_i)$. Since at most εN vertices of A are atypical to B_0 and $|N(w_i, A_0)| \geq (d-\varepsilon)\sqrt{d}N > 4\sqrt{\varepsilon}N + \varepsilon N$, we can choose $\phi(r_i)$ from $N(w_i, A_0)$ such that it is typical to B_0.

Let p_j, $j > i$, be a parent-vertex in T_i. If $T_j \in F_b$, then by Definition 6.2 item 3, we have $p_j = r_i$. Then (6.9) holds by our choice of $\phi(r_i)$. Otherwise $T_j \in F_a$. Then the distance between r_i and p_j is odd (at least 1). Assume that ϕ embeds the subtree of T_i containing p_j into $\{X, Y\}$, and say, $\phi(p_j) \in X$. Then $X \sim A$ since the ancestor of p_j in $Level_1(T_i)$ is also embedded into X. Since $p_j \xrightarrow{2\varepsilon N} X$ and at most εN vertices from X are atypical to A_0, we can choose $\phi(p_j)$ to be a vertex typical to A_0. Therefore (6.9) holds.

Part 2. Let S be the set of all parent-vertices $p_i \in V(F_0)$ such that $r_i \in V(F_a)$. Then $|S| \leq c(F_a) \leq \varepsilon N$. By the definition of $F_0 \to (A, \mathcal{C}, \mathcal{M}_0)$, ϕ maps S to $\{C \in \mathcal{C} : C \sim A\}$.

Suppose we want to embed $T_i \in F_0$ (the cases when $T_i \in F_b$ and when $T_i \in F_1$ are similar to Part 1). The embedding of r_i is the same as in Part 1. Consider a parent-vertex $p_j \in V(T_i)$ such that $T_j \in F_a$ (otherwise $p_j = r_i$ and (6.9) automatically holds). Thus $p_j \in S$. By (5.9), ϕ maps $p_j \xrightarrow{2\varepsilon N} C$ for some cluster $C \in \mathcal{C}$ such that $C \sim A$. We can therefore choose $\phi(p_j)$ to be a vertex typical to A_0 such that (6.9) holds. □

Lemma 6.5 gives more sufficient conditions for embedding a tree. Its proof needs the following simple fact (stated in [2] without a proof).

Fact 6.4. *Let $\{a_i\}_{i=1}^m, \{b_i\}_{i=1}^m$ be two finite sequences such that $0 \leq a_i, b_i \leq \Delta$ for all i. Suppose that $\sum a_i = a$ and $\sum b_i = b$. Let s, t be positive real numbers such that $\frac{s}{a} + \frac{t}{b} \leq 1$. Then there is a partition of $[m]$ into I_1 and I_2 such that*

$$\sum_{i \in I_1} a_i > s - \Delta, \quad \text{and} \quad \sum_{i \in I_2} b_i > t - \Delta.$$

Proof. We first reorder the two sequences such that $c_i = \frac{a_i}{a} - \frac{b_i}{b}$ is non-increasing. Then $\sum_{i=1}^j c_i \geq 0$ for any j because $\sum_{i=1}^m c_i = 0$. Choose $j \in [m]$ such that $s - \Delta < \sum_{i=1}^j a_i \leq s$. Then

$$\sum_{i>j} \frac{b_i}{b} = 1 - \sum_{i=1}^{j} \frac{b_i}{b} \geq 1 - \sum_{i=1}^{j} \frac{a_i}{a} \geq 1 - \frac{s}{a} \geq \frac{t}{b},$$

which gives $\sum_{i>j} b_i \geq t$. □

Lemma 6.5. *Let A and B be two adjacent clusters of size N with subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0|, |B_0| \geq \sqrt{d}N$. Let $\mathcal{M}$ be a cluster-matching on $\mathcal{V} \setminus \{A, B\}$. Given a tree T' of size at most n, then T' can be embedded to $A_0 \cup B_0 \cup \bigcup_{X \in V(\mathcal{M})} X$ such that $Rt(F) \to A_0 \cup B_0$ if either of the following conditions holds.*

1. *There are an ordered εN-forest $F = F_a \cup F_b$ of T' and a partition $\mathcal{M}_a \cup \mathcal{M}_b$ of $\mathcal{M}$ such that*

$$||F_a|| \leq \overline{\deg}(A, \mathcal{M}_a) - 3\gamma n \quad \text{and} \quad ||F_b|| \leq \overline{\deg}(B, \mathcal{M}_b) - 3\gamma n, \tag{6.10}$$

2. $||T'|| \le \min\{\overline{\deg}(A,\mathcal{M}), \overline{\deg}(B,\mathcal{M})\} - 8\gamma n$.

Proof. *Part 1.* By (6.8), $|R_a|, |R_b| < \varepsilon N$. So by (6.10), we can apply Lemma 5.8 Part 1 to embed $F_a \to (A, \mathcal{M}_a)$ and $F_b \to (B, \mathcal{M}_b)$. Next we apply Lemma 6.3 Part 1 embedding T' to G such that $Rt(F) \to A_0 \cup B_0$.

Part 2. Let $F = F_a \cup F_b$ be an ordered εN-forest of T'. By Part 1, it suffices to have (6.10). Let $f_a = ||F_a||$ and $f_b = ||F_b||$. Then $f_a + f_b \le ||T'||$. Let $s = f_a + 4\gamma n$ and $t = f_b + 4\gamma n$. Suppose that $\mathcal{M} = \{e_i\}_{i\in I}$. Let $a_i = \overline{\deg}(A, e_i)$, $b_i = \overline{\deg}(B, e_i)$, $a = \sum a_i$, and $b = \sum b_i$. We have $0 \le a_i, b_i \le \Delta := 2N$, and $a, b \ge ||T'|| + 8\gamma n$. Then

$$\frac{s}{a} + \frac{t}{b} \le \frac{f_a + 4\gamma n + f_b + 4\gamma n}{||T'|| + 8\gamma n} \le 1.$$

Fact 6.4 thus provides a partition of $\mathcal{M}$ into $\mathcal{M}_a$ and $\mathcal{M}_b$ such that $\overline{\deg}(A, \mathcal{M}_a) \ge f_a + 4\gamma n - 2N > f_a + 3\gamma n$, and $\overline{\deg}(B, \mathcal{M}_b) \ge f_b + 4\gamma n - 2N > f_b + 3\gamma n$, which gives (6.10). □

6.4 Tutte's one-factor theorem

In this subsection we apply Tutte's one-factor theorem to prove Claim 6.7, which provides a large matching in G_r. This lemma was proved in [2] without introducing the set O, whose role can only be seen in Section 6.5.3, where we need the matching M to cover not only the neighbors of O but also the neighbors of $N(O) := \bigcup_{u\in O} N(u)$. When M is a matching and $u \notin V(M)$, we let $M^1(u) = \{(x,y) \in M : \deg(u, \{x,y\}) = 1\}$ and $M^2(u) = \{(x,y) \in M : \deg(u, \{x,y\}) = 2\}$.

Lemma 6.6. *Let H be a graph on $2k$ vertices and c be a real number such that $0 < c < 1$ and $ck \ge 1$. Suppose L is the set of vertices of H with degree greater than $(1-c)k$. If $|L| \ge (1-c)k$ and L is not independent, then there is either a matching in H that misses at most $2ck+1$ vertices of H or a matching M and a set $O \subseteq V(H)$ such that*

- *$L \cap O$ contains two adjacent vertices,*
- *all but at most one vertex of $N(O)$ are covered by M,*
- *for any $u \in O$, all but at most one vertex covered by $M^2(u)$ are also contained in O.*

Proof. We apply the Gallai–Edmonds decomposition to H. Let S denote the usual cut-set such that the following holds: every even component has a complete matching; every odd component has a matching covering all but one vertex x_i; and there is a matching $\{s_i x_i : i = 1, \dots, |S|\}$ from S to $|S|$ odd components, where $s_i \in S$ and each x_i is from a different odd component. Let M be the union of these matchings. Then

$$|M| = |S| + \sum_C \left\lfloor \frac{|C|}{2} \right\rfloor, \tag{6.11}$$

where the sum is over all components C of $H - S$. It suffices to prove the following claim.

Claim. *Either $|V(M)| \geq 2(1-c)k - 1$, or there is a component C in $H - S$ that contains two adjacent vertices of L.*

The former case of the claim proves our lemma immediately. Suppose the latter holds. Let $O = V(C)$. Since $N(O) \subseteq O \cup S$, by the definition of M, all but at most one vertex in $N(O)$ are covered by M. In addition, for any $u \in O$ and any $xy \in M^2(u)$, we have $x, y \in O$, unless $x = x_i \in O$ and $y = s_i \in S$.

We now prove this claim. If no component of $H - S$ contains any vertex of L, then $L \subseteq S$ and consequently $(1-c)k \leq |L| \leq |S|$. Using (6.11), we obtain the desired bound $|V(M)| \geq 2|S| \geq 2(1-c)k$. On the other hand, if there are two components $C_1, C_2 \in H-S$ and two vertices $v_1, v_2 \in L$ such that $v_i \in C_i$, then $(1-c)k \leq \deg(v_i) \leq |C_i| - 1 + |S|$ for $i = 1, 2$. Consequently $2(1-c)k \leq |C_1| + |C_2| + 2|S| - 2$. Using (6.11), we again derive that $|V(M)| \geq 2|S| + |C_1| + |C_2| - 2 \geq 2(1-c)k$.

We may therefore assume there is one component C of $H - S$ such that $V(C) \cap L \neq \emptyset$ and $V(C') \cap L = \emptyset$ for all other components C' of $H - S$. If there are two adjacent vertices in $V(C) \cap L$, then we are done. Otherwise, letting $a = |V(C) \cap L|$ and $b = |V(C) \setminus L|$, we have $(1-c)k \leq |L| = a + |S|$. Furthermore, for any $v \in V(C) \cap L$, we have $(1-c)k \leq \deg(v) \leq b + |S|$. Consequently $2|S| + |C| = 2|S| + a + b \geq 2(1-c)k$. By (6.11), we have $|V(M)| \geq 2|S| + |C| - 1 \geq 2(1-c)k - 1$. □

We apply Lemma 6.6 to the reduced graph G_r and obtain the following claim.

Claim 6.7. *The reduced graph G_r contains a set $\mathcal{O} \subseteq \mathcal{V}$ and a matching $\mathcal{M}$ such that the following holds.*

1. *There are $A, B \in \mathcal{L} \cap \mathcal{O}$ with $A \sim B$.*
2. *For any $U \in \mathcal{O}$, all but at most $9\sqrt{d}k$ neighbors of U are covered by $\mathcal{M}$.*
3. *For any $U \in \mathcal{O}$, all but at most one cluster from $\mathcal{M}^2(U)$ are also contained in $\mathcal{O}$.*

Proof. Claim 6.1 implies that the reduced graph G_r satisfies the conditions of Lemma 6.6 with $L = \mathcal{L}$ and $c = 4\sqrt{d}$, where $ck = 4\sqrt{d}k \gg 1$ follows from (6.1) and (6.4). By Lemma 6.6, G_r either contains a matching covering all but at most $2(4\sqrt{d})k+1 < 9\sqrt{d}k$ clusters, or a matching $\mathcal{M}$ and a set $\mathcal{O}$ satisfying the three properties of the lemma. The latter case immediately yields the three desired assertions. In the former case, we let $\mathcal{O} = V(G_r)$. It is easy to see that the three assertions holds; in particular, the first assertion follows from Claim 6.1 Part 3, which says that $\mathcal{L}$ contains two adjacent clusters. □

6.5 Embedding a tree of size n

In this subsection we finish the proof of Theorem 3.3.

Let T be a tree of size n. Recall that G is a $2n$-vertex graph satisfying $\ell(G) \geq (1-\varepsilon)n$ and G is not in **EC1** or **EC2** with parameter α. Assume that T *cannot be embedded in G and our goal is to conclude a contradiction.*

Let $F = F_a \cup F_b$ be an εN-forest of T. Then $R := Rt(F)$ is partitioned into R_a and R_b satisfying (6.8), which implies that $c_f := |R| \le 2\varepsilon N$. Let $p_2, \dots, p_{c_f}$ denote the parent-vertices and $f_a := ||F_a||$ and $f_b := ||F_b||$. Without loss of generality, assume that $f_a \ge f_b$. Since $f_a + f_b = ||F|| = n + 1 - c_f$ and $c_f \ge 1$, we have $f_b \le \frac{n}{2}$.

By Claim 6.7, the reduced graph G_r contains a set $\mathcal{O}$, two adjacent clusters $A, B \in \mathcal{L} \cap \mathcal{O}$, and a cluster-matching $\mathcal{M}$. For any cluster $X \in \mathcal{L} \cap \mathcal{O}$, including A, B, Claim 6.1 Part 1 says that $\overline{\deg}(X) \ge (1 - 4d)n$. By item 2 in Claim 6.7,

$$\overline{\deg}(X, \mathcal{M}) \ge \overline{\deg}(X) - 9\sqrt{d}kN \ge (1 - 4d)n - 9\sqrt{d}kN \ge (1 - 10\sqrt{d})n. \tag{6.12}$$

Thus, by Lemma 6.5 Part 2 with $A_0 = A$ and $B_0 = B$, any tree of size at most $(1 - 10\sqrt{d})n - 8\gamma n$ can be embedded into G.

We divide the rest of proof into three subsections. In Section 6.5.1 we study the structure of F and conclude that most trees in $F - Rt(F)$ have at least two vertices, and reasonably many trees in $F_a - Rt(F_a)$ have ratio not close to 0 or 1. In Section 6.5.2 we partition $\mathcal{V}$ into $\mathcal{V}_1 \cup \mathcal{V}_2$ such that $|\mathcal{V}_1| \approx |\mathcal{V}_2|$ and $\mathcal{V}_1$ is covered by regular pairs $e \in \mathcal{M}$ such that $d(A, e) \approx 2$. In Section 6.5.3, we show that there are not many dense regular pairs between $\mathcal{V}_1$ and $\mathcal{V}_2$, and therefore there are not many edges of G between the two vertex sets covered by the clusters of $\mathcal{V}_1, \mathcal{V}_2$. This implies that G is in **EC2**, a contradiction. Throughout the proof, a complication occurs when f_b is very small; we have to use different strategies for the cases when f_b is small and when f_b is large.

6.5.1 Structure of F

Let us analyze the structure of F carefully. We first observe that there are not many leaves of F in $Level_1(F)$. Let $Leaf_1(F)$ denote the set of leaves of F that are located in $Level_1(F)$. Define $\tilde{F} = F - Leaf_1(F)$ and $\tilde{F}_a = F_a - Leaf_1(F)$.

Claim 6.8. *$||\tilde{F}|| \ge (1 - 12\sqrt{d})n$, and $||\tilde{F}_a|| > n/2 - 12\sqrt{d}n$.*

Proof. We first show that $|Leaf_1(F)| \le 11\sqrt{d}n + c_f$. By Definition 6.2, F is obtained from T by removing edges $r_i p_i$, $2 \le i \le c_f$. Then a vertex in $Leaf_1(F)$ is either a leaf in T or a parent-vertex p_i. We may therefore partition $Leaf_1(F)$ into $W_1 \cup W_1'$, where W_1 is the set of the leaves of T located in $Level_1(F)$, and W_1' is the set of parent-vertices that are contained in $Leaf_1(F)$. Clearly $|W_1'| \le c_f \le 2\varepsilon N$. If $|W_1| \ge 11\sqrt{d}n$, then because of (6.12), $T - W_1$ can be embedded by Lemma 6.5 Part 2 with A_0, B_0 as the set of large vertices in A, B, respectively (the definition of $\mathcal{L}$ implies that $|A_0|, |B_0| \ge 2\sqrt{d}N$). The vertices in W_1 can be added greedily at last. Thus we assume that $|W_1| < 11\sqrt{d}n$. Since $||F|| = n + 1 - c_f$ and $c_f \ll \sqrt{d}n$,

$$||\tilde{F}|| = ||F|| - |Leaf_1(F)|| \ge n + 1 - c_f - 11\sqrt{d}n - c_f > (1 - 12\sqrt{d})n.$$

Since $||F_a|| \ge ||F||/2$,

$$||\tilde{F}_a|| = ||F_a|| - |Leaf_1(F) \ge \frac{n + 1 - c_f}{2} - 11\sqrt{d}n - c_f > \frac{n}{2} - 12\sqrt{d}n. \quad \square$$

Next we show that reasonably many trees in $F - Rt(F)$ have ratio not close to 0 or 1 by using the assumption that G is not in **EC1**. Let us first recall a simple fact on trees.

Fact 6.9. *Given a tree T, if $V(T)$ can be partitioned into a nonempty subset U_1 and an independent subset U_2, then U_2 contains at least $|U_2| - |U_1| + 1$ leaves. In particular, any tree with at least two vertices contains at least $||T_{even}| - |T_{odd}|| + 1$ leaves.*

Proof. Let a vertex $x \in U_1$ be the root (here we need $U_1 \neq \emptyset$). Let U_2' be the set of non-leaf vertices in U_2. Since each vertex in U_2' has at least one child in $U_1 \setminus \{x\}$ (using the fact that U_2 is independent) and the sets of children are disjoint, we have $|U_1| - 1 \geq |U_2'|$ and consequently the number of leaves in U_2 is at least $|U_2| - |U_1| + 1$. For the second assertion, assume that $v(T) \geq 2$. Then both of its partition sets T_{even} and T_{odd} are nonempty. Letting U_2 be the larger set of T_{even} and T_{odd}, then U_2 contains at least $||T_{even}| - |T_{odd}|| + 1$ leaves. □

Claim 6.10. *Let $\alpha_0 = \alpha/16$ and $F^2 = \{T \in F - Rt(F) : \alpha_0 < Ratio(T) < 1 - \alpha_0\}$. Then $v(F^2) > \alpha_0 n$.*

Proof. Let $F^1 := \tilde{F} \setminus F^2$. Then $v(F^1) + v(F^2) = ||\tilde{F}|| \geq (1 - 12\sqrt{d})n$ by Claim 6.8. Suppose to the contrary, that $v(F^2) \leq \alpha_0 n$ and consequently $v(F^1) \geq (1 - 12\sqrt{d} - \alpha_0)n$.

Consider a tree $T \in F^1$. The definition of $\tilde{F}$ implies that $v(T) \geq 2$. By Fact 6.9, T contains at least $||T_{even}| - |T_{odd}|| + 1$ leaves. Since $Ratio(T) \leq \alpha_0$ or $Ratio(T) \geq 1 - \alpha_0$, the tree T has at least $(1 - 2\alpha_0)v(T)$ leaves. The total number of leaves in F^1 is thus at least

$$(1 - 2\alpha_0)(1 - 12\sqrt{d} - \alpha_0)n > (1 - 2\alpha_0)(1 - 2\alpha_0)n = (1 - 4\alpha_0)n + 4\alpha_0^2 n.$$

Since F is a obtained from T by removing $c_f - 1$ edges, F has at most $2(c_f - 1)$ more leaves than T. Since $c_f \leq 2\varepsilon N$, we have $4{\alpha_0}^2 n > 2c_f + 1$. Then T has at least $(1 - 4\alpha_0)n + 1$ leaves, or at most $4\alpha_0 n$ non-leaf vertices.

On the other hand, the set L of large vertices of G contains at least $(1 - \varepsilon)n$ vertices. Let V_1 be a set of size n containing at least $(1 - \varepsilon)n$ vertices of L. Let $L_1 := V_1 \cap L$. Since G is not in **EC1** with parameter α, we have $d(V_1, V \setminus V_1) < 1 - \alpha$. Consequently

$$e(L_1, V \setminus L_1) = e(L_1, V \setminus V_1) + e(L_1, V_1 \setminus L_1) \leq (1 - \alpha)n^2 + \varepsilon n^2$$

and

$$e(L_1, L_1) = e(L_1, V) - e(L_1, V \setminus L_1) \geq (1 - \varepsilon)n^2 - (1 - \alpha + \varepsilon)n^2 > \alpha n^2/2.$$

Note that $e(L_1, L_1) = 2e(G[L_1])$, where $G[L_1]$ is the induced subgraph on L_1. Hence the average degree of $G[L_1]$ is at least $e(L_1, L_1)/|L_1| \geq \alpha n/2$. By a well-known fact in graph theory, $G[L_1]$ has a subgraph G_0 such that $\delta(G_0) \geq \alpha n/4 = 4\alpha_0 n$. We may therefore embed all non-leaf vertices of T into G_0 using the greedy algorithm. Since the vertices in L_1 have degree at least n, we can add all the leaves to complete the embedding of T by the greedy algorithm. This contradicts our assumption that $T \not\to G$. □

6.5.2 Partition $\mathcal{V}$ into two almost equal sets

The purpose of this subsection is to prove the following lemma, which shows that, among other things, there are about $k/2$ edges $e \in \mathcal{M}$ such that $\overline{\deg}(A, e) \approx 2$.

Lemma 6.11. *Let $\mathcal{O}, \mathcal{M}$ be given as in Claim 6.7. For any adjacent clusters $A, B \in \mathcal{O} \cap \mathcal{L}$, there is a sub-matching $\mathcal{M}_{in} \subset \mathcal{M}$ such that $\mathcal{M}_{in}$, $\mathcal{V}_1 := V(\mathcal{M}_{in})$ and $\mathcal{V}_2 := \mathcal{V} - \mathcal{V}_1$ satisfy the following properties.*

(i) $d(A, X), d(A, Y) > 1 - 2\eta$ *and* $\overline{\deg}(A, e) > 2 - 3\eta$ *for every* $e = \{X, Y\} \in \mathcal{M}_{in}$.

(ii) $\overline{\deg}(A, \mathcal{M}_{in}) > (1 - 8\eta)n$.

(iii) $(1 - 8\eta)k \le |\mathcal{V}_1| \le k$, *and consequently* $k \le |\mathcal{V}_2| < (1 + 8\eta)k$.

(iv) $\mathcal{V}_1 \subseteq \mathcal{O}$.

(v) *If* $f_d \ge d^{\frac{1}{4}} n$, $\overline{\deg}(B, \mathcal{M}_{in}) > (1 - 9\eta)n$.

(vi) *If* $f_d < d^{\frac{1}{4}} n$, *then there exists a matching* $\mathcal{M}_b \subset \mathcal{M} \setminus \mathcal{M}_{in}$ *such that*

$$|\mathcal{M}_b| \le 2d^{\frac{1}{4}} k \quad \text{and} \quad f_b + 3\gamma n \le \overline{\deg}(B, \mathcal{M}_b) < f_b + 3\gamma n + 2N. \tag{6.13}$$

In order to prove Lemma 6.11, we need the next few lemmas.

Lemma 6.12. *Suppose that $\overline{\deg}(B, \mathcal{M}) \ge (1 - 10\sqrt{d})n$ for some cluster B. If $f_b < d^{\frac{1}{4}} n$, then there exists a matching $\mathcal{M}_b \subset \mathcal{M}$ such that (6.13) holds.*

Proof. We arrange the edges $e \in \mathcal{M}$ in the decreasing order of $d(B, e)$ and denote them by $e_1, \dots, e_m$. Let j_0 be the smallest j such that $\sum_{i=1}^{j} d(B, e_i)N \ge f_b + 3\gamma n$ – such j exists because

$$\sum_{i=1}^{m} d(B, e_i)N = \overline{\deg}(B, \mathcal{M}) \ge (1 - 10\sqrt{d})n > d^{\frac{1}{4}} n + 3\gamma n \ge f_b + 3\gamma n.$$

Since $d(B, e_{j_0}) \le 2N$, we have $\sum_{i=1}^{j_0} d(B, e_i)N \le f_b + 3\gamma n + 2N$ (otherwise j_0 is not the smallest). Since $\{d(B, e_i) : i = 1, \dots, m\}$ is a decreasing sequence and $m \le k$, we have

$$\sum_{i=1}^{j_0} \frac{d(B, e_i)}{j_0} \ge \sum_{i=1}^{m} \frac{d(B, e_i)}{m} \ge (1 - 10\sqrt{d}) \frac{n}{Nk}.$$

Consequently

$$j_0 \le \frac{\sum_{i=1}^{j_0} d(B, e_i)}{(1 - 10\sqrt{d})\frac{n}{Nk}} \le \frac{d^{\frac{1}{4}} n + 3\gamma n + 2N}{(1 - 10\sqrt{d})n} k \le 2d^{\frac{1}{4}} k \quad \text{by using (6.1) and (6.5).}$$

Thus $\mathcal{M}_b := \{e_1, \dots, e_{j_0}\}$ satisfies (6.13). ☐

Lemma 6.13. *Suppose that G_r contains two adjacent clusters A, B and a cluster-matching $\mathcal{M}$ on $\mathcal{V} \setminus \{A, B\}$ such that*

$$\overline{\deg}(A, \mathcal{M}), \overline{\deg}(B, \mathcal{M}) \geq (1 - 10\sqrt{d})n. \tag{6.14}$$

If $f_b \geq d^{\frac{1}{4}} n$ and $T \not\subset G$, then $|\overline{\deg}(A, \mathcal{M}') - \overline{\deg}(B, \mathcal{M}')| < 15 d^{\frac{1}{4}} n$ for any *sub-matching $\mathcal{M}' \subseteq \mathcal{M}$.*

Proof. After removing some edges in G_r if necessary, we may assume that $\overline{\deg}(A, \mathcal{M}) = \overline{\deg}(B, \mathcal{M}) = (1 - 10\sqrt{d})n$. Define $\mathcal{M}^+ = \{e \in \mathcal{M} : d(A, e) > d(B, e)\}$ and $\mathcal{M}^- = \mathcal{M} - \mathcal{M}^+$. Write $a^+ = \overline{\deg}(A, \mathcal{M}^+)$, $a^- = \overline{\deg}(A, \mathcal{M}^-)$, $b^+ = \overline{\deg}(B, \mathcal{M}^+)$ and $b^- = \overline{\deg}(B, \mathcal{M}^-)$. We thus have $a^+ > b^+$, $b^- \geq a^-$, and $a^+ + a^- = b^+ + b^- = (1 - 10\sqrt{d})n$. By definition, $a^+ - b^+ = b^- - a^- = \max_{\mathcal{M}' \subseteq \mathcal{M}} |\overline{\deg}(A, \mathcal{M}') - \overline{\deg}(B, \mathcal{M}')|$.

Suppose that $f_a \geq f_b \geq d^{\frac{1}{4}} n$ and $a^+ - b^+ \geq 15 d^{\frac{1}{4}} n$. Our goal is derive $T \subset G$ by using Lemma 6.5 Part 1. Without loss of generality, we assume that $b^- \geq a^+$ (otherwise we exchange A and B). Then $b^- - b^+ = b^- - a^+ + a^+ - b^+ \geq 15 d^{\frac{1}{4}} n$. Since $b^- + b^+ = (1 - 10\sqrt{d})n$ and $f_b \leq n/2$, we have

$$b^- \geq (1 + 15 d^{\frac{1}{4}} - 10\sqrt{d}) \frac{n}{2} \overset{(6.1)}{>} \frac{n}{2} + 3\gamma n \geq f_b + 3\gamma n.$$

We now partition $\mathcal{M}$ into $\mathcal{M}_a$ and $\mathcal{M}_b$ as follows. Put the edges of M^- in the decreasing order of $\frac{d(B,e) - d(A,e)}{d(B,e)}$ and denote them by $e_1, \ldots, e_m$. Let j_0 be the smallest $j \geq 1$ such that $\sum_{i=1}^{j} d(B, e)N \geq f_b + 3\gamma n$ (j_0 exists because $\sum_{i=1}^{m} d(B, e)N = b^- > f_b + 3\gamma n$). Let $\mathcal{M}_b = \{e_1, \ldots, e_{j_0}\}$. Since $d(B, e)N \leq 2N$ for any e, we have

$$f_b + 3\gamma n \leq \overline{\deg}(B, \mathcal{M}_b) < f_b + 3\gamma n + 2N.$$

It is easy to see that if $\{\frac{a_i}{b_i}\}_{i=1}^m$ is a decreasing sequence, then for any $1 \leq j_0 \leq m$, we have

$$\frac{\sum_{i=1}^{j_0} a_i}{\sum_{i=1}^{j_0} b_i} \;\geq\; \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m} b_i}. \tag{6.15}$$

In fact, this follows from repeatedly applying the fact

$$\forall x_1, x_2, y_1, y_2 > 0, \quad \frac{x_1}{y_1} \geq \frac{x_2}{y_2} \quad \Rightarrow \quad \frac{x_1}{y_1} \geq \frac{x_1 + x_2}{y_1 + y_2} \geq \frac{x_2}{y_2}.$$

Applying (6.15) to $\left\{\frac{d(B,e_i) - d(A,e_i)}{d(B,e_i)}\right\}_{i=1}^{m}$, we obtain that

$$\frac{\overline{\deg}(B, \mathcal{M}_b) - \overline{\deg}(A, \mathcal{M}_b)}{\overline{\deg}(B, \mathcal{M}_b)} \geq \frac{\overline{\deg}(B, \mathcal{M}^-) - \overline{\deg}(A, \mathcal{M}^-)}{\overline{\deg}(B, \mathcal{M}^-)} = \frac{b^- - a^-}{b^-} \geq \frac{15 d^{\frac{1}{4}} n}{b^-}.$$

Consequently

$$\overline{\deg}(B, \mathcal{M}_b) - \overline{\deg}(A, \mathcal{M}_b) \geq \overline{\deg}(B, \mathcal{M}_b) \frac{15 d^{\frac{1}{4}} n}{b^-} > f_b \, \frac{15 d^{\frac{1}{4}} n}{n} \geq 15\sqrt{d} n,$$

where the last inequality uses the hypothesis $f_b > d^{\frac{1}{4}}n$.

Let $\mathcal{M}_a = \mathcal{M} - \mathcal{M}_b$. Then

$$\begin{aligned}\overline{\deg}(A, \mathcal{M}_a) &= \overline{\deg}(A, \mathcal{M}) - \overline{\deg}(A, \mathcal{M}_b) \\ &\geq (1 - 10\sqrt{d})n + 15\sqrt{d}n - (f_b + 3\gamma n + 2N) \\ &= (n - f_b) + (15\sqrt{d}n - 10\sqrt{d}n - 3\gamma n - 2N) \\ &> f_a + 3\gamma n.\end{aligned}$$

Thus $\mathcal{M}_a$ and $\mathcal{M}_b$ satisfy (6.10). Applying Lemma 6.5, Part 1, we derive that $T \subset G$. □

Lemma 6.14. *Suppose that G_r contains two adjacent clusters A, B and a cluster-matching $\mathcal{M}$ on $\mathcal{V} \setminus \{A, B\}$ such that* (6.14) *holds. Then $T \subset G$ if either of the following conditions holds.*

1. *There exist a root-subforest F_0 of F_a or F_b and a sub-matching $\mathcal{M}_0 \subset \mathcal{M}$ such that*

$$F_0 \to (A, \mathcal{M}_0), \quad \text{and} \quad ||F_0|| \geq \eta^3 n + \overline{\deg}(A, \mathcal{M}_0). \tag{6.16}$$

Furthermore, if $f_b \leq d^{\frac{1}{4}}n$, then exists $\mathcal{M}_b \subset \mathcal{M} \setminus \mathcal{M}_0$ that satisfies (6.13).

2. *There exists $\varepsilon_1, \varepsilon_2$ such that $\{d, \varepsilon_1\} \ll \varepsilon_2$. There is a partition of $\mathcal{M} = \mathcal{M}_{in} + \mathcal{M}_{out}$ such that $\overline{\deg}(A, \mathcal{M}_{in}) \geq (1 - \varepsilon_1)n$. There are sub-matchings $\mathcal{M}_0 \subset \mathcal{M}_{in}$ and $\mathcal{M}_2 \subset \mathcal{M}_{out}$ and a root-subforest F_0 of F_a such that*

$$F_0 \to (A, V(\mathcal{M}_0), \mathcal{M}_2), \quad \text{and} \quad ||F_0|| \geq \varepsilon_2 n + \overline{\deg}(A, \mathcal{M}_0). \tag{6.17}$$

Furthermore, if $f_b \leq d^{\frac{1}{4}}n$, then exists $\mathcal{M}_b \subset \mathcal{M}_{out} \setminus \mathcal{M}_2$ that satisfies (6.13).

Proof. *Part 1.* First assume that $F_0 \subseteq F_a$. Let $F_1 = F_a - E(F_0)$. Then $F_0 \cup F_1$ is a root-partition of F_a. Our goal is to partition $\mathcal{M} \setminus \mathcal{M}_0$ into $\mathcal{M}_1 \cup \mathcal{M}_2$ such that $F_1 \to (A, \mathcal{M}_1)$ and $F_b \to (B, \mathcal{M}_2)$. Together with (6.16), this implies that $F_a \xrightarrow{-4\sqrt{\varepsilon}N} (A, \mathcal{M}_a)$ by Proposition 5.7, where $\mathcal{M}_a = \mathcal{M}_0 \cup \mathcal{M}_1$. We then apply Lemma 6.3 Part 1 to obtain $T \subset G$.

We now separate cases based on the value of f_b.

Case 1a: $f_b \geq d^{\frac{1}{4}}n$. Since $\overline{\deg}(A, \mathcal{M}) \geq (1 - 10\sqrt{d})n$, $\overline{\deg}(A, \mathcal{M}_0) \leq ||F_0|| - \eta^3 n$, and $||F_0|| + ||F_1|| \leq n$, we have

$$\overline{\deg}(A, \mathcal{M} \setminus \mathcal{M}_0) \geq (1 - 10\sqrt{d})n - (||F_0|| - \eta^3 n) \geq ||F_1|| + 3\gamma n,$$

where the last inequality also uses $\gamma \ll d \ll \eta$. We can thus find a sub-matching $\mathcal{M}_1$ of $\mathcal{M} \setminus \mathcal{M}_0$ such that

$$||F_1|| + 3\gamma n \leq \overline{\deg}(A, \mathcal{M}_1) < ||F_1|| + 3\gamma n + 2N. \tag{6.18}$$

Let $\mathcal{M}_2 = \mathcal{M} \setminus (\mathcal{M}_0 \cup \mathcal{M}_1)$. By (6.14), (6.16), and (6.18),

$$\begin{aligned}\overline{\deg}(A, \mathcal{M}_2) &= \overline{\deg}(A, \mathcal{M}) - \overline{\deg}(A, \mathcal{M}_0) - \overline{\deg}(A, \mathcal{M}_1) \\ &\geq (1 - 10\sqrt{d})n - (||F_0|| - \eta^3 n) - (||F_1|| + 3\gamma n + 2N) \\ &\geq f_b + (\eta^3 - 15\sqrt{d})n,\end{aligned}$$

where the last inequality follows from $||F_1|| + ||F_0|| + f_b \leq n$, $N \leq \varepsilon n$ and (6.1). Since $f_b \geq d^{\frac{1}{4}} n$, by Lemma 6.13, we have $a^+ - b^+ \leq 15 d^{\frac{1}{4}} n$ (otherwise $T \subset G$ and we are done). Using (6.1), we obtain that

$$\overline{\deg}(B, \mathcal{M}_2) \geq \overline{\deg}(A, \mathcal{M}_2) - 15 d^{\frac{1}{4}} n \geq f_b + (\eta^3 - 15\sqrt{d})n - 15 d^{\frac{1}{4}} n \geq f_b + 3\gamma n. \quad (6.19)$$

By Lemma 5.8 Part 1, (6.18) and (6.19) imply that $F_1 \to (A, \mathcal{M}_1)$ and $F_b \to (B, \mathcal{M}_2)$, respectively.

Case 1b: $f_b < d^{\frac{1}{4}} n$. By assumption, there exists $\mathcal{M}_b \subset \mathcal{M} \setminus \mathcal{M}_0$ such that $|\mathcal{M}_b| \leq 2 d^{\frac{1}{4}} k$ and $f_b + 3\gamma n \leq \overline{\deg}(B, \mathcal{M}_b) < f_b + 3\gamma n + 2N$. Then $F_b \to (B, \mathcal{M}_b)$ by Lemma 5.8 Part 1. It remains to show that $F_1 \to (A, \mathcal{M}_1)$, where $\mathcal{M}_1 = \mathcal{M} \setminus (\mathcal{M}_0 \cup \mathcal{M}_b)$. Since $|\mathcal{M}_b| \leq 2 d^{\frac{1}{4}} k$, trivially $\overline{\deg}(A, \mathcal{M}_b) \leq 2N 2 d^{\frac{1}{4}} k \leq 4 d^{\frac{1}{4}} n$. By (6.14) and (6.16),

$$\begin{aligned}\overline{\deg}(A, \mathcal{M}_1) &= \overline{\deg}(A, \mathcal{M}) - \overline{\deg}(A, \mathcal{M}_0) - \overline{\deg}(A, \mathcal{M}_b) \\ &\geq (1 - 10\sqrt{d})n - (||F_0|| - \eta^3 n) - 4 d^{\frac{1}{4}} n \\ &\geq ||F_1|| + 3\gamma n.\end{aligned}$$

Then $F_1 \to (A, \mathcal{M}_1)$ follows from Lemma 5.8 Part 1.

The case when $F_0 \subseteq F_b$ can be handled similarly. Since $f_b \geq ||F_0|| \geq \eta^3 n \geq d^{\frac{1}{4}} n$, we can follow the procedure in Case 1a. More precisely, letting $F_1 = F_b - E(F_0)$, we first find a sub-matching $\mathcal{M}_1$ of $\mathcal{M} \setminus \mathcal{M}_0$ satisfying (6.18) and then derive $\overline{\deg}(B, \mathcal{M}_2) > f_a + 3\gamma n$. Lemma 5.8 Part 1 thus gives $F_1 \to (A, \mathcal{M}_1)$ and $F_a \to (B, \mathcal{M}_2)$. Together with $F_0 \to (A, \mathcal{M}_0)$, we obtain that $F_b \overset{-4\sqrt{\varepsilon}N}{\longrightarrow} (A, \mathcal{M}_b)$ by Proposition 5.7, where $\mathcal{M}_b = \mathcal{M}_0 \cup \mathcal{M}_1$. We finally apply Lemma 6.3 Part 1 to derive $T \subset G$.

Part 2. We proceed as in Part 1. Let $F_1 = F_a - E(F_0)$. First consider the case when $f_b \geq d^{\frac{1}{4}} n$. Since $\overline{\deg}(A, \mathcal{M}_{in}) \geq (1 - \varepsilon_1)n$ and $\overline{\deg}(A, \mathcal{M}_0) \leq ||F_0|| - \varepsilon_2 n$, we have

$$\overline{\deg}(A, \mathcal{M}_{in} \setminus \mathcal{M}_0) \geq (1 - \varepsilon_1)n - (||F_0|| - \varepsilon_2 n) = (n - ||F_0||) + (\varepsilon_2 - \varepsilon_1)n \geq ||F_1|| + 3\gamma n$$

by using $\{\gamma, \varepsilon_1\} \ll \varepsilon_2$. We thus find a sub-matching $\mathcal{M}_1$ of $\mathcal{M}_{in} \setminus \mathcal{M}_0$ satisfying (6.18). By letting $\mathcal{M}_2 = \mathcal{M}_{in} \setminus (\mathcal{M}_0 \cup \mathcal{M}_1)$, we derive that $\overline{\deg}(A, \mathcal{M}_2) \geq f_b + (\varepsilon_2 - \varepsilon_1 - 5\gamma)n$ and finally $\overline{\deg}(B, \mathcal{M}_2) \geq f_b + 3\gamma n$ as in Case 1a.

When $f_b < d^{\frac{1}{4}} n$, we let $\mathcal{M}_1 = \mathcal{M} \setminus (\mathcal{M}_0 \cup \mathcal{M}_b)$ and derive that $\overline{\deg}(A, \mathcal{M}_1) \geq ||F_1|| + 3\gamma n$ as in Case 1b. □

By Claim 6.8, most trees in $F_a - Rt(F_a)$ have at least two vertices. By Claim 6.10, at least $\alpha_0 n$ vertices are contained in the trees of $F - Rt(F)$ with ratio between α_0 and $1 - \alpha_0$. These facts and Lemma 5.8, Parts 2 and 3, lead to the following lemma, which will be also used in Claim 6.18 later.

Lemma 6.15. *Suppose that G_r contains two adjacent clusters A, B and a cluster-matching $\mathcal{M}$ on $\mathcal{V} \setminus \{A, B\}$ such that* (6.14) *holds. Let $\mathcal{M}_{unbal} = \{\{X, Y\} \in \mathcal{M} : |d(A, X) - d(A, Y)| \geq \eta\}$ and*

$$\mathcal{M}_{nonex} = \{\{X, Y\} \in \mathcal{M} : \eta \leq d(A, X) \leq 1 - \eta \text{ and } \eta \leq d(A, Y) \leq 1 - \eta\}.$$

If $|\mathcal{M}_{unbal}| \geq \eta k$ or $|\mathcal{M}_{nonex}| \geq \eta k$, then $T \subset G$.

Proof. By Lemma 6.14 Part 1, it suffices to show that there exist a root-subforest F_0 of F_a or F_b and a sub-matching $\mathcal{M}_0$ of $\mathcal{M}_{unbal}$ or $\mathcal{M}_{nonex}$ such that (6.16) holds, and if $f_b < d^{\frac{1}{4}} n$, there also exists $\mathcal{M}_b \subset \mathcal{M} \setminus \mathcal{M}_0$ that satisfies (6.13).

Case 1: $|\mathcal{M}_{unbal}| \geq \eta k$. If $f_b \geq d^{\frac{1}{4}} n$, we pick a matching $\mathcal{M}_0 \subseteq \mathcal{M}_{unbal}$ of size $\eta k/2$. If $f_b < d^{\frac{1}{4}} n$, then by Lemma 6.12, there exists a sub-matching $\mathcal{M}_b \subset \mathcal{M}$ satisfying (6.13). Since $|\mathcal{M}_b| \leq 2d^{\frac{1}{4}} k$ and $2d^{\frac{1}{4}} \leq \eta/2$, we can still pick a matching $\mathcal{M}_0 \subseteq (\mathcal{M}_{unbal} \setminus \mathcal{M}_b)$ of size $\eta k/2$.

Recall that $F^2 = \{T \in F - Rt(F) : \alpha_0 < Ratio(T) < 1 - \alpha_0\}$. Claim 6.10 says that $v(F^2) \geq cn$. Let $F_a^2 = F^2 \cap F_a$ and $F_b^2 = F^2 \cap F_b$. Without loss of generality, assume that $v(F_a^2) \geq \alpha_0 n/2$. By (6.1), we have $\alpha_0 \geq 4\eta$ and thus

$$\frac{\alpha_0}{2} n > 2N|\mathcal{M}_0| + \eta^3 n \geq \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n.$$

Since any tree in F_a^2 has at most εN vertices, we can find a sub-forest $\hat{F}_0$ of F_a^2 such that

$$\overline{\deg}(A, \mathcal{M}_0) + \eta^3 n \leq v(\hat{F}_0) < \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n + \varepsilon N.$$

By adding the vertices in $Rt(F_a)$ adjacent to the roots of $\hat{F}_0$, we extend $\hat{F}_0$ to a root-subforest F_0 of F. Then $||F_0|| = v(\hat{F}_0)$. Since $\alpha_0 \geq 4\eta$ and $\varepsilon \ll \gamma \ll \eta$,

$$||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n + \varepsilon N < \overline{\deg}(A, \mathcal{M}_0) + \alpha_0 \eta N |\mathcal{M}_0| - 3\gamma n.$$

By Lemma 5.8 Part 2, we derive $F_0 \to (A, \mathcal{M}_0)$ and consequently (6.16).

Case 2: $|\mathcal{M}_{nonex}| \geq \eta k$. As in Part 1, we can pick a sub-matching $\mathcal{M}_0 \subseteq \mathcal{M}_{nonex}$ of size $\eta k/2$ such that if $f_b < d^{\frac{1}{4}} n$, there also exists $\mathcal{M}_b \subset \mathcal{M} \setminus \mathcal{M}_0$ that satisfies (6.13). By Claim 6.8, $||\tilde{F}_a|| \geq n/2 - 12\sqrt{d}n > 2N|\mathcal{M}_0| + \eta^3 n$. Since $\tilde{F}_a$ is an εN-forest, we may find a root-subforest F_0 of $\tilde{F}_a$ (thus a root-subforest F_0 of F_a) such that

$$\overline{\deg}(A, \mathcal{M}_0) + \eta^3 n \leq ||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n + \varepsilon N.$$

Hence $||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \eta N |\mathcal{M}_0| - 3\gamma n$. By Lemma 5.8 Part 3, we obtain $F_0 \to (A, \mathcal{M}_0)$ and consequently (6.16). □

We are ready to prove Lemma 6.11 now.

Proof of Lemma 6.11. Define $\mathcal{M}_{unbal}, \mathcal{M}_{nonex}$ as in Lemma 6.15, which gives that $|\mathcal{M}_{unbal}|, |\mathcal{M}_{nonex}| \leq \eta k$. Let $\mathcal{M}_{small} = \{\{X, Y\} \in \mathcal{M} \setminus \mathcal{M}_{unbal} : d(A, X) < \eta$ or $d(A, Y) < \eta\}$. Consider $\{X, Y\} \in \mathcal{M}_{small}$. One of $d(A, X)$ and $d(A, Y)$ is smaller than η and $|d(A,X) - d(A,Y)| < \eta$. Consequently $d(A,X) + d(A,Y) < 3\eta$ and hence $\overline{\deg}(A, \mathcal{M}_{small}) < 3\eta Nk$.

If $f_b < d^{\frac{1}{4}}n$, then we apply Lemma 6.12 and find a sub-matching $\mathcal{M}_b \subset \mathcal{M}$ satisfying (6.13). Since $|\mathcal{M}_b| \le 2d^{\frac{1}{4}}k$, trivially $\overline{\deg}(A, \mathcal{M}_b) \le 4d^{\frac{1}{4}}n$. Let

$$\mathcal{M}'_{in} = \begin{cases} \mathcal{M} - \mathcal{M}_{unbal} - \mathcal{M}_{nonex} - \mathcal{M}_{small} & \text{if } f_b \ge d^{\frac{1}{4}}n \\ \mathcal{M} - \mathcal{M}_{unbal} - \mathcal{M}_{nonex} - \mathcal{M}_{small} - \mathcal{M}_b & \text{if } f_b < d^{\frac{1}{4}}n. \end{cases}$$

Consider $e = \{X, Y\} \in \mathcal{M}'_{in}$. We have $|d(A,X)-d(A,Y)| < \eta$ and, by the definition of $\mathcal{M}_{nonex}$, either $d(A,X) > 1-\eta$ or $d(A,Y) > 1-\eta$. Consequently $d(A,X), d(A,Y) > 1-2\eta$ and $\overline{\deg}(A, e) > 2 - 3\eta$.

Recall that $\mathcal{M}^2(A)$ is the set of those $\{X, Y\} \in \mathcal{M}$ such that $d(A,X), d(A,Y) > 0$. Thus $\mathcal{M}_{in} \subseteq \mathcal{M}^2(A)$. Then, by Claim 6.7 Part 3, at most one cluster in $V(\mathcal{M}_{in})$ may not be in $\mathcal{O}$. Let $e_1 \in \mathcal{M}'_{in}$ denote the edge containing this cluster if it exists (otherwise $e_1 = \emptyset$). Let $\mathcal{M}_{in} = \mathcal{M}'_{in} - \{e_1\}$ if $|\mathcal{M}'_{in} - \{e_1\}| \le k/2$; otherwise let $\mathcal{M}_{in}$ be a sub-matching of $\mathcal{M}'_{in} - \{e_1\}$ of size $\lfloor k/2 \rfloor$.

This definition of $\mathcal{M}_{in}$ implies (i), (iv), and (vi) immediately. If $|\mathcal{M}_{in}| = \lfloor k/2 \rfloor$, then we have $\overline{\deg}(A, \mathcal{M}_{in}) \ge (2-3\eta)N\lfloor k/2 \rfloor > (1-8\eta)n$ because $\overline{\deg}(A, e) > 2 - 3\eta$ for each $e \in \mathcal{M}_{in}$. Otherwise $\mathcal{M}_{in} = \mathcal{M}'_{in} - \{e_1\}$; by the definition of $\mathcal{M}'_{in}$,

$$\overline{\deg}(A, \mathcal{M}_{in}) > (1 - 10\sqrt{d})n - \eta k 2N - \eta k 2N - 3\eta Nk - 4d^{\frac{1}{4}}n - 2N > (1-8\eta)n.$$

We thus have (ii) in either case. If $f_b \ge d^{\frac{1}{4}}n$, then by Lemma 6.13, $\overline{\deg}(B, \mathcal{M}_{in}) > (1-8\eta)n - 15d^{\frac{1}{4}}n \ge (1-9\eta)n$, which give (v).

Let $\mathcal{V}_1 = V(\mathcal{M}_{in})$ and $\mathcal{V}_2 = \mathcal{V} - \mathcal{V}_1$. Then $(1-8\eta)k \le (1-8\eta)n/N \le |\mathcal{V}_1| \le k$, and consequently $(1-2\eta)k \le |\mathcal{V}_2| < (1+8\eta)k$. Hence (iii) holds. $\square$

6.5.3 Edges between $\mathcal{V}_1$ and $\mathcal{V}_2$

Let $\mathcal{M}_{in}, \mathcal{V}_1, \mathcal{V}_2$ be given by Lemma 6.11 with properties (i) – (vi). Let $\mathcal{M}_{out} := \mathcal{M} - \mathcal{M}_{in}$. Let V_i denote the set of vertices of G contained in the clusters in $\mathcal{V}_i$ for $i = 1, 2$. Items (ii) and (iii) together imply that $(1-8\eta)n \le \overline{\deg}(A, \mathcal{M}_{in}) \le |V_1| \le n$, which means that both $|V_1|$ and $|V_2|$ are very close to n. The goal of this subsection is to show that $e(V_1, V_2)$ is very small and thus G is in **EC2**.

More precisely, if $e_{G_r}(\mathcal{V}_1, \mathcal{V}_2) \le \rho k^2$ for ρ satisfying (6.1), then

$$e_{G''}(V_1, V_2) \le dN^2|\mathcal{V}_1||\mathcal{V}_2| + \sum_{X \in \mathcal{V}_1, Y \in \mathcal{V}_2, X \sim Y} N^2 \le (\rho + d)n^2, \tag{6.20}$$

which implies that $e_G(V_1, V_2) < 2\rho n^2$. After adding or removing at most $8\eta n$ vertices to or from V_1 such that $|V_1| = |V_2| = n$, we still have $e(V_1, V_2) < 3\rho n^2$, which contradicts the assumption that **EC2** does not hold.

We therefore assume that

$$e_{G_r}(\mathcal{V}_1, \mathcal{V}_2) > \rho k^2. \tag{6.21}$$

Our next claim says that not many trees in $F_a - Rt(F_a)$ have more than two vertices. The following is its proof idea. If a cluster $X \in \mathcal{V}_1$ has many neighbors in $\mathcal{M}_{out}$, then we may use Lemma 5.9 Part 2 to embed a tree $T_i \in F_a \to (A, X, \mathcal{M}_{out})$ such that $Rt(T_i) \to A$, $Level_1(T_i) \to X$, and $Level_{\geq 2}(T_i) \to \mathcal{M}_{out}$. When T_i has more than 3 vertices, this embedding is more efficient than embedding T_i into $A \cup \mathcal{M}_{in}$. If many trees in F_a have more than 3 vertices, then we obtain a subforest $\tilde{F}_a$ satisfying (6.17) in Claim 6.14.

Claim 6.16. *Let $F_3 = \{T \in F_a - Rt(F_a) : v(T) \geq 3\}$ and $\rho_0 = \rho/10$. Then $v(F_3) < 3\rho_0 n$.*

Proof. Suppose instead, that $v(F_3) \geq 3\rho_0 n$. By Lemma 6.11 (vi), if $f_d < d^{\frac{1}{4}} n$, then there exists a matching $\mathcal{M}_b \subset \mathcal{M}_{out}$ satisfying (6.13). Let

$$\mathcal{M}_2 = \begin{cases} \mathcal{M}_{out} & \text{if } f_b \geq d^{\frac{1}{4}} n \\ \mathcal{M}_{out} \setminus \mathcal{M}_b & \text{if } f_b < d^{\frac{1}{4}} n. \end{cases}$$

Let $\mathcal{V}_1'$ be the set of clusters $C \in \mathcal{V}_1$ such that $\deg_{G_r}(C, \mathcal{V}_2) \geq 9\rho_0 k$. Then $|\mathcal{V}_1'| \geq \rho_0 k$, otherwise $e(\mathcal{V}_1, \mathcal{V}_2) < \rho_0 k |\mathcal{V}_2| + |\mathcal{V}_1| 9\rho_0 k \leq 10\rho_0 k^2$, contradicting (6.21). Let $\mathcal{C}$ be a subset of $\mathcal{V}_1'$ of size $\rho_0 k$, and $\mathcal{M}_0$ be the (minimum) sub-matching of $\mathcal{M}_{in}$ that covers $\mathcal{C}$. Then $|\mathcal{M}_0| \leq |\mathcal{C}| = \rho_0 k$. We know that $\mathcal{C} \subset \mathcal{O}$ from Lemma 6.11 (iv). Consider a cluster $C \in \mathcal{C}$. By Claim 6.7 Part 2, all but at most $9\sqrt{d}k$ neighbors in $\mathcal{V}_2$ of C are covered by $\mathcal{M}_{out}$. If $\mathcal{M}_b$ exists, then $\deg_{G_r}(C, V(\mathcal{M}_b)) \leq 4d^{\frac{1}{4}} k$ because $|\mathcal{M}_b| \leq 2d^{\frac{1}{4}} k$. Since $\deg_{G_r}(C, \mathcal{V}_2) \geq 9\rho_0 k$, we have

$$\deg_{G_r}(C, V(\mathcal{M}_2)) \geq \deg_{G_r}(C, \mathcal{V}_2) - 9\sqrt{d}k - 4d^{\frac{1}{4}} k \geq 8\rho_0 k. \tag{6.22}$$

Since $v(F_3) \geq 3\rho_0 n$, $\overline{\deg}(A, \mathcal{M}_0) \leq 2N|\mathcal{M}_0| \leq 2\rho_0 n$ and every tree in F_3 has at most εN vertices, we can find a root-subforest F_0 of F_a such that $F_0 - Rt(F_0) \subseteq F_3$ and

$$\overline{\deg}(A, \mathcal{M}_0) + \rho_0 \frac{n}{2} \leq ||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \rho_0 \frac{n}{2} + \varepsilon N. \tag{6.23}$$

It remains to show that $F_0 \to (A, \mathcal{C}, \mathcal{M}_2)$ because then we can apply Claim 6.14 Part 2 with $\varepsilon_1 = 8\eta$ and $\varepsilon_2 = \rho_0/2$ to embed $T \to G$. Let $m = \min_{C \in \mathcal{C}} |\{e \in \mathcal{M}_2 : d(C, e) > 0\}|$. We have $m \geq \deg_{G_r}(C, V(\mathcal{M}_2))/2 \geq 4\rho_0 k$ by (6.22). Together with (6.23), this gives $||F_0|| \leq (1-\gamma)mN$. Since every tree in $F_0 - Rt(F_0)$ has at least three vertices, we have $|Level_1(F_0)| \leq ||F_0||/3 \leq (5\rho_0 \frac{n}{2} + \varepsilon N)/3$. By using Lemma 6.11 (i) and $|\mathcal{C}| = \rho_0 k$, we have

$$\overline{\deg}(A, \mathcal{C}) - 2\gamma|\mathcal{C}|N \geq (1 - 2\eta - 2\gamma)N|\mathcal{C}| \overset{(6.1)}{\geq} \frac{5\rho_0 n/2 + \varepsilon N}{3} \geq |Level_1(F_0)|.$$

We thus apply Lemma 5.9 Part 2 to obtain $F_0 \to (A, \mathcal{C}, \mathcal{M}_2)$. □

Recall that $\tilde{F}_a$ is the subforest of F_a obtained by removing all the leaves in $Level_1(F_a)$, and $||\tilde{F}_a|| > n/2 - 12\sqrt{d}n$ by Claim 6.8. A *root-2-path* in F is a path of length 2 having one end in $Rt(F)$. Claim 6.16 implies that most vertices of $\tilde{F}_a$ are covered by root-2-paths.

Let $\mathcal{S}_1 = \{Y : \{X, Y\} \in \mathcal{M}_{in}$ for some $X \in \mathcal{L}\}$, the set of clusters whose partners in $\mathcal{M}_{in}$ are large clusters. Since no regular pair runs between two small clusters, all the small clusters of $\mathcal{V}_1$ are contained in $\mathcal{S}_1$ (though $\mathcal{S}_1$ may contain large clusters as well). Let $\mathcal{L}_1 = \mathcal{V}_1 \setminus \mathcal{S}_1$. Since their partners in $\mathcal{M}_{in}$ are small clusters, all the clusters in $\mathcal{L}_1$ are large and located in different regular pairs of $\mathcal{M}_{in}$.

Claim 6.17. $e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) < 16\rho k^2$.

Proof. By (vi) in Lemma 6.11, if $f_d < d^{\frac{1}{4}}n$, then there exists a matching $\mathcal{M}_b \subset \mathcal{M}_{out}$ satisfying (6.13). Let

$$\mathcal{V}_2' = \begin{cases} \mathcal{V}_2 & \text{if } f_b \geq d^{\frac{1}{4}}n \\ \mathcal{V}_2' \setminus V(\mathcal{M}_b) & \text{if } f_b < d^{\frac{1}{4}}n. \end{cases}$$

We may assume that there are at least $10\rho k$ clusters in $\mathcal{S}_1$ that have degree at least $5\rho k$ in $\mathcal{V}_2'$. For instead, at most $10\rho k$ clusters in $\mathcal{S}_1$ have degree at least $5\rho k$ in $\mathcal{V}_2'$. Since $|V(\mathcal{M}_b)| \leq 4d^{\frac{1}{4}}k$ (if exists), at most $10\rho k$ clusters in $\mathcal{S}_1$ have degree at least $5\rho k + 4d^{\frac{1}{4}}k$ in $\mathcal{V}_2$. By using $|\mathcal{S}_1| \leq |\mathcal{V}_1| \leq k$ and $|\mathcal{V}_2| \leq (1 + 8\eta)k$, we derive

$$e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) < 10\rho k|\mathcal{V}_2| + |\mathcal{S}_1|(5\rho + 4d^{\frac{1}{4}})k \leq 10\rho k(1 + 8\eta)k + (5\rho + 4d^{\frac{1}{4}})k^2 \leq 16\rho k^2,$$

we are done. We then pick $5\rho k$ such clusters that are located in different pairs of $\mathcal{M}_{in}$ and denote this cluster-set by $\mathcal{S}_0$. Let $\mathcal{M}_0$ be the minimum sub-matching of $\mathcal{M}_{in}$ covering $\mathcal{S}_0$. Let $\mathcal{L}_0 = V(\mathcal{M}_0) \setminus \mathcal{S}_0$ be the partner set of $\mathcal{S}_0$. The definition of $\mathcal{S}_1$ implies that $\mathcal{L}_0 \subset \mathcal{L}$. Since $\deg(C, \mathcal{V}_2') \geq 5\rho k = |\mathcal{S}_0|$ for all $C \in \mathcal{S}_0$, for each element of $\mathcal{S}_0$ we may choose a distinct neighbor in $\mathcal{V}_2'$ thus forming a new matching $\mathcal{M}_0'$ that covers $\mathcal{S}_0$. Let $\mathcal{M}' = \mathcal{M}_{in} - \mathcal{M}_0 + \mathcal{M}_0'$. Then $\mathcal{M}'$ and $\mathcal{M}_b$ are disjoint matchings.

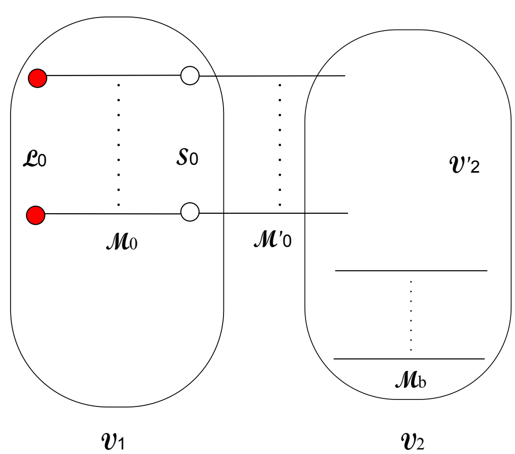

Figure 2: $\mathcal{M}' = \mathcal{M}_{in} - \mathcal{M}_0 + \mathcal{M}_0'$

By Claims 6.8 and 6.16, there are at least $(\frac{n}{2} - 12\sqrt{d} - 3\rho_0 n)/2 > n/8$ root-2-paths in F_a. Since there are $c_f - 1 < 2\varepsilon N$ parent-vertices, we can pick $4\rho n$ root-2-paths which contain no parent-vertices (hence these paths may be embedded at any time). Let Z be the set of the mid-points and leaves in these paths. Then $|Z| = 8\rho n$. Let $T' = T - Z$. Then T' is a tree with $n - 8\rho n$ edges. Below we first embed T' into $A \cup B \cup V(\mathcal{M}')$ and then embed Z by using $\mathcal{L}_0$.

We claim that T' or its εN-forest F' satisfies the conditions of Lemma 6.5, thus $T' \subset G$ follows. First assume that $f_b \geq d^{\frac{1}{4}} n$. By (ii) and (v) in Lemma 6.11, we have $\overline{\deg}(A, \mathcal{M}_{in}) \geq (1 - 8\eta)n$, and $\overline{\deg}(B, \mathcal{M}_{in}) \geq (1 - 9\eta)n$. We thus derive

$$\overline{\deg}(A, \mathcal{M}') \geq \overline{\deg}(A, \mathcal{M}_{in}) - \overline{\deg}(A, \mathcal{L}_0) \geq (1 - 8\eta - 5\rho)n,$$

and similarly $\overline{\deg}(B, \mathcal{M}_{in}) \geq (1 - 9\eta - 5\rho)n$. Since $||T'|| = (1 - 8\rho)n$ and $\gamma \ll \eta \ll \rho$, we have $||T'|| \leq \min\{\overline{\deg}(A, \mathcal{M}'), \overline{\deg}(B, \mathcal{M}')\} - 8\gamma n$, as desired by Lemma 6.5 Part 2. Now assume that $f_b < d^{\frac{1}{4}} n$. Note that $F' = F'_a \cup F_b$ with $F'_a = F_a - Z$. We have $||F'_a|| \leq ||T'|| \leq \overline{\deg}(A, \mathcal{M}') - 8\gamma n$. By (6.13), we have $||F'_b|| \leq \overline{\deg}(B, \mathcal{M}') - 3\gamma n$. Since $\mathcal{M}'$ and $\mathcal{M}_b$ are disjoint, we are under the condition of Lemma 6.5 Part 1.

We next embed all the mid-points in Z into the clusters of $\mathcal{L}_0$ and embed all the leaves in Z at last by the greedy algorithm. By definition, each large cluster contains at least $2\sqrt{d}N$ *large* vertices, whose degrees in G are at least n. By Claim 6.1, at least $(1 - \sqrt{\varepsilon})N$ vertices have degree at least $(1 - 5d)n$ in G – we call them *near-large* vertices. For each $X \in \mathcal{L}_0$, we take two *disjoint* subsets $P_X, Q_X \subset X$ such that P_X consists of $2\sqrt{d}N$ large vertices and Q_X consists of $(1 - 2\sqrt{d} - \sqrt{\varepsilon})N$ near-large vertices. By Proposition 4.5, at most $\sqrt{\varepsilon}N$ vertices of A are atypical to $\{P_X : X \in \mathcal{L}_0\}$; at most $\sqrt{\varepsilon}N$ vertices of A are atypical to $\{Q_X : X \in \mathcal{L}_0\}$. Let $A_0 \subset A$ consist of all large vertices that are typical to both $\{P_X : X \in \mathcal{L}_0\}$ and $\{Q_X : X \in \mathcal{L}_0\}$. Then $|A_0| \geq 2\sqrt{d}N - 2\sqrt{\varepsilon}N > \sqrt{d}N$. Lemma 6.5 says that we can embed $Rt(F_a)$ to A_0 while embedding T'. This means if $u \in A_0$ is the image of a root in F_a, there exist subsets $\mathcal{L}'_0, \mathcal{L}''_0 \subseteq \mathcal{L}_0$ such that $|\mathcal{L}'_0|, |\mathcal{L}''_0| \geq (1 - \sqrt{\varepsilon})|\mathcal{L}_0|$ and

$$\begin{aligned} \deg(u, P_X) &\geq (d(A, X) - \varepsilon)|P_X| \quad \text{for all } X \in \mathcal{L}'_0, \\ \deg(u, Q_X) &\geq (d(A, X) - \varepsilon)|Q_X| \quad \text{for all } X \in \mathcal{L}''_0. \end{aligned}$$

By Lemma 6.11 (i), we have $d(A, X) \geq 1 - 2\eta$ for $X \in \mathcal{L}_0$. We partition the to-be-embedded $4\rho n$ root-2-paths into two groups, with $(4\rho - 5d)n$ paths in group 1 and $5dn$ paths in group 2. We embed the mid-points of the paths in group 1 into $\bigcup_{X \in \mathcal{L}''_0} N(u, Q_X)$, and the mid-points of the paths in group 2 into $\bigcup_{X \in \mathcal{L}'_0} N(u, P_X)$ for some $u \in A_0$ (note that P_X and Q_X are disjoint). This is possible because

$$\begin{aligned} \sum_{X \in \mathcal{L}''_0} \deg(u, Q_X) &\geq |\mathcal{L}''_0|(1 - 2\eta - \varepsilon)|Q_X| \\ &\geq (1 - \sqrt{\varepsilon})5\rho k(1 - 2\eta - \varepsilon)(1 - 2\sqrt{d} - \sqrt{\varepsilon})N \\ &> (4\rho - 5d)n, \end{aligned}$$

$$\sum_{X \in \mathcal{L}'_0} \deg(u, P_X) \geq |\mathcal{L}'_0|(1 - 2\eta - \varepsilon)|P_X| \geq (1 - \sqrt{\varepsilon})5\rho k(1 - 2\eta - \varepsilon)2\sqrt{d}N > 5dn.$$

To finish the embedding, we choose an unoccupied (distinct) neighbor to be the leaf for each of the $(4\rho - 5d)n$ vertices embedded in $Q_X, X \in \mathcal{L}_0''$. This is possible because each vertex in Q_X has degree at least $(1-5d)n$. Finally, we attach one leaf to each of the $5dn$ vertices embedded in $P_X, X \in \mathcal{L}_0'$. □

Let G_r' be the subgraph of G_r containing all regular pairs between $\mathcal{V}_1$ and $\mathcal{V}_2$ with density at least 2η. We claim that $e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2)$ is small.

Claim 6.18. $e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2) < 16\rho_1 k^2$, *where* $\rho_1 = \rho^{1/3}$.

Proof. We assume that there is a subset $\mathcal{L}_0 \subseteq \mathcal{L}_1$ of size $8\rho_1 k$ such that every $C \in \mathcal{L}_0$ has at least $8\rho_1 k$ G_r'-neighbors in $\mathcal{V}_2$ (neighbors in $\mathcal{V}_2$ with respect to G_r'). Otherwise $e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2) < 8\rho_1 k|\mathcal{L}_1| + |\mathcal{V}_2| 8\rho_1 k \le 8\rho_1 k(2k)$, and we are done. By the definition of $\mathcal{L}_1$, the clusters in $\mathcal{L}_1$ must be large and located in different regular pairs in which the other ends (*partners*) are small clusters. Let $\mathcal{S}_0$ be the set of the partners of $\mathcal{L}_0$. Then $\mathcal{S}_0$ is a subset of $\mathcal{S}_1$. Our goal is to derive that $e_{G_r}(\mathcal{S}_0, \mathcal{V}_2) \ge 16\rho k^2$, which contradicts Claim 6.17.

Fix a cluster $C \in \mathcal{L}_0$. We find a set $\mathcal{N}_C \subseteq N_{G_r}(C) \cap \mathcal{L} \cap \mathcal{O}$ of size $|\mathcal{N}_C| \ge 3\rho_1 k$ as follows. From (iv) in Lemma 6.11, we know that $C \in \mathcal{O}$. By Claim 6.7 Part 2, $\mathcal{M}$ contains all but at most $9\sqrt{d}k$ G_r-neighbors of C in $\mathcal{V}_2$. Consequently $\mathcal{M}_{out}$ contains all but at most $9\sqrt{d}k$ G_r'-neighbors of C in $\mathcal{V}_2$. Let $\mathcal{M}_C$ be the minimum sub-matching of $\mathcal{M}_{out}$ that covers all the G_r'-neighbors of C in $\mathcal{M}_{out}$. Then

$$\deg_{G_r'}(C, V(\mathcal{M}_C)) = \deg_{G_r'}(C, V(\mathcal{M}_{out})) \ge \deg_{G_r'}(C, \mathcal{V}_2) - 9\sqrt{d}k \ge 8\rho_1 k - 9\sqrt{d}k. \quad (6.24)$$

Now let $\tilde{\mathcal{M}}_C$ be the set of $\{X, Y\} \in \mathcal{M}_C$ such that $|d(C,X) - d(C,Y)| \ge \eta$. Since $C \in \mathcal{L} \cap \mathcal{O}$, we have $\overline{\deg}(C, \mathcal{M}) \ge (1 - 10\sqrt{d})n$ from (6.12). Since C and A are adjacent, we can apply Lemma 6.15 with $A = C$ and $B = A$: since $T \not\subset G$, we have $|\tilde{\mathcal{M}}_C| < \eta k$. Let $\mathcal{M}_C' = \mathcal{M}_C \setminus \tilde{\mathcal{M}}_C$. Then $|\mathcal{M}_C'| \ge |\mathcal{M}_C| - \eta k$. For any $\{X, Y\} \in \mathcal{M}_C'$, by the definition of G_r', one of $d(C,X)$ and $d(C,Y)$ is at least 2η, consequently the other density is at least η. This implies that $\mathcal{M}_C' \subseteq \mathcal{M}^2(C)$, namely, for every $\{X, Y\} \in \mathcal{M}_C'$ both X and Y are adjacent to C *in* G_r. By Claim 6.7 Part 3, all but at most one cluster in $V(\mathcal{M}_C')$ are members of $\mathcal{O}$. We therefore take a set $\mathcal{N}_C \subset \mathcal{L} \cap \mathcal{O}$ by picking one large cluster from each edge of $\mathcal{M}_C'$ unless this large cluster is *not* in $\mathcal{O}$. Consequently

$$\begin{aligned} |\mathcal{N}_C| &= |\mathcal{M}_C'| - 1 \ge |\mathcal{M}_C| - \eta k - 1 \\ &\ge \frac{1}{2}\deg_{G_r'}(C, V(\mathcal{M}_C)) - \eta k - 1 \\ &\overset{(6.24)}{\ge} \frac{1}{2}(8\rho_1 k - 9\sqrt{d}k) - \eta k - 1 > 3\rho_1 k. \end{aligned} \quad (6.25)$$

Let $\mathcal{N} = \cup_{C \in \mathcal{L}_0} \mathcal{N}_C$ (then $\mathcal{N} \subset \mathcal{V}_2 \cap \mathcal{L} \cap \mathcal{O}$). Define a bipartite graph H on $\mathcal{L}_0 \cup \mathcal{N}$ such that $C \in \mathcal{L}_0$ is adjacent to $D \in \mathcal{N}$ if and only if C, D are adjacent in G_r'. Let $\mathcal{N}_0$ be the set of $D \in \mathcal{N}$ such that $\deg_H(D) \ge 12\rho_1^2 k$. Since $|\mathcal{L}_0| = 8\rho_1 k$, (6.25) implies that

$$24{\rho_1}^2 k^2 = |\mathcal{L}_0|\, 3\rho_1 k \le |E(H)| \le |\mathcal{N}_0|\, 8\rho_1 k + |\mathcal{N}|\, 12\rho_1^2 k.$$

By using $|\mathcal{N}| \leq |\mathcal{V}_2| \leq (1+8\eta)k$, we obtain that $|\mathcal{N}_0| \geq 3(1-8\eta)\rho_1 k/2$. It suffices to show that $\deg_{G_r}(D, \mathcal{S}_0) \geq 11{\rho_1}^2 k$ for every $D \in \mathcal{N}_0$ because then we have (by using $\rho_1^3 = \rho \gg \eta$)

$$e_{G_r}(\mathcal{N}_0, \mathcal{S}_0) > \frac{3}{2}\,(1-8\eta)\,\rho_1 k\; 11\rho_1^2 k > 16\rho k^2.$$

a contradiction to Claim 6.17.

Fix a cluster $D \in \mathcal{N}_0$, and assume that $D \in \mathcal{N}_C$ for some $C \in \mathcal{L}_0$. Since $D, C \in \mathcal{L} \cap \mathcal{O}$ and $D \sim C$, we may apply Lemma 6.15 with $A = D$ and $B = C$. As a result, at most ηk pairs $\{X, Y\} \in \mathcal{M}_{in}$ satisfy $d(D, X) \geq \eta$ and $d(D, Y) = 0$. The definition of $\mathcal{N}_0$ implies that D has at least $12\rho_1^2 k$ G'_r-neighbors in $\mathcal{L}_0$. Since $\mathcal{S}_0$ is the partner set of $\mathcal{L}_0$ in $\mathcal{M}_{in}$, it follows that D has at least $12{\rho_1}^2 k - \eta k > 11{\rho_1}^2 k$ G_r-neighbors in $\mathcal{S}_0$. In other words, $\deg_{G_r}(D, \mathcal{S}_0) \geq 11{\rho_1}^2 k$, as desired. □

From Claim 6.17 and 6.18, we conclude that

$$e_{G'_r}(\mathcal{V}_1, \mathcal{V}_2) \leq e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) + e_{G'_r}(\mathcal{L}_1, \mathcal{V}_2) < 16\rho k^2 + 16\rho_1 k^2 < 32\rho_1 k^2.$$

Using the same arguments as in (6.20) and $d < \eta$, we derive that $e_{G''}(V_1, V_2) < (32\rho_1 + 2\eta)n^2$. Since $\alpha \geq 32\rho_1 + 2\eta$, it follows that G is in **EC2** with parameter α, contradiction. We have thus completed the proof of Theorem 3.3. □

7 The extremal cases

In this section we prove Proposition 3.1 and Theorem 3.2. The proof of Proposition 3.1 is straightforward, but a proof of Theorem 3.2 is far from trivial. To prove it, we first define and handle a particular extremal case (denoted by **EC3**), in which the embedding of T mainly takes place in one partition set V_1 and then show that the assumption of Theorem 3.2, **EC2** actually implies **EC3**.

We first list a few facts to be used in both proofs.

Fact 7.1. *Let $0 < c < 1$ and G_1 be a graph of order n containing two disjoint vertex sets A and B. If $e(A, B) \geq (1-c)|A||B|$, then there exists a subset $B' \subseteq B$ such that*

$$|B'| \geq (1-\sqrt{c})|B|,\ \ \delta(B', A) \geq (1-\sqrt{c})|A|$$

Proof. Let $B' = \{u \in B : \deg(u, A) \geq (1-\sqrt{c})|A|\}$ and $m = |B \setminus B'|$. Because

$$(1-c)|A||B| \leq e(A, B) \leq m\left(1-\sqrt{c}\right)|A| + \left(|B| - m\right)|A|,$$

which implies that $m \leq \sqrt{c}|B|$. □

The naive greedy algorithm is the main tool of for embedding trees, as seen in Fact 1.1. Furthermore, given a tree T, if a graph G_1 contains disjoint vertex sets A and B such that $\delta(A, B) \geq |T_{odd}|$, $\delta(B, A) \geq |T_{even}|$, then $T \subset G_1$. In particular, we can start our embedding by mapping any vertex $a \in A$ to any vertex $u \in T_{even}$ or any vertex $b \in B$ to any vertex $v \in T_{odd}$ (denoted by $a \to u$ and $b \to v$). The following fact gives a few variants of this embedding.

Fact 7.2. *Let G_1 be a graph with two disjoint vertex sets A and B. Then G_1 contains a tree T if any of the following conditions holds.*

1. $\delta(A,B), \delta(B,A) \geq \min\{|T_{even}|, |T_{odd}|\}$, *and* $\delta(A,V) \geq e(T)$.

2. *T has a vertex-partition $U_1 + U_2$ such that*

   - *U_2 is independent (but $U_1 \neq \emptyset$ is not necessarily independent);*
   - $\min\{\delta(A,B), \delta(A,A), \delta(B,A)\} \geq |U_1|$, *and* $\delta(A,V) \geq e(T)$.

3. *T has a vertex-partition $U_1 + U_2$ such that*

   - *U_2 is independent;*
   - $\delta(A,A), \delta(B,A) \geq |U_1|$, $\delta(A,B) \geq |\tilde{U}_2|$, *and* $\delta(A,V) \geq e(T)$, *where $\tilde{U}_2 \subseteq U_2$ is a set that contains all the nonleaf vertices of U_2.*

Furthermore, when embedding T to G_1, we may map any vertex $x \in U_1$ to any vertex $a \in A$ or alternatively any $y \in \tilde{U}_2$ to any $b \in B$.

Proof. *Part 1.* Without loss of generality, assume that $|T_{even}| < |T_{odd}|$. Assume that $v(T) \geq 2$ otherwise $T \subset G_1$ is trivial. Applying Fact 6.9, we know that there are at least $|T_{odd}| - |T_{even}| + 1$ leaves in T_{odd}. We are thus able to put all the nonleaf vertices of T_{odd} into B, and all the vertices of T_{even} into A by the greedy algorithm. Since $\delta(A,V) \geq e(T)$, we can add the leaves of T_{odd} greedily.

Part 2. The proof is similar to Part 1, the only difference is that we need $\delta(A,A) \geq |U_1|$ when embedding U_1 because U_1 may not be independent.

Part 3. We first embed U_1 to A and $\tilde{U}_2$ to B by the greedy algorithm starting with $x \to a$ or $y \to b$. Since the vertices in $U_2 \setminus \tilde{U}_2$ are leaves, we can add them by the greedy algorithm. □

Proposition 7.3 follows from Fact 7.1 easily.

Proposition 7.3. *Suppose $\theta \leq \frac{1}{100}$ and $n \geq 100$. Let G_1 be a graph of order n with a vertex set X such that $|X - \frac{n}{2}| \leq \theta n$ and $\delta(X, V(G_1)) \geq n - \theta n$. Then there exists $Y \subseteq V(G_1) \setminus X$ such that*
(i) $\delta(X,Y) \geq |Y| - \theta n$, $\delta(Y,X) \geq |X| - \sqrt{\theta} n$,
(ii) $\delta(X,Y), \delta(Y,X) \geq \lceil n/2 \rceil - \sqrt{\theta} n$.

Proof. Let $Y' = V(G_1) \setminus X$. Since $\delta(X,V) \geq n - \theta n$, we have $\delta(X,Y') \geq |Y'| - \theta n > (1-3\theta)|Y'|$ (because $n < 3|Y'|$). Hence $e(X,Y') > (1-3\theta)|X||Y'|$. By Fact 7.1, there is a subset $Y \subseteq Y'$ such that $\delta(Y,X) \geq (1-\sqrt{3\theta})|X|$ and $|Y| \geq (1-\sqrt{3\theta})|Y'|$. Since

$|Y'| \geq n/2 - \theta n$, then $|Y| \geq (1-\sqrt{3\theta})(n/2-\theta n)$. Since $\delta(X, V(G_1)) \geq n - \theta n$, we have

$$\begin{aligned}\delta(X,Y) &\geq |Y| - \theta n \\ &\geq (1-\sqrt{3\theta})(n/2-\theta n) - \theta n \\ &> \left(\frac{1}{2} - \frac{\sqrt{3\theta}}{2} - 2\theta\right) n \\ &\geq \left\lceil \frac{n}{2} \right\rceil - \sqrt{\theta} n,\end{aligned}$$

where the last inequality holds because $\sqrt{3\theta}/2 + 2\theta < \sqrt{\theta}$ or $\theta \leq (1-\frac{\sqrt{3}}{2})^2$, and $n \geq 100$. With $|X| \geq n/2 - \theta n$, the same computation shows that $\delta(Y,X) \geq (1-\sqrt{3\theta})|X| \geq \lceil n/2 \rceil - \sqrt{\theta} n$. Finally $\delta(Y,X) \geq (1-\sqrt{3\theta})|X| \geq |X| - \sqrt{\theta} n$ because $|X| \leq n/2 + \theta n$ and $\theta < \frac{1}{\sqrt{3}} - \frac{1}{2}$. □

7.1 Extremal Case 1 (EC1)

In the proof below and later proofs, we often use the trivial fact that for any vertex x, an integer s, and two sets $A \subseteq B$, if $\deg(x, B) \geq |B| - s$, then $\deg(x, A) \geq |A| - s$.

Proof of Proposition 3.1. Given $0 < \sigma < 1$, let c be a real number such that $\sqrt[4]{c} + 2\sqrt{c} < (1-\sqrt[4]{c})\sigma$ (thus $\sqrt{c} < \sigma$) and n_0 be the smallest integer n that satisfies

$$\left((1-\sqrt[4]{c})\sigma - \sqrt[4]{c} - 2\sqrt{c}\right) n \geq 1 \tag{7.1}$$

Suppose that $n \geq n_0$. Let G be a $2n$-vertex graph such that $|L| \geq 2\sigma n$, where L is the set of vertices of degree at least n, and $V(G) = V_1 + V_2$ with $|V_1| = |V_2|$ and $d(V_1, V_2) \geq 1 - c$. Without loss of generality, we assume that $|V_1 \cap L| \geq \sigma n$. Since $e(V_1, V_2) > (1-c)|V_1||V_2|$, we may apply Fact 7.1 to obtain $V_1' \subseteq V_1$ such that $|V_1'| \geq (1-\sqrt{c})n$ and

$$\delta(V_1', V_2) \geq (1-\sqrt{c})n. \tag{7.2}$$

Next we separate two cases based on the values of $t_e = |T_{even}|$ and $t_o = |T_{odd}|$.

Case a). $\min\{t_e, t_o\} \leq ((1-\sqrt[4]{c})\sigma - \sqrt{c})\, n$.

Let $A = L \cap V_1'$. Since $|V_1'| \geq |V_1| - \sqrt{c}n$, we have $|A| \geq \sigma n - \sqrt{c} n$. Since $|V_2| = n$, (7.2) implies that $e(A, V_2) \geq (1-\sqrt{c})|V_2||A|$. Applying Fact 7.1 again, we find $B \subseteq V_2$ such that $|B| \geq (1-\sqrt[4]{c})n$ and

$$\delta(B, A) \geq (1-\sqrt[4]{c})|A| \geq (1-\sqrt[4]{c})(\sigma - \sqrt{c})n > ((1-\sqrt[4]{c})\sigma - \sqrt{c})n.$$

On the other hand, (7.2) can be written as $\delta(V_1', V_2) \geq |V_2| - \sqrt{c}n$, which implies that

$$\delta(A, B) \geq |B| - \sqrt{c}n \geq (1 - \sqrt[4]{c} - \sqrt{c})n > ((1-\sqrt[4]{c})\sigma - \sqrt{c})n$$

by using $\sigma \leq 1$. Since $\delta(A,B), \delta(B,A) \geq \min\{t_e, t_o\}$, we have $T \subset G$ from Fact 7.2 Part 1.

Case b). $\min\{t_e, t_o\} > ((1-\sqrt[4]{c})\sigma - \sqrt{c})\, n$.

Since $t_e + t_o = v(T) = n+1$, we have $\max\{t_e, t_o\} < (1-\sigma(1-\sqrt[4]{c}) + \sqrt{c})n + 1$. By (7.2), $e(V_1', V_2) \geq (1-\sqrt{c})|V_1'||V_2|$. We apply Fact 7.1 again to obtain a set $V_2' \subseteq V_2$ such that $|V_2'| \geq (1-\sqrt[4]{c})n$ and

$$\delta(V_2', V_1') \geq (1-\sqrt[4]{c})|V_1'| \geq (1-\sqrt[4]{c})(1-\sqrt{c})n > (1-\sqrt{c}-\sqrt[4]{c})n.$$

We have $\delta(V_1', V_2') \geq |V_2'| - \sqrt{c}n \geq (1-\sqrt[4]{c}-\sqrt{c})n$ from (7.2). The assumption (7.1) implies that

$$(1-\sqrt[4]{c}-\sqrt{c})n > (1-\sigma(1-\sqrt[4]{c}) + \sqrt{c})n + 1,$$

and consequently $\delta(V_1', V_2'), \delta(V_2', V_1') \geq \max\{t_e, t_o\}$. We then apply the greedy algorithm to embed T into G. □

7.2 Extremal Case 2 (EC2)

In this section, we prove Theorem 3.2 and also complete the proof of Theorem 1.9. Recall that a graph G is **EC2** with parameter α if there is a partition $V(G) = V_1 + V_2$ such that $|V_1| = |V_2| = n$, and $d(V_1, V_2) \leq \alpha$.

We say that G is in the *Extremal Case 3* (**EC3**) with parameter θ if

- $V = V_1 + V_2, |V_1| = |V_2| = n$,
- There exists $A \subseteq V_1$ such that $|A| \geq n/2$, $\delta(A, V) \geq n$, and $\delta(A, V_1) \geq (1-\theta)n$.

Theorem 3.2 immediately follows from the next two lemmas. Note that we only need $\ell(G) \geq n/2 + 1$ for Lemma 7.4, which is much weaker than $\ell(G) \geq n$ provided by Theorem 3.2.

Lemma 7.4. *There exist $\theta_0 > 0$ and $n_0 \in \mathbf{N}$ such that for any $\theta \leq \theta_0$ and $n \geq n_0$, if a $2n$-vertex graph G with $\ell(G) \geq n/2 + 1$ is in* **EC3** *with parameter θ, then $G \supset \mathcal{T}_n$.*

Lemma 7.5. *Let G be a graph on $2n$ vertices with $\ell(G) \geq n$. If G is in* **EC2** *with parameter α, then either $G \supset \mathcal{T}_n$ or G is in* **EC3** *with parameter $\theta \leq 40\sqrt[4]{\alpha} + \sqrt{\alpha}$.*

We are ready to prove Theorem 1.9 now.

Proof of Theorem 1.9. Let c be given by Proposition 3.1 with $\sigma = 1/4$, and let θ_0 be from Lemma 7.4. Let $\beta > 0$ be given as in Theorem 1.9. We may assume that $\beta < 1$ (otherwise there is nothing to prove). Now set $\alpha = \min\{c, \theta_0^2, \beta^2/9\}$.

Let $\varepsilon = \varepsilon(\alpha)$ be given by Theorem 3.3. Let $0 < \zeta \leq 1/2$ such that $\zeta \leq \varepsilon$ and $2\zeta \leq \sqrt{\alpha} - 3\alpha$ (note that $\sqrt{\alpha} > 3\alpha$ because $\alpha < 1/9$). Suppose that G is a $2n$-vertex graph for sufficiently large n such that $\ell(G) \geq (1-\zeta)n$ and $G \not\supset \mathcal{T}_n$. Since $\ell(G) \geq (1-\varepsilon)n$, Theorem 3.3 implies that G is in either of the two extreme cases with parameter α. Since $\zeta \leq 1/2$, then $\ell(G) \geq n/2$. If G is in **EC1** with parameter α $(\leq c)$, then by the choice of c, we can apply Proposition 3.1 to get $G \supset \mathcal{T}_n$, a contradiction. This implies that G is in **EC2** with parameter α, namely, $V(G)$ can be evenly partitioned into V_1 and V_2 such that $d(V_1, V_2) \leq \alpha$.

Let L be the set of vertices in G of degree at least n. We claim that $|V_i \cap L| < \frac{n}{2} + \sqrt{\alpha} n$ for $i = 1, 2$. Suppose instead, say $|V_1 \cap L| \geq \frac{n}{2} + \sqrt{\alpha} n$. Let V_1' be the set of $x \in V_1$ such that $\deg(x, V_2) \geq \sqrt{\alpha} n$. Then $|V_1'| \leq \sqrt{\alpha} n$ (otherwise $d(V_1, V_2) > \alpha$). Let $A' = (V_1 \cap L) \setminus V_1'$. Since $|V_1 \cap L| \geq \frac{n}{2} + \sqrt{\alpha} n$, we have $|A'| \geq n/2$. Consequently G is in **EC3** with parameter $\sqrt{\alpha}$ ($\leq \theta_0$). Lemma 7.4 thus implies that $G \supset \mathcal{T}_n$, a contradiction.

Since $|V_1 \cap L| + |V_2 \cap L| = |L| \geq (1 - \zeta)n$, we conclude that

$$\frac{n}{2} - \zeta n - \sqrt{\alpha} n \leq (1 - \zeta)n - \left(\frac{n}{2} + \sqrt{\alpha} n\right) < |V_i \cap L| < \frac{n}{2} + \sqrt{\alpha} n. \tag{7.3}$$

Let $A = V_1 \cap L$. We have

$$e(A, V_1) \geq |A|n - e(A, V_2) \geq |A|n - e(V_1, V_2) \geq |A|n - \alpha n^2.$$

After adding at most αn^2 edges, every $x \in A$ is adjacent to all other vertices in V_1. By (7.3), $G[V_1]$ becomes H_n after adding or removing at most $(\sqrt{\alpha} + \zeta)n^2$ more edges. Similarly we may change at most $\alpha n^2 + (\sqrt{\alpha} + \zeta)n^2$ edges to transform $G[V_2]$ into H_n. After deleting αn^2 edges between V_1 and V_2, we finally transform G into $2H_n$. The total number of changed edges is at most

$$2(\alpha n^2 + (\sqrt{\alpha} + \zeta)n^2) + \alpha n^2 \leq 3\sqrt{\alpha} n^2 \leq \beta n^2$$

by using $3\alpha + 2\zeta \leq \sqrt{\alpha}$ and $3\sqrt{\alpha} \leq \beta$. □

7.2.1 Proof of Lemma 7.4

In this subsection we prove Lemma 7.4. Let $\theta_0 = (\frac{1}{1782})^2$. Suppose that $0 < \theta \leq \theta_0$ and n is sufficiently large. Let $G = (V, E)$ be a $2n$-vertex graph with $|L| \geq n/2 + 1$, where $L := \{x \in V : \deg(x) \geq n\}$. Assume that G is in **EC3**, that is, $V(G)$ can be evenly partitioned into $V_1 \cup V_2$ such that V_1 contains a set $A \subseteq V_1 \cap L$ with $|A| \geq n/2$ and $\delta(A, V_1) \geq |V_1| - \theta n$. We assume that $|A| = \lceil n/2 \rceil$ (otherwise consider a subset of A). Hence

$$|A| = \lceil n/2 \rceil, \quad \delta(A, V) \geq n, \quad \delta(A, A) \geq |A| - \theta n. \tag{7.4}$$

Let $B = V_1 \setminus A$. Applying Proposition 7.3 with $G_1 = G[V_1]$ and $X = A$, we obtain a subset $B_1 \subseteq B$ such that

$$\delta(A, B_1) \geq |B_1| - \theta n,\ \delta(B_1, A) \geq |A| - \sqrt{\theta} n,\ \delta(A, B_1), \delta(B_1, A) \geq \lceil n/2 \rceil - \sqrt{\theta} n. \tag{7.5}$$

The rest of our proof is divided into two cases according to the number of leaves in T.

Case 1: Embedding trees with at least $33\sqrt{\theta} n$ leaves.

We need some definitions. For a tree T, the gap $g(T)$ is defined as $||T_{odd}| - |T_{even}||$.

Definition 7.6. *Let T be a tree of size n such that $V(T) = U_1 + U_2$.*

1). $U_1 + U_2$ is called an ideal *partition if*

1. $|U_1| \leq |U_2|$,

2. U_2 is independent,

3. U_1 contains at least $5\sqrt{\theta}n$ leaves, and U_2 contains at least $2\sqrt{\theta}n$ leaves.

2). $U_1 + U_2$ is called a near-ideal *partition if*

1. $|U_1| = n/2 + 1$ and $|U_2| = n/2$ (so n is even),

2. U_2 is independent,

3. U_1 contains at least $5\sqrt{\theta}n$ leaves, and U_2 contains at least $2\sqrt{\theta}n$ leaves.

4. There exists a leaf $z \in U_1$ whose parent $y \in U_2$ has degree 2.

The following two lemmas are main ingredients in our proof. We postpone their proofs to the end.

Lemma 7.7. *Let T be a tree with n edges and at least $33\sqrt{\theta}n$ leaves. Then either $g(T) \geq 2\sqrt{\theta}n + 1$ or T has an ideal partition or T has a near-ideal partition.*

Lemma 7.8. *Suppose $0 \leq l < n$ is an integer. Let T be a tree of size at most n, with a partition $V(T) = U_1 + U_2$ such that U_1 contains at least $5l$ leaves and U_2 is independent. Let $\tilde{U}_2$ be a subset of U_2 such that all the vertices in $U_2 \setminus \tilde{U}_2$ are leaves (though $\tilde{U}_2$ may contains leaves as well). If a graph G contains two disjoint vertex sets X and Y such that*
(i) $\delta(X, X), \delta(Y, X) \geq |X| - l$, $\delta(X, Y) \geq \max\{|Y| - l, |\tilde{U}_2|\}$,
(ii) $|X| \geq |U_1|$, $\delta(X, V(G)) \geq e(T)$,
then $T \subset G$. Furthermore, for any $x \in U_1$ and any $a \in X$, we can map $x \to a$; alternatively, for any leaf $y \in \tilde{U}_2$ and any $b \in Y$, we can map $y \to b$.

Proof of Lemma 7.4 for trees with at least $33\sqrt{\theta}n$ leaves. Let T a tree with n edges and at least $33\sqrt{\theta}n$ leaves. By Lemma 7.7, either $g(T) \geq 2\sqrt{\theta}n + 1$ or T has an ideal partition or a near-ideal partition.

Case 1: $g(T) \geq 2\sqrt{\theta}n + 1$. This implies that

$$\min\{|T_{even}|, |T_{odd}|\} \leq \frac{1}{2}(n + 1 - (2\sqrt{\theta}n + 1)) = \frac{n}{2} - \sqrt{\theta}n.$$

Together with (7.5), it gives $\delta(A, B_1), \delta(B_1, A) \geq \min\{|T_{even}|, |T_{odd}|\}$. As $\delta(A, V) \geq n$, we can thus apply Fact 7.2 Part 1 to get $T \subset G$.

Case 2: T has an ideal partition $U_1 + U_2$. Then U_2 is independent, and U_1 contains at least $5\sqrt{\theta}n$ leaves. Since $|U_1| + |U_2| = n + 1$ and $|U_1| \leq |U_2|$, we have

$$|U_1| \leq \lceil n/2 \rceil \leq |A|. \tag{7.6}$$

Let W_2 be the set of all leaves in U_2. By the definition of ideal partitions, $|W_2| \geq 2\sqrt{\theta}n$. On the other hand, $|W_2| \geq |U_2| - |U_1| + 1$ by Fact 6.9. Using $|U_1| + |U_2| = n + 1$, we obtain $n + |W_2| \geq 2|U_2|$. Define $\tilde{U}_2 := U_2 \setminus W_2$. Then $|\tilde{U}_2| = |U_2| - |W_2| \leq \frac{n+|W_2|}{2} - |W_2|$. By using $|W_2| \geq 2\sqrt{\theta}n$ and (7.5), we derive that

$$|\tilde{U}_2| \leq \frac{n}{2} - \sqrt{\theta}n \leq \delta(A, B_1). \tag{7.7}$$

Because of (7.4), (7.5), (7.6), and (7.7), we can apply Lemma 7.8 with $l = \lfloor \sqrt{\theta}n \rfloor$,[11] $X = A$ and $Y = B_1$ to embed T to G.

Case 3: T has a near-ideal partition $U_1 + U_2$. In this case n is even, $|U_1| = n/2 + 1$, and $|U_2| = n/2$. Let W_2 be the set of leaves in U_2. Then $|W_2| \geq 2\sqrt{\theta}n$. By item 4 in the definition of near-ideal partitions: *there exists a leaf $z \in U_1$ such that its parent $y \in U_2$ has degree* 2. Let $x = p(y)$. Then $x \in U_1$ since U_2 is independent.

We need to make some preparation in G. Let $B_2 = B \setminus B_1$. Since $|V_1| = n$ and $\delta(A, V) \geq n$, then $\delta(A, V_2) \geq 1$; in particular, some vertex $v_2 \in V_2$ has at least one neighbor in A. If a vertex $v_1 \in B_2$ has no neighbor in A, then we may switch v_1 and v_2. Repeating this if necessary, we now assume that $\delta(B_2, A) \geq 1$. Since such switches do not change A and B_1, (7.5) still holds.

Let $L_2 = L \cap V_2$. We claim that either $E(A, L_2) \neq \emptyset$ or $E(B, V_2) \neq \emptyset$. In fact, since $|L| \geq n/2 + 1$ and $|A| = n/2$, either $B \cap L \neq \emptyset$ or $L_2 \neq \emptyset$. If $B \cap L \neq \emptyset$, then $E(B, V_2) \neq \emptyset$ because any vertex in L has at least n neighbors and $|V_1| = n$. If $L_2 \neq \emptyset$, then $E(V_1, L_2) \neq \emptyset$ for the same reason. It follows that either $E(A, L_2) \neq \emptyset$ or $E(B, L_2) \neq \emptyset$.

We consider three cases $E(A, L_2) \neq \emptyset$, $E(B_1, V_2) \neq \emptyset$, and $E(B_1, V_2) \neq \emptyset$ separately.

Case 3a: $E(A, L_2) \neq \emptyset$.

Suppose that a vertex $v_0 \in L_2$ is adjacent to a vertex $a \in A$. Let $T' = T \setminus \{y, z\}$ and $G' = G \setminus \{v_0\}$. Then $V(T')$ has a partition $U_1' + U_2'$ with $U_1' = U_1 \setminus \{z\}$ and $U_2' = U_2 \setminus \{y\}$. We have $|U_1'| = n/2 = |A|$. Let $\tilde{U}_2 = U_2' \setminus W_2$. Then $|\tilde{U}_2| \leq \frac{n}{2} - 1 - 2\sqrt{\theta}n$ because $|W_2| \geq 2\sqrt{\theta}n$. Since $v_0 \notin A \cup B_1$, (7.5) still holds in G'. Since $e(T') = n - 2$ we can replace (7.4) with

$$|A| = n/2, \quad \delta(A, V(G')) \geq n - 1 \geq e(T'), \quad \delta_{G'}(A, A) \geq |A| - \theta n. \tag{7.8}$$

With $l = \lfloor \sqrt{\theta}n \rfloor$, $X = A$, and $Y = B_1$, conditions (i) and (ii) in Lemma 7.8 hold in G'. We then apply Lemma 7.8 to embed T' to G' such that $x \to a$. Next map y to v_0, and finally add the leaf z by using $\deg(v_0) \geq n$.

Case 3b: $E(B_1, V_2) \neq \emptyset$

Suppose that a vertex $b \in B_1$ is adjacent to a vertex $v_0 \in V_2$. Let $T' = T \setminus \{z\}$ and $G' = G \setminus \{v_0\}$. Then $V(T')$ has a partition $U_1' + U_2$ with $U_1' = U_1 \setminus \{z\}$. Then $|U_1'| = n/2 = |A|$. Let $\tilde{U}_2 = U_2 \setminus W_2$. Then $|\tilde{U}_2| \leq \frac{n}{2} - 2\sqrt{\theta}n$. We know that (7.5) holds in G'. Since $e(T') = n - 1$, (7.8) holds as well. With $l = \lfloor \sqrt{\theta}n \rfloor$, $X = A$, and $Y = B_1$, we can apply Lemma 7.8 embedding T' to G'. Note that y is a leaf of T' and $y \in \tilde{U}_2$ (because

[11] In (7.4) and (7.5), we can add floors to θn and $\sqrt{\theta}n$ because all other terms in the inequalities are integers.

y loses its only child z in T'). We thus require $y \to b$ when embedding T' to G'. Finally we map z to v_0.

Case 3c: $E(B_1, V_2) \neq \emptyset$.

Suppose that a vertex $b \in B_2$ is adjacent to a vertex $v_0 \in V_2$. Recall that $\delta(B, A) \geq 1$. Let $a \in A$ be a neighbor of b. Let $T' = T \setminus \{y, z\}$ and $G' = G \setminus \{b, v_0\}$. Since $v_0, b \notin A \cup B_1$, (7.5) still holds in G'. Since $\delta_{G'}(A, V) \geq n-2 = e(T')$, (7.8) holds. We apply Lemma 7.8 to embed T' to G' such that $x \to a$ as in Case 3a. Then map y to b and z to v_0. □

We now prove Lemmas 7.7 and 7.8.

Let T be a rooted tree T and $x \in V(T)$. Recall that $T(x)$ is the maximal subtree of T containing x but not $p(x)$. Given $C \subset C(x)$, the subtree obtained from $T(x)$ by removing all $T(y), y \in C$ is called a *natural subtree rooted at* x. A natural subtree T' of T has the property that $T - T'$ is also a tree. The following simple fact on natural subtrees is needed for proving Lemma 7.7 and Claim 7.12.

Fact 7.9. *Let T be a rooted tree with $v(T)$ vertices and $w(T)$ leaves.*

1. For any positive integer $k \leq v(T)$, there is a natural subtree T' such that $\frac{k}{2} \leq v(T) < k$. In this case, we call T' a $[k/2, k]$-subtree.

2. For any positive integer $k \leq w(T)$, there exists a natural subtree with m leaves such that $k/2 \leq m < k$.

Proof. For $x \in V(T)$, write $t(x)$ for $v(T(x))$. In the partial order defined by T with $Rt(T)$ as the highest element, we find the lowest vertex x such that $t(x) \geq \frac{k}{2}$. Then $t(y) < \frac{k}{2}$ for every $y \in C(x)$. If $t(x) < k$, then $T(x)$ is the desired natural subtree. Otherwise, from $T(x)$, we repeat removing the subtree $T(y)$ for $y \in C(x)$ until the remaining subtree has order less than k. We know the size of this tree is at least $k/2$ because the last removed $y \in C(x)$ satisfies $t(y) < \frac{k}{2}$ and the subtree right before removing $T(y)$ has order at least k.

Part 2 can be proved similarly. □

Given a tree with a vertex-partition $U_1 + U_2$, *flipping* a vertex set S (which may intersect both U_1 and U_2) mean moving the vertices of S from one partition set to the other one. It results in a new partition $U_1' + U_2'$ with $U_1' = (U_1 \setminus S) \cup (U_2 \cap S)$ and $U_2' = (U_2 \setminus S) \cup (U_1 \cap S)$.

In the proof of Lemma 7.7, unless $g(T)$ is large, we find a natural subtree T_0 rooted at r_0 such that both T_0 and $T - T_0$ have many leaves and then flip T_0 or $T_0 - r_0$ in the default partition (T_{even}, T_{odd}). In most cases, the resulting partition is an ideal partition. In the remaining cases, we obtain a near-ideal partition.

Proof of Lemma 7.7. Without loss of generality, assume that $|T_{odd}| \geq |T_{even}|$. Let $g = |T_{odd}| - |T_{even}|$ (then $g \geq 0$). If $g \geq 2\sqrt{\theta}n + 1$, then we are done. We may thus assume that

$$g \leq 2\sqrt{\theta}n + 1. \tag{7.9}$$

Since $|T_{odd}| + |T_{even}| = n + 1$, g has the same parity as $n + 1$. Denote sets of leaves in T_{even} and T_{odd} by W_e, W_o, respectively. Thus $|W_e| + |W_o| \geq 33\sqrt{\theta}n$. If $|W_e| \geq 5\sqrt{\theta}n$ and

$|W_o| \geq 2\sqrt{\theta}n$, then $T_{even} + T_{odd}$ is an ideal partition, and we are done. Otherwise we have either $|W_o| < 2\sqrt{\theta}n$ or $|W_e| < 5\sqrt{\theta}n$.

Case a) $|W_o| < 2\sqrt{\theta}n$.

Then $|W_e| > 31\sqrt{\theta}n$. We flip $2\sqrt{\theta}n - |W_o|$ vertices of W_e and their parents (not moving other vertices under the parents). Let U_1 and U_2 be the resulting sets obtained from T_{even} and T_{odd}, respectively. Clearly U_2 is independent and $|U_1| \leq |T_{even}| \leq |U_2|$. In addition, U_2 contains $2\sqrt{\theta}n$ leaves, and U_1 contains more than $5\sqrt{\theta}n$ leaves. Therefore $U_1 + U_2$ is an ideal partition.

Case b) $|W_e| < 5\sqrt{\theta}n$.

Applying Fact 7.9, we find a natural subtree T_0 rooted at r_0 with m leaves, where $11\sqrt{\theta}n \leq m < 22\sqrt{\theta}n$. Then $T_1 := T - T_0$ is also a subtree and contains at least $11\sqrt{\theta}n$ leaves. Since $|W_e| < 5\sqrt{\theta}n$, each of T_0 and T_1 contains at least $11\sqrt{\theta} - 5\sqrt{\theta} = 6\sqrt{\theta}n$ vertices of W_o.

Let $g_i = |V(T_i) \cap T_{odd}| - |V(T_i) \cap T_{even}|$ for $i = 0, 1$. Then

$$g_0 + g_1 = \begin{cases} g - 1 & \text{if } r_0 \in T_{even} \\ g + 1 & \text{if } r_0 \in T_{odd}. \end{cases}$$

If $g_0 \geq g/2$ and $r_0 \in T_{even}$, then we flip T_0. Let U_2 and U_1 be the resulting sets generated from T_{even} and T_{odd}, respectively. Then

$$|U_1| - |U_2| = |T_{odd}| - |T_{even}| - 2(|V(T_0) \cap T_{odd}| - |V(T_0) \cap T_{even}|) = g - 2g_0 \leq 0,$$

and only U_1 contains internal edges. In addition, U_1 contains at least $6\sqrt{\theta}n$ leaves (from T_1), and U_2 contains at least $6\sqrt{\theta}n$ leaves (from T_0). Therefore $U_1 + U_2$ is an ideal partition. If $g_0 \leq g/2$ and $r_0 \in T_{odd}$, then we also flip T_0 and obtain an ideal partition similarly.

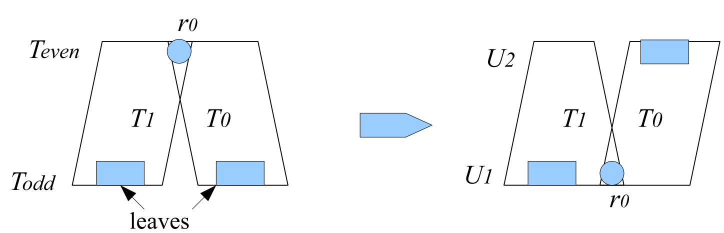

Figure 3: Flipping T_0 when $g_0 \geq g/2$, $r_0 \in T_{even}$ and in Case 1)

If $g_0 \leq g/2 - 1$ and $r_0 \in T_{even}$, or $g_0 \geq g/2 + 1$ and $r_0 \in T_{odd}$, we can also obtain an ideal partition by flipping $T_0 \setminus \{r_0\}$.

If $g \equiv n+1 \pmod 2$ is even, then these are all the cases and we are done. Now assume that g is odd (then n is even). The only remaining cases are

Case 1) $g_0 = \frac{g-1}{2}$ *and* $r_0 \in T_{even}$ *(thus* $g_1 = \frac{g-1}{2}$*),*

Case 2) $g_0 = \frac{g+1}{2}$ *and* $r_0 \in T_{odd}$ *(thus* $g_1 = \frac{g+1}{2}$*).*

We flip T_0 in these cases. In Case 1), let U_2 and U_1 be the resulting sets generated from T_{even} and T_{odd}; while in Case 2), let U_1 and U_2 be the resulting sets generated from T_{even} and T_{odd}. It is easy to see that[12] $|U_1| = \frac{n}{2}+1$, $|U_2| = \frac{n}{2}$, and U_2 is independent. Furthermore, U_1 and U_2 each contains more than $6\sqrt{\theta}n$ leaves. By Definition 7.6, in order to call U_1+U_2 a near-ideal partition, we need to find a leaf $z \in U_1$ whose parent $y \in U_2$ has degree 2.

Below we show that there exists a leaf $z \in U_1$, whose parent $y \in U_2$ has exactly one nonleaf neighbor. This suffices because if $\deg(y) = 2$, then $U_1 + U_2$ is a near-ideal partition; otherwise y has another leaf neighbor z' (in U_1 since U_2 is independent), then we flip z, z' and y and the resulting partition becomes ideal.

Suppose we were in Case 1). Then $V(T_1) \cap T_{odd} \subset U_1$. Therefore it suffices to show that there exists a leaf $z \in V(T_1) \cap T_{odd}$ such that its parent $p(z)$ has exactly one nonleaf neighbor. Note that $p(z) \in U_2$ unless $p(z) = r_0$; but $p(z) \neq r_0$ because r_0 has at least two nonleaf neighbors (one from T_0 and one from T_1).[13] Let $W_o^1 = W_o \cap V(T_1)$. We has shown that $|W_o^1| \geq 6\sqrt{\theta}n$. Suppose that for every $v \in W_o^1$, its parent $p(v)$ has at least two nonleaf neighbors. Let $T_1' = T_1 - W_o^1$. Then two trees T_1' and T_1 have the same number of leaves in T_{even}. We now use Fact 6.9 to find a lower bound for this number. Let $T_{even}^1 = V(T_1) \cap T_{even}$ and $T_{odd}^1 = V(T_1) \cap T_{odd}$. The tree T_1' has the bipartition $(T_{even}^1, T_{odd}^1 - W_o^1)$. We know $|T_{odd}^1| - |T_{even}^1| = g_1 = \frac{g-1}{2} \leq \sqrt{\theta}n$ from (7.9). By Fact 6.9, the number of leaves of T_1' in T_{even} is at least

$$|T_{even}^1| - (|T_{odd}^1| - |W_o^1|) + 1 \geq -\sqrt{\theta}n + 6\sqrt{\theta}n + 1 = 5\sqrt{\theta}n + 1.$$

All but at most one leaf of T_1 in T_{even} are leaves of T (the exception is r_0). Therefore T has at least $5\sqrt{\theta}n$ leaves in T_{even}, contradicting $|W_e| < 5\sqrt{\theta}n$.

In Case 2), we define $W_o^0 = W_o \cap V(T_0)$ and $T_0' = T_0 - W_o^0$. Following the same arguments except that $g_0 = \frac{g+1}{2}$ replaces $g_0 = \frac{g-1}{2}$, we conclude that T_0' has at least $5\sqrt{\theta}n$ leaves in T_{even}. Since $r_0 \notin T_{even}$, all these leaves are leaves of T. Thus T has at least $5\sqrt{\theta}n$ leaves in T_{even}, a contradiction. □

Given a vertex set C in a tree, let $p(C)$ denote the union of parents $p(x)$ for all $x \in C$.

Proof of Lemma 7.8. Let W_1 be the set of leaves in U_1 not including x. Then $|W_1| \geq 5l - 1$. Let $\hat{W}_1$ be the set of leaves in U_1 with parent in U_2.

Claim. $|\hat{W}_1| \geq 4l$.

Proof. For instead, at least l leaves in U_1 have their parents in U_1. We move these leaves to U_2 and let U_1', U_2' denote the resulting sets. Then $|U_1'| = |U_1| - l$ and U_2' is independent. Let $\tilde{U}_2$ be the given subset of U_2, which contains all the nonleaf vertices of U_2' and a leaf y. Then $\tilde{U}_2$ is also a subset of U_2'. Conditions (i) and (ii) imply that

$$\delta(X,X), \delta(Y,X) \geq |X| - l \geq |U_1| - l = |U_1'|, \quad \delta(X,Y) \geq |\tilde{U}_2|, \quad \delta(X,V) \geq e(T).$$

[12] For example, in Case 1), we use $|U_1| - |U_2| = g - 2g_0 = 1$.

[13] Otherwise either T_0 or T_1 is a star but this is impossible because $g_i < g \leq 2\sqrt{\theta}n + 1$ and T_i has at least $11\sqrt{\theta}n$ leaves.

Applying Fact 7.2 Part 3, we can embed $T \to G$ such that $x \to a$ or alternatively $y \to b$. $\square$

Let $W_1' = \{v \in \hat{W}_1 : v \text{ is the unique leaf among the children of } p(v)\}$. First assume that $|W_1'| < 2l$. Let $W_1'' := \hat{W}_1 \setminus W_1'$. Then $|p(W_1'')| \leq |W_1''|/2$. By the claim above, $|W_1'| + |W_1''| = |\hat{W}_1| \geq 4l$, thus $|W_1''| > 2l$. We flip $p(W_1'') \cup W_1''$ and let $U_1' + U_2'$ denote the resulting sets. Then U_2' is independent and $|U_1'| = |U_1| - |W_1''| + |p(W_1'')| \leq |U_1| - l$ because $|W_1''| - |p(W_1'')| > l$. Let $\tilde{U}_2' = \tilde{U}_2 - p(W_1'')$. Then $|\tilde{U}_2'| < |\tilde{U}_2|$ and $y \in \tilde{U}_2'$ because y is a leaf and $p(W_1'')$ contains no leaves. We then apply Fact 7.2 Part 3 to embed $T \to G$ such that either $x \to a$ or $y \to b$.

Now assume that $|W_1'| \geq 2l$. Since any two leaves in W_1' have different parents, we have $|p(W_1')| = |W_1'|$. Since

$$\delta(X, X), \delta(Y, X) \geq |X| - l > |U_1| - 2l \geq |U_1 \setminus W_1'|, \quad \text{and} \quad \delta(X, Y) \geq |\tilde{U}_2|,$$

we can apply the greedy algorithm to embed $U_1 \setminus W_1'$ into X and $\tilde{U}_2$ into Y such that either $x \to a$ or $y \to b$. Note that we do not embed $W_2 := U_2 \setminus \tilde{U}_2$ at this moment. Next, let Y' be the set of images of $p(W_1')$. Since $|X| \geq |U_1| \geq |U_1 \setminus W_1'| + |W_1'|$, we can find a set $X' \subset X$ of $|W_1'|$ unoccupied vertices. Then $|X'| = |Y'| \geq 2l$. Since $\delta(X, Y) \geq |Y| - l$ and $\delta(Y, X) \geq |X| - l$, in the bipartite subgraph $G[Y', X']$, we have $\delta(Y', X') \geq |X'| - l \geq |X'|/2$, and $\delta(X', Y') \geq |Y'| - l \geq |Y'|/2$. The well-known marriage theorem thus provides a perfect matching from Y' to X', which in turn gives an embedding of W_1'. Finally, since W_2 is a set of leaves and $p(W_2)$ was embedded to X, we can add all the leaves in W_2 greedily. $\square$

Case 2. Embedding trees with at most $33\sqrt{\theta}n$ leaves

In this case we need a lemma which generalizes the naive greedy algorithm and postpone its proof to the end. Given a graph G, we write $G = (X, Y; E)$ if it is bipartite with partition sets X and Y.

Lemma 7.10. *Let $T = (U_1, U_2; E(T))$ be a tree with at most l leaves such that $|U_1|, |U_2| \geq 26l$. Let $G = (X, Y; E)$ be a bipartite graph satisfying*

1. $|X| \geq |U_1|, |Y| \geq |U_2|$,
2. $Y = Y_1 + Y_2$, $\delta(X, Y_1) \geq |Y_1| - l$, $\delta(Y_1, X) \geq |X| - l$,
3. *$|Y_2| \leq l$, and G contains $|Y_2|$ vertex-disjoint 2-paths, each of which consists of one vertex of Y_2 as mid-point and two vertices of X as end-points.*

Suppose that $z \in U_1$ and $a \in X$ such that a is not contained in the given 2-paths. Then T can be embedded to G such that $U_1 \to X$, $U_2 \to Y$, and $z \to a$.

We now prove Lemma 7.4 for trees that do not have many leaves. In this case we do not need the assumption $\ell(G) \geq n/2 + 1$.

Proof of Lemma 7.4: T has at most $33\sqrt{\theta}n$ leaves. Recall that V_1 contains two disjoint subsets A, B_1 satisfying (7.4) and (7.5).

Let $\mathcal{F}$ be a maximum family of vertex-disjoint 2-paths with mid-points in $V(G)\setminus(A\cup B_1)$ and both end-points in A. Let B_2 be the set of the mid-points from $\min\{|\mathcal{F}|, n-|A|-|B_1|\}$ paths of $\mathcal{F}$. Let $B = B_1 \cup B_2$ $V_1' = A\cup B$, and $V_2' = V\setminus V_1'$. As $|B| \geq |B_1| \geq \lceil n/2\rceil - \sqrt{\theta}n$, we have $n-\sqrt{\theta}n \leq |V_1'| \leq n$ and consequently $|V_2'| \leq n+\sqrt{\theta}n$.

We claim that $|V_1'| \geq n-1$ or equivalently $|B| \geq \lfloor n/2\rfloor - 1$. Suppose to the contrary, that

$$|V_1'| \leq n-2. \tag{7.10}$$

The definition of B_2 thus implies that $|\mathcal{F}| < n-|A|-|B_1| \leq \sqrt{\theta}n$. Let A' be the set of the vertices of A that are *not* end-points of $\mathcal{F}$. Then $|A'| > n/2 - 2\sqrt{\theta}n$. For any vertex $v\in A'$, as $\deg(v)\geq n$, we have $\deg(v, V_2') \geq n - |V_1'| + 1 \geq 3$ by using (7.10). The neighborhoods in V_2' of the vertices of A' must be disjoint, otherwise it yields a new 2-path which is vertex-disjoint from $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$. But this implies that

$$3\left(\frac{n}{2} - 2\sqrt{\theta}n\right) \leq \sum_{v\in A'} \deg(v, V_2') \leq |V_2'| \leq n+\sqrt{\theta}n,$$

a contradiction.

In summary, $G[V_1']$ satisfies Conditions 2 and 3 of Lemma 7.10 with $X = A$, $Y_1 = B_1$, $Y_2 = B_2$, and any $l \geq \sqrt{\theta}n$. We also know that $A\subseteq L$, $A = \lceil n/2\rceil$ and $\lfloor n/2\rfloor - 1 \leq |B| \leq \lfloor n/2\rfloor$.

Let T be a tree with n edges and at most $33\sqrt{\theta}n$ leaves. Without loss of generality, assume that $|T_{even}| \leq |T_{odd}|$. Then $|T_{even}| \leq \lceil n/2\rceil = |A|$. We also assume that $|T_{even}| > \lceil n/2\rceil - \sqrt{\theta}n$ otherwise Fact 7.2 Part 1 provides an embedding of T. Let T_{odd}' be the set of non-leaf vertices in T_{odd}. Let T' be the induced subtree of T on $T_{even}\cup T_{odd}'$. Then T' has at most $33\sqrt{\theta}n$ leaves and partition sizes $|T_{even}| > \lceil n/2\rceil - \sqrt{\theta}n$ and $|T_{odd}'| \geq n/2 - 33\sqrt{\theta}n$. We have $|T_{even}|, |T_{odd}'| \geq 26(33\sqrt{\theta}n)$ as long as $\frac{n}{2} \geq 27(33\sqrt{\theta}n)$ or $\theta \leq (\frac{1}{1782})^2$.

If $|T_{odd}'| \leq |B|$, then with $l = \lfloor 33\sqrt{\theta}n\rfloor$, all the conditions of Lemma 7.10 are satisfied. We can apply Lemma 7.10 to embed T' into $G[V_1']$ such that $T_{even}\to A$ and $T_{odd}'\to B$. Finally we add the leaves in T_{odd} greedily and complete the embedding of T.

Now assume that $|T_{odd}'| > |B|$. By Proposition 6.9, T_{odd} has at least $|T_{odd}| - |T_{even}| + 1$ leaves. Then $|T_{odd}'| \leq |T_{even}| - 1 \leq \lceil n/2\rceil - 1$. Since $\lfloor n/2\rfloor - 1 \leq |B| \leq \lfloor n/2\rfloor$, we have $|T_{odd}'| > |B|$ only if n is odd and $|B| = \frac{n-3}{2}$ and $|T_{even}| = |T_{odd}| = \frac{n+1}{2}$.

In this case if either T_{even} or T_{odd} has at least two leaves, then we can apply Lemma 7.10 as well (by letting U_2 be T_{even}' or T_{odd}'). Otherwise T has at most two leaves. Then T is a path. Let $\tilde{A}$ be the set of the vertices of A that are *not* on the 2-paths covering B_2'. Fix a vertex $a\in\tilde{A}$. Since $\deg(a,\tilde{A}) \geq |\tilde{A}| - \theta n > 0$, we can find a neighbor $v\in\tilde{A}$ of a. Let P be a path on $n-2$ vertices with leaves $x, y\in P_{even}$. Then $|P_{even}| = (n-1)/2$ and $|P_{odd}| = (n-3)/2$. Let $A' = A\setminus\{v\}$; then $|A'| = (n-1)/2 = |P_{even}|$. All conditions of Lemma 7.10 hold with $U_1 = P_{even}$, $U_2 = P_{odd}$, $X = A'$, $Y = B$, and $l = \lceil\sqrt{\theta}n\rceil$. We apply Lemma 7.10 to embed P to $G[V_1'\setminus\{v\}]$ such that $x\to a$. Suppose that the other leaf y is mapped to $w\in A\setminus\{v\}$. We then extend P to a path on $n+1$ vertices by connecting a and v and adding a neighbor of v and a neighbor of w greedily.

We thus complete the proof of Lemma 7.4. □

To prove Lemma 7.10, we need some properties of trees with a small number of leaves. Given a 2-path uvw, we call it an S-2-*path* if the mid-point $v \in S$, and call it a *special 2-path* if furthermore, all vertices in $N(\{u,w\})$ have degree at most two (consequently $\deg(v)=2$).

Proposition 7.11. *Let T be a tree with l leaves.*

1. *$\sum_{x\in U^3}(\deg(x)-2) = l-2$, where $U^3 = \{x \in V(T) : \deg(x) \geq 3\}$. In particular, $|U^3| \leq l-2$.*
2. *$|N(S)| \leq 2|S| + l - 2$ for any subset $S \subset V(T)$.*
3. *Let $T = (U_1, U_2; E)$ be a tree such that $|U_1|, |U_2| \geq 26l$. Fix a vertex $z \in U_1$. Then T contains $5l$ special U_2-2-paths $P_1, \ldots, P_{5l}$ and $4l$ U_1-2-paths such that all these paths are vertex-disjoint and do not contain z.*

Proof. Define $U^i = \{x \in V(T) : \deg(x) = i\}$ for $i = 1, 2$. Hence $U^1 \cup U^2 \cup U^3$ is a partition of $V(T)$.

Part 1: $\sum_{x\in V(T)}(\deg(x)-2) = 2e(T) - 2v(T) = -2$. On the other hand,

$$\sum_{x\in V(T)}(\deg(x)-2) = -l + \sum_{x\in U^3}(\deg(x)-2),$$

which implies that $\sum_{x\in U^3}(\deg(x)-2) = l-2$.

Part 2: We partition S into S_1, S_2 and S_3 such that $S_i = \{x \in S : \deg(x) = i\}$ for $i = 1, 2$, and $S_3 = \{x \in S : \deg(x) \geq 3\}$. Then

$$\begin{aligned}|N(S)| &\leq \sum_{x\in S}\deg(x) = |S_1| + 2|S_2| + \sum_{x\in S_3}\deg(x)\\ &= |S_1| + 2|S_2| + 2|S_3| + \sum_{x\in S}(\deg(x)-2)\\ &\leq 2|S| + (l-2), \quad \text{by Part 1.}\end{aligned}$$

Part 3: Let $U_j^i = U^i \cap U_j$ for $i = 1, 2, 3$ and $j = 1, 2$. Define two subsets

$$U_2' = U_2^2 \setminus N(U_1^3 \cup \{z\}) \quad \text{and} \quad U_1' = (U_1^2 \setminus \{z\}) \setminus N(U_2^3).$$

We claim that $|U_2'| \geq |U_2| - 4l$. In fact,

$$\begin{aligned}|U_2'| &\geq |U_2^2| - |N(U_1^3 \cup \{z\})|\\ &\geq |U_2| - |U_2^1| - |U_2^3| - 2(|U_1^3|+1) - (l-2) \quad \text{by Part 2}\\ &\geq |U_2| - |U^1| - 2|U^3| - l\\ &\geq |U_2| - l - 2(l-2) - l \quad \text{by Part 1}\\ &> |U_2| - 4l.\end{aligned}$$

Similar arguments show that $|U_1'| \geq |U_1| - 4l$.

Next we observe that for any subset $D \subseteq U_1'$ or $D \subseteq U_2'$, we can find at least $|D|/3$ vertex-disjoint D-2-paths. Below we prove this for $D \subseteq U_2'$. Let $x_1y_1z_1, \ldots, x_my_mz_m$ be m D-2-paths for some $m < |D|/3$ (then $x_i, z_i \in U_1$). By the definition of U_2', we have $|N(x_i) \cup N(z_i)| \leq 3$ for all i, and consequently there exists $y \in D$ such that $y \notin \bigcup_{i=1}^m N(x_i) \cup N(z_i)$. In other words, $y \notin \{y_1, \ldots, y_m\}$ and $N(y)$ is disjoint from $\{x_1, z_1, \ldots, x_m, z_m\}$. Hence y together with $N(y)$ (of size two) form a D-2-path that is vertex-disjoint from the existing U_2'-2-paths.

Furthermore consider $U_2'' = U_2' \setminus N^2(U_2^3)$, where $N^2(U_2^3) := N(N(U_2^3))$ is the set of the second-neighbors of U_2^3. Then every vertex $x \in U_2''$, its (two) neighbors, and its (at most two) second-neighbors all have degree at most two. Therefore every U_2''-2-path is a special U_2-2-path. Applying Part 1 and Part 2, we obtain that

$$|N^2(U_2^3)| \leq 2|N(U_2^3)| + (l-2) \leq 2(2|U_2^3| + l - 2) + l - 2 \leq 7(l-2),$$

and consequently $|U_2''| \geq |U_2'| - 7(l-2) \geq |U_2| - 11l$. Since $|U_2| \geq 26l$, we can find $|U_2''|/3 \geq (|U_2| - 11l)/3 \geq 5l$ vertex-disjoint U_2''-2-paths $P_1, \ldots, P_{5l}$.

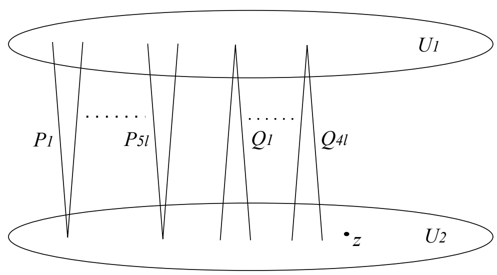

Figure 4: Proposition 7.11, Part 3

Finally let $\tilde{U}_1' = U_1' \setminus \cup_{i=1}^{5l} V(P_i)$. Then $|\tilde{U}_1'| \geq |U_1'| - 2(5l) \geq |U_1| - 14\,l$. Since $|U_1| \geq 26l$, we can find $|\tilde{U}_1'|/3 \geq (|U_1| - 14\,l)/3 \geq 4l$ vertex-disjoint $\tilde{U}_1'$-2-paths $Q_1, \ldots, Q_{4l}$. Since the mid-points of $P_1, \ldots, P_{5l}$ have degree two, the end-points of $P_1, \ldots, P_{5l}$ are all their neighbors. For each $x \in \tilde{U}_1'$, since $x \notin \cup_{i=1}^{5l} V(P_i)$, x is not adjacent to any mid-point of $P_1, \ldots, P_{5l}$. Therefore all $P_1, \ldots, P_{5l}, Q_1, \ldots, Q_{4l}$ are vertex-disjoint. In addition, our definition of U_1', U_2' guaranteed that z is not contained in any P_i or Q_i. □

Proof of Lemma 7.10. Let $k := |Y_2| \leq l$ and denote the given Y_2-2-paths by $O_1, \ldots, O_k$. Let $X' = X \setminus \cup_{i=1}^k V(O_i)$. By Proposition 7.11, T contains $4l + k$ special U_2-2-paths $P_1, \ldots, P_{4l+k}$ and $4l$ U_1-2-paths $Q_1, \ldots, Q_{4l}$ such that all the paths are vertex-disjoint and do not contain z. Let z be the root of T. For each $i = 1, \ldots, k$, let t_i be the end-point of $V(P_{4l+i})$ closer to z, and let $s_i = p(t_i)$ and $r_i = p(s_i)$ be its parent and grand-parent, respectively. (Note that s_i, r_i exist because $t_i, z \in U_1$ and $z \neq t_i$.) Since each P_i is a special U_2-2-path, we have $\deg(s_i) \leq 2$ and therefore $\deg(s_i) = 2$.

Let F be the forest obtained from T by removing the mid-points and the edges of P_i, Q_i for $i = 1, \ldots, 4l$. Then F has two partition sets F_e and F_o with $|F_e| = |U_1| - 4l$

and $|F_o| = |U_2| - 4l$, and F contains F_o-2-paths $P_{4l+1}, \dots, P_{4l+k}$. For $i = 1, \dots, k$, let $P'_{4l+i} := s_i P_{4l+i}$ (the 3-edge path obtained by extending P_{4k+i} to include s_i). We now embed F into $X \cup Y$ in three steps.

1. First embed $P_{4l+1}, \dots, P_{4l+k}$ to $O_1, \dots, O_k$
2. Next embed $F - \bigcup_{i=1}^{k} V(P'_{4k+i})$ to $X' \cup Y_1$ such that $z \to a$.
3. Finally embed $s_1, \dots, s_k$.

The embedding in Step 1 is obvious. The embedding in Step 2 follows from the greedy algorithm, in which we first embed $z \to a$ and then use

$$\delta(Y_1, X') \geq |X'| - l = |X| - 2k - l \geq |U_1| - 3l > |F_e|, \quad \text{and}$$

$$\delta(X', Y_1) \geq |Y_1| - l \geq |Y| - 2l \geq |U_2| - 2l > |F_o|.$$

In Step 3, we embed s_i for $1 \leq i \leq k$ as follows. Suppose that t_i is embedded to u_i in Step 1. If $r_i \notin V(F)$[14], then we simply map s_i to an unoccupied vertex in $N(u_i, Y_1)$. Otherwise assume that $r_i \in V(F)$ is mapped to some vertex v_i in Step 2. Then we map s_i to an unoccupied vertex in $N(u_i, Y_1) \cap N(v_i, Y_1)$. This is always possible because

$$|N(u_i, Y_1) \cap N(v_i, Y_1)| \geq |Y_1| - 2l \geq |U_2| - 3l > |F_o|.$$

It remains to embed the mid-points of $P_1, \dots, P_{4l}, Q_1, \dots, Q_{4l}$. Since $|X| \geq |U_1| = |F_e| + 4l$, we can find a subset $\tilde{X} \subset X$ containing $4l$ unoccupied vertices. For $i = 1, \dots, 4l$, let $p_i, q_i \in Y_1$ be the images of the end-vertices of Q_i. We form a bipartite graph $\tilde{B}$ on $\tilde{X}$ and $\tilde{Y} := \{p_i q_i : i = 1, \dots, 4l\}$ in which two vertices $x \in \tilde{X}$ and $p_i q_i \in \tilde{Y}$ are adjacent if and only if x is adjacent to both p_i and q_i. Since $\delta(Y_1, X) \geq |X| - l$, we have $\delta_{\tilde{B}}(\tilde{Y}, \tilde{X}) \geq 4l - 2l = |\tilde{X}|/2$. On the other hand, $\delta(X, Y_1) \geq |Y_1| - l$ implies that $\delta_{\tilde{B}}(\tilde{X}, \tilde{Y}) \geq 4l - l > |\tilde{Y}|/2$. By the marriage theorem, there exists a perfect matching between $\tilde{X}$ and $\tilde{Y}$ in $\tilde{B}$. We accordingly add $4l$ $\tilde{X}$-2-paths to F. We repeat this process to embed the mid-points of $P_1, \dots, P_{4l}$ and thus complete the embedding of T. $\square$

7.2.2 Proof of Lemma 7.5

In this subsection we prove Lemma 7.5 and thus complete the proof of Theorem 3.2. Let G be a $2n$-vertex graph G in **EC2** with parameter α, *i.e.*, $V(G)$ can be partitioned into $V_1 \cup V_2$ such that $|V_1| = |V_2| = n$ and $d(V_1, V_2) \leq \alpha$. Let L be the set of vertices of degree at least n. By assumption $|L| \geq n$. Assume that $\mathcal{T}_n \not\subset G$. Our goal is to show that G is in **EC3** with parameter $40\alpha^{\frac{1}{4}} + \sqrt{a}$. Now let $\alpha_1 = 40\alpha^{\frac{1}{4}}$.

Claim 7.12. *There is no vertex $v \in L$ such that* $\deg(v, V_1), \deg(v, V_2) \geq \alpha_1 n$.

[14] This means that r_i is the mid-point of some Q_j.

Proof. Suppose instead, there exists $v_0 \in L$ such that $\deg(v_0, V_1), \deg(v_0, V_2) \ge \alpha_1 n$. Without loss of generality, assume that $\deg(v_0, V_1) \ge \frac{n}{2}$.

For $i = 1, 2$, let A_i be the set of $x \in V_i \cap L$ such that $\deg(x, V_j) \le \sqrt{\alpha} n$ for $j \ne i$ (then $v_0 \notin A_i$). Thus $\delta(A_i, V_i) \ge (1 - \sqrt{\alpha})n$. Since $d(V_1, V_2) \le \alpha$, we have $|A_i| \ge |V_i \cap L| - \sqrt{\alpha} n$. If $|V_i \cap L| \ge n/2 + \sqrt{\alpha} n$ for any i, then $|A_i| \ge n/2$ and consequently G is in **EC3** with parameter $\sqrt{\alpha}$. We may thus assume that $|V_i \cap L| < \frac{n}{2} + \sqrt{\alpha} n$ for $i = 1, 2$. Since $|L| \ge n$, this implies that $|V_i \cap L| > n/2 - \sqrt{\alpha} n$ for $i = 1, 2$. Consequently

$$\frac{n}{2} + \sqrt{\alpha} n > |V_i \cap L| \ge |A_i| \ge |V_i \cap L| - \sqrt{\alpha} n > \frac{n}{2} - 2\sqrt{\alpha} n.$$

Applying Proposition 7.3 with $\theta = 2\sqrt{\alpha}$, we obtain $B'_i \subseteq V_i \setminus A_i$ such that $\delta(A_i, B_i)$, $\delta(B_i, A_i) \ge \frac{n}{2} - \sqrt{2}\alpha^{\frac{1}{4}} n$. Let $B_i = B'_i \setminus \{v_0\}$. We have

$$\delta(A_i, B_i), \delta(B_i, A_i) \ge \frac{n}{2} - \sqrt{2}\alpha^{\frac{1}{4}} n - 1 \ge \frac{n}{2} - 2\alpha^{\frac{1}{4}} n. \tag{7.11}$$

In addition,

$$\delta(A_i, A_i) \ge |A_i| - \sqrt{\alpha} n \ge \frac{n}{2} - 3\sqrt{\alpha} n \ge \frac{n}{2} - 2\alpha^{\frac{1}{4}} n. \tag{7.12}$$

Let T be a tree of size n. We will show that $T \subset G$. If T has a partition $U_1 + U_2$ such that $|U_1| \le \frac{n}{2} - 2\alpha^{\frac{1}{4}} n$ and U_2 is independent, then because of (7.12) and (7.11), $T \subset G$ follows from Fact 7.2 Part 2. We thus assume that T has no such partition.

Applying Fact 7.9 Part 1, we find an $[\frac{\alpha_1}{4} n, \frac{\alpha_1}{2} n]$-subtree T' rooted at r. Then $F = T - V(T')$ is a forest and $F \cup \{r\}$ spans a tree. We map r to v_0. Since $v_0 \in L$, all leaves that are adjacent to r can be added at the end. Let F' be the subforest obtained from F after removing all isolated vertices. Our goal is to map $T' - r$ to $A_2 \cup B_2$ and F' to $A_1 \cup B_1$ (note that $v_0 \notin A_1 \cup B_1 \cup A_2 \cup B_2$).

The embedding of $T' - r$ is easy. From (7.11), we derive that $|A_2 \cup B_2| \ge n - 4\alpha^{\frac{1}{4}} n$. Together with $\deg(v_0, V_2) \ge \alpha_1 n$, this implies that $\deg(v_0, A_2 \cup B_2) > \alpha_1 n - 4\alpha^{\frac{1}{4}} n > \frac{\alpha_1}{2} n$. Since $\deg_{T'}(r) \le \frac{\alpha_1}{2} n$, we are able to map $N_{T'}(r)$ to $A_2 \cup B_2$. Let $G_2 = G[A_2 \cup B_2]$. We have $\delta(G_2) \ge \frac{n}{2} - 2\alpha^{\frac{1}{4}} n > e(T')$ from (7.11). The remaining vertices in $T' - r$ thus can be embedded in G_2 by the greedy algorithm.

We now show how to embed F'. Since F' contains no isolated vertices,

$$|Rt(F') \le |V(F')|/2 \le (n - \frac{\alpha_1}{4} n)/2 = \frac{n}{2} - \frac{\alpha_1}{8} n.$$

Since $\deg(v_0, V_1) \ge n/2$ and $|A_1 \cup B_1| \ge n - 4\alpha^{\frac{1}{4}} n$, we have $\deg(v_0, A_1 \cup B_1) \ge \frac{n}{2} - 4\alpha^{\frac{1}{4}} n \ge \frac{n}{2} - \frac{\alpha_1}{8} n$ (here we need $\alpha_1 \ge 32\alpha^{\frac{1}{4}}$). Therefore we can map $Rt(F')$ to $N(v_0, A_1 \cup B_1)$. Let (X_1, Y_1) be the bipartition of F' such that the roots embedded to A_1 are in X_1, and the roots embedded to B_1 are in Y_1. If $\max\{|X_1|, |Y_1|\} \le \frac{n}{2} - 2\alpha^{\frac{1}{4}} n$, then we can embed F' to $A_1 \cup B_1$ by the greedy algorithm. Otherwise, without loss of generality, assume that $|X_1| > \frac{n}{2} - 2\alpha^{\frac{1}{4}} n$. Suppose that $T' - r$ has the bipartition (X_2, Y_2) with $|X_2| \ge |Y_2|$. Let $U_1 = Y_1 \cup Y_2 \cup \{r\}$ and $U_2 = X_1 \cup X_2$. Clearly U_2 is independent.

We claim that $|U_1| \le \frac{n}{2} - 2\alpha^{\frac{1}{4}}n$, contrary to our earlier assumption on T. In fact, since $|X_1| + |Y_1| = v(F') = n + 1 - v(T')$, we have

$$|Y_1| \le n + 1 - v(T') - \left(\frac{n}{2} - 2\alpha^{\frac{1}{4}}n\right) \le \frac{n}{2} - v(T') + 2\alpha^{\frac{1}{4}}n + 1.$$

Since $|Y_2| \le (v(T') - 1)/2$, it follows that

$$\begin{aligned} |U_1| &\le |Y_1| + \frac{v(T')-1}{2} + 1 \le \frac{n}{2} - \frac{v(T')}{2} + 2\alpha^{\frac{1}{4}}n + \frac{3}{2} \\ &\le \frac{n}{2} - 3\alpha^{\frac{1}{4}}n + \frac{3}{2} \quad \text{because } v(T') \ge \frac{\alpha_1}{4}n \ge 10\alpha^{\frac{1}{4}}n \\ &< \frac{n}{2} - 2\alpha^{\frac{1}{4}}n. \quad \Box \end{aligned}$$

Proof of Lemma 7.5. Let $L^1 = \{v \in L : \deg(v, V_1) > \alpha_1 n\}$ and $L^2 = \{v \in L : \deg(v, V_2) > \alpha_1 n\}$. Claim 7.12 implies that $L^1 \cap L^2 = \emptyset$. Since $\delta(L, V) \ge n$ and $2\alpha_1 n < n$, $L^1 \cup L^2$ is a partition of L. Thus $\delta(L^1, V_1) \ge (1 - \alpha_1)n$, and $\delta(L^2, V_2) \ge (1 - \alpha_1)n$. Let $L^i_j = L^i \cap V_j$, for $1 \le i, j \le 2$. Since $d(V_1, V_2) \le \alpha$ and $\alpha_1 \ge \sqrt{\alpha}$, we have $|L^1_2|, |L^2_1| < \sqrt{\alpha}n$. Let $V'_1 = (V_1 \cup L^1_2) \setminus L^2_1$ and $V'_2 = (V_2 \cup L^2_1) \setminus L^1_2$. Then $L^i \subseteq V'_i$ and $|V'_i| \ge n/2 - \sqrt{\alpha}n$ for $i = 1, 2$. We move at most $\sqrt{\alpha}n$ vertices of $V \setminus L$ between V'_1 and V'_2 such that $|V'_1| = |V'_2| = n$. Without loss of generality, assume that $|L^1| \ge n/2$. Since $\delta(L^1, V'_1) \ge (1 - \alpha_1)n - \sqrt{\alpha}n$, we conclude that G is in **EC3** with parameter $\alpha_1 + \sqrt{\alpha}$, with partition sets $V'_1 + V'_2$ and $A = L^1$. $\Box$

8 Concluding Remarks

- What is the smallest $m = m(n, n/2)$ such that *every* n-vertex graph with at least m vertices of degree at least $n/2$ contains *all* trees on n edges as subgraphs? We have shown that this number is between $n/2 - \sqrt{n} - 1$ and $n/2$. We feel that lower bound is closer to the truth. To verify it, because of the robustness of Theorem 3.3, it suffices to improve our proof of the extremal cases.
- The techniques proving the Extremal Case 3 can be applied to prove the $k \ge (1 - \varepsilon)v(G)$ case of the Komlós-Sós Conjecture (exactly) for sufficiently small ε. Since the aim of this paper is to prove the $(n/2 - n/2 - n/2)$ Conjecture, we do not generalize our proof for this purpose.

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Note added in proof. After the first version of this paper was written and publicized in 2002, more work has been done on the Komlós-Sós Conjecture (Conjecture 1.4). Piguet and Stein [15] recently proved an approximate version of the conjecture. More recently Piguet and Hladký [12] and independently Cooley [6] combined the ideas from the present paper and [15] to prove Conjecture 1.4 for all $k = \Omega(n)$.

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A Appendix

We prove Lemma 5.4 Part 3. The following corollary of Lemma 5.1 gives a sufficient condition for embedding a forest of small trees into two adjacent clusters with prescribed location of roots.

Corollary A.1. *Let $X, Y \in \mathcal{V}$ be two adjacent clusters containing subsets $P \subseteq X_1 \subseteq X$ and $Q \subseteq Y_1 \subseteq Y$. Let F be a forest consisting of trees of order at most εN. If (U_1, U_2) is a bipartition of F with $R_1 := Rt(F) \cap U_1$ and $R_2 := Rt(F) \cap U_2$ such that*

$$\begin{array}{ll} |U_1| \le |X_1| - (\gamma + 3\varepsilon)N, & |U_2| \le |Y_1| - (\gamma + 3\varepsilon)N, \\ |R_1| \le |P| - 3\varepsilon N, & |R_2| \le |Q| - 3\varepsilon N, \end{array} \tag{A.1}$$

then we can embed F with $U_1 \xrightarrow{2\varepsilon N} X_1$, $U_2 \xrightarrow{2\varepsilon N} Y_1$, $R_1 \xrightarrow{2\varepsilon N} P$, and $R_2 \xrightarrow{2\varepsilon N} Q$.

Proof. We describe an algorithm of embedding trees in F by applying Lemma 5.1 repeatedly while mapping as many non-root vertices as possible to $X \setminus P$ and $Y \setminus Q$. Assume that trees $T_1, \ldots, T_{i-1}$ from F have been embedded such that $\bigcup_{j<i} V(T_j) \cap U_1 \to X_1$ and $\bigcup_{j<i} V(T_j) \cap U_2 \to Y_1$. Let X_1^*, Y_1^*, P^*, Q^* denote the sets of available vertices in X_1, Y_1, P, Q, respectively, at this moment. The assumption (A.1) implies that

$$|X_1^*|, |Y_1^*| \ge (\gamma + 3\varepsilon)N. \tag{A.2}$$

Without loss of generality, suppose the next tree T_i in F has its root at U_1. Let $X_0 = X_1^* \setminus P$ if $|X_1^* \setminus P| \ge \gamma N$; otherwise $X_0 = X_1^*$. Similarly we define Y_0. In order to embed $T_i \to P^* \cup X_0 \cup Y_0$ by Lemma 5.1, we need to verify that $|P^*| \ge 3\varepsilon N$ and $|X_0|, |Y_0| \ge \gamma N$. It is easy to see that, for example, $|X_0| \ge \gamma N$ follows from the definition of X_0 and (A.2). Since $|R_1| \le |P| - 3\varepsilon N$, $|P^*| \le 3\varepsilon N$ is only possible when P contains images of non-root vertices. This implies that $X_1 \setminus P$ has fewer than γN vertices available before embedding T_{i-1}. Together with $|P^*| \le 3\varepsilon N$, this implies that $|X_1^*| < (\gamma + 3\varepsilon)N$, a contradiction. □

The proof of Lemma 5.4 Part 3 is somewhat technical. The main difficulty is that when embedding the first tree T_1 of F, the image u of the second root r_2 has not been decided yet so we can *not* purposely avoid $P := N(u, X)$ or $Q := N(u, Y)$ as in the proof of Corollary A.1. Certainly we want to map $Rt(F)$ to the vertices that are typical to the sets of available vertices in X and Y. Nevertheless we may not be able to embed an *ordered* F to $C \cup X \cup Y$ even if $F^o := F - Rt(F)$ has a bipartition similar to the one in Corollary A.1:

$$\begin{array}{ll} |U_1| \le |X| - (\gamma + 3\varepsilon)N, & |U_2| \le |Y| - (\gamma + 3\varepsilon)N, \\ |R_1| \le d(C, X)N - \varepsilon N - 3\varepsilon N, & |R_2| \le d(C, Y)N - \varepsilon N - 3\varepsilon N. \end{array} \tag{A.3}$$

Let us give an example. Construct a tripartite random graph on three sets C, X, Y of size N such that each edge appears with probability $1/3$ independently. Suppose that F consists of two εN-trees with roots r_1 and r_2. Accordingly $F^o = F - \{r_1, r_2\}$ is partitioned into two forests F_1 and F_2, each of which consists of trees of order at most εN. Suppose that $v(F_1) = v(F_2) = (1-2\gamma)N$ and all trees in F_1, F_2 have ratio $1/2$ (for example, they are paths of even order). Furthermore, assume that $|Rt(F_1)| = \frac{N}{6}$ and $|Rt(F_2)| = (\frac{1}{2} - 2\gamma)N$. After embedding r_1 to C and F_1 to $X \cup Y$, the sets X^* and Y^* of the remaining vertices are of size about $N/2$ (because $Ratio(F_1) = 1/2$). However, for each vertex $u \in C$, we have $\deg(u, X^* \cup Y^*) = |X^* \cup Y^*|/3 \approx \frac{N}{3} < (\frac{1}{2} - 2\gamma)N = |Rt(F_2)|$. There is no enough space for $Rt(F_2)$ no matter how we map r_2 in C. On the other hand, let (U_1, U_2) be a bipartition of F^o such that the roots of F_1 and F_2 are distributed evenly. Let $R_i = V(F^o) \cap U_i$. Then

$$|R_1| = |R_2| = \frac{1}{2}\left(\frac{N}{6} + \left(\frac{1}{2} - 2\gamma\right)N\right) = \frac{N}{3} - \gamma N \le \frac{N}{3} - 4\varepsilon N,$$

and $|U_1| = |U_2| = (1 - 2\gamma)N \le N - (\gamma + 3\varepsilon)N$. Thus (A.3) holds.

Let X, Y be adjacent clusters with $P \subseteq X$ and $Q \subseteq Y$, we write $F \to (P, Q; X, Y)$ if $F \xrightarrow{2\varepsilon N} X \cup Y$ with $Rt(F) \xrightarrow{2\varepsilon N} P \cup Q$.

Lemma A.2. *Let $X, Y \in \mathcal{V}$ be two adjacent clusters with subsets $P \subseteq X_1 \subseteq X$ and $Q \subseteq Y_1 \subseteq Y$. Assume that $|X_1| \le |Y_1|$. Let F be a forest consisting of trees of order between 2 and εN (inclusive). Then $F \to (P, Q; X_1, Y_1)$ if*

$$v(F) \le \min\Big\{2|P| + 2|Q| - 12\varepsilon N,\ \min\{|P|, |Q|\} + |X_1| - (2\gamma + 7\varepsilon)N\Big\}. \tag{A.4}$$

Furthermore, let X_1^*, Y_1^* denote the sets of available vertices in X_1, Y_1 after F is embedded, and $X_1' := X_1 - X_1^*$ and $Y_1' := Y_1 - Y_1^*$. Then one of the following holds.

Case 1: $||X_1^*| - |Y_1^*|| \le \max\{||X_1| - |Y_1||, \varepsilon N\}$.

Case 2: $|X_1'|, |Y_1'| \ge |P| - 3\varepsilon N$.

Case 3: $|X_1'|, |Y_1'| \ge |Q| - 3\varepsilon N$.

Proof. We show that there is a bipartition of $V(F)$ into U_1 and U_2, with $R_1 = Rt(F) \cap U_1$ and $R_2 = Rt(F) \cap U_2$ satisfying (A.1). Then $F \to (P, Q; X_1, Y_1)$ follows from Corollary A.1.

Suppose that $F = \{T_1, \dots, T_s\}$. For every $i \le s$, we have $|(T_i)_{even}|, |(T_i)_{odd}| \le \varepsilon N - 1$ because $v(T_i) \le \varepsilon N$, . Fix $i \le s$. By distributing the roots of $T_1, \dots, T_i$ properly, we obtain a bipartition (U_1^i, U_2^i) of $T_1 \cup \dots \cup T_i$ such that $|U_1^i| \le |U_2^i| < |U_1^i| + \varepsilon N$. Let $R_1^i = Rt(F) \cap U_1^i$ and $R_2^i = Rt(F) \cap U_2^i$. Now we consider three possibilities.

(1). $|R_1^s| \le |P| - 3\varepsilon N$ and $|R_2^s| \le |Q| - 3\varepsilon N$.

Let $U_1 = U_1^s$, $U_2 = U_2^s$, $R_1 = R_1^s$, and $R_2 = R_2^s$. Then

$$|U_1| \le |U_2| < |U_1| + \varepsilon N. \tag{A.5}$$

We claim that (U_1, U_2) is a bipartition of F satisfying (A.1). Clearly R_1 and R_2 satisfy (A.1). Using (A.5) and $|U_1| + |U_2| = v(F) \le 2|X_1| - (2\gamma + 7\varepsilon)N$, we derive that $2|U_2| \le 2|X_1| - (2\gamma + 6\varepsilon)N$, and consequently $|U_1| \le |U_2| \le |X_1| - (\gamma + 3\varepsilon)N$.

We observe that Case 1 holds. In fact, $|Y_1| \ge |X_1|$ implies that $|Y_1^*| + |Y_1'| \ge |X_1^*| + |X_1'|$. Since $|X_1'| = |U_1|$ and $|Y_1'| = |U_2|$, we have $|Y_1^*| - |X_1^*| \ge |U_1| - |U_2| > -\varepsilon N$ by (A.5). On the other hand, $|Y_1^*| - |X_1^*| = (|Y_1| - |X_1|) - (|U_2| - |U_1|) \le |Y_1| - |X_1|$ by (A.5). Putting them together, we obtain $||X_1^*| - |Y_1^*|| \le \max\{|Y_1| - |X_1|, \varepsilon N\}$.

(2). There exists $i < s$ such that $|R_1^i| = |P| - 3\varepsilon N$.

In this case, after constructing (U_1^i, U_2^i), we add the remaining trees of F to U_1^i and U_2^i such that all roots are in U_2^i. Let (U_1, U_2) denote the resulting bipartition and $R_i = Rt(F) \cap U_i$ for $i = 1, 2$. We claim that U_1, U_2, R_1 and R_2 satisfy (A.1). First, we have $|R_1| = |R_1^i| = |P| - 3\varepsilon N$. Second, it is impossible to have $|R_2| > |Q| - 3\varepsilon N$ because it implies that $|R_1| + |R_2| > |P| + |Q| - 6\varepsilon N$. Since every tree in F has at least 2 vertices, this yields that $v(F) \ge 2(|R_1| + |R_2|) > 2(|P| + |Q| - 6\varepsilon N)$, contrary to (A.4). Since $|R_1^i| = |P| - 3\varepsilon N$ and every tree in F has at least 2 vertices, we have $|U_1^i|, |U_2^i| \ge |P| - 3\varepsilon N$. Together with

$$|U_1| + |U_2| = v(F) \le |P| + |X_1| - (\gamma + 6\varepsilon)N \le |P| + |Y_1| - (\gamma + 6\varepsilon)N,$$

it gives that $|U_2| \le |Y_1| - (\gamma + 3\varepsilon)N$ and $|U_1| \le |X_1| - (\gamma + 3\varepsilon)N$.

Furthermore, we are in Case 2 because $|X_1'|, |Y_1'| \ge \min\{|U_1^i|, |U_2^i|\} \ge |P| - 3\varepsilon N$.

(3). There exists $i < s$ such that $|R_2^i| = |Q| - 3\varepsilon N$.

In this case we add the remaining trees of F to U_1^i and U_2^i such that all their roots are in U_1^i. The rest of the proof is similar to **(2)** except that we derive Case 3, $|X_1'|, |Y_1'| \ge |Q| - 3\varepsilon N$, instead. □

Proof of Lemma 5.4 Part 3. Suppose that $F = \{T_1, \ldots, T_s\}$ has roots $r_1, \ldots, r_s$ and satisfies

$$||F|| \le (d_x + d_y + \lambda - 2\gamma - 13\varepsilon)N. \tag{A.6}$$

We embed trees $T_1, \ldots, T_s$ to $C \cup X \cup Y$ in order. Let $X^0 = Y^0 = \emptyset$. For $i = 1, \ldots, s$, let X^i and Y^i denote the sets of occupied vertices in X and Y, respectively, after embedding $T_1, \ldots, T_i$. Our goal is to prove the following claim.

Claim. *For $i = 1, \ldots, s$, let $u_i \in C$ be an unoccupied vertex in C that is typical to $X - X^{i-1}$ and $Y - Y^{i-1}$. Then we can embed T_i such that $Rt(T_i) \to u_i$ and $T_i - r_i \xrightarrow{2\varepsilon N} (X - X^{i-1}) \cup (Y - Y^{i-1})$. In addition, one of the following holds.*

Case a) $||X^i| - |Y^i|| < \varepsilon N$.

Case b) $|X^i|, |Y^i| \ge d_x N - 5\varepsilon N$.

Case c) $|X^i|, |Y^i| \ge d_y N - 5\varepsilon N$.

The claim immediately implies $F \to (C, \{X, Y\})$: by the regularity of (C, X) and (C, Y), all but at most $s - 1 + 2\varepsilon N < 3\varepsilon N$ of C can be chosen as u_i for $1 \le i \le s$. Furthermore, since $X^0 = Y^0 = \emptyset$, u_1 can be any vertex in C that is typical to X and Y.

Let us prove this claim by induction on i. To facilitate our induction, we start with $i = 0$: there is nothing to embed; we are in Case a) because $|X^0| = |Y^0| = 0$. Suppose the claim holds for $i - 1$ for some $i \ge 1$. Then $\{T_1, \ldots, T_{i-1}\} \to (C, \{X, Y\})$ such that Case a), b) or c) holds for $i - 1$. Let $X_1 = X - X^{i-1}$ and $Y_1 = Y - Y^{i-1}$ denote the sets of available vertices in X and Y. Without loss of generality, assume that $|X_1| \le |Y_1|$.

We first map the root r_i of T_i to the given vertex $u_i \in C$. Then we attempt to embed the forest $F_i := T_i - \{r_i\}$ by Lemma A.2 to $X_1 \cup Y_1$ with $P := N(u, X_1)$, and $Q := N(u, Y_1)$. For convenience, write $x_0 = |X^{i-1}|$, $y_0 = |Y^{i-1}|$, $x_1 = |X_1|$, and $y_1 = |Y_1|$ (so $x_0 + x_1 = y_0 + y_1 = N$). Since u_i is typical to X_1 and Y_1, we have

$$|P| \ge d_x x_1 - \varepsilon N \quad \text{and} \quad |Q| \ge d_y y_1 - \varepsilon N. \tag{A.7}$$

If (A.4) holds for F_i, then $F_i \xrightarrow{2\varepsilon N} X_1 \cup Y_1$ by Lemma A.2. Otherwise

$$v(F_i) > \min\Big\{2|P| + 2|Q| - 12\varepsilon N, \min\{|P|, |Q|\} + |X_1| - (2\gamma + 7\varepsilon)N\Big\},$$

which leads to two possible cases.

Case I. $v(F_i) > \min\{|P|, |Q|\} + |X_1| - (2\gamma + 7\varepsilon)N$.

First assume that $|P| \ge |Q|$. By (A.7), it follows that

$$v(F_i) > |Q| + |X_1| - (2\gamma + 7\varepsilon)N \ge d_y y_1 + x_1 - (2\gamma + 8\varepsilon)N \tag{A.8}$$

Consequently

$$\begin{aligned} ||F|| &\ge x_0 + y_0 + v(F_i) \\ &> x_0 + y_0 + d_y y_1 + x_1 - (2\gamma + 8\varepsilon)N \\ &\ge N + d_y N - (2\gamma + 8\varepsilon)N \quad \text{by using} \quad d_y y_1 + y_0 \ge d_y y_1 + d_y y_0 = d_y N \\ &\ge (d_x + d_y + \lambda)N - (2\gamma + 8\varepsilon)N \quad \text{by using} \quad d_x \le 1 - \lambda, \end{aligned}$$

contrary to (A.6).

Second assume that $|P| < |Q|$. Using (A.7) we have

$$v(F_i) > |P| + |X_1| - (2\gamma + 7\varepsilon)N \ge d_x x_1 + x_1 - (2\gamma + 8\varepsilon)N.$$

Consequently $||F|| > x_0 + y_0 + x_1 + x_1 d_x - (2\gamma + 8\varepsilon)N$. Now we proceed under the three cases defined in the claim.

- Under Case a), $y_0 \ge x_0 - \varepsilon N$, we have $||F|| > N + d_x N - (2\gamma + 9\varepsilon)N$.
- Under Case b), $y_0 \ge d_x N - 5\varepsilon N$, we have $||F|| > N + d_x N - (2\gamma + 13\varepsilon)N$.
- Under Case c), $y_0 \ge d_y N - 5\varepsilon N$, we have $||F|| > N + d_y N - (2\gamma + 13\varepsilon)N$.

Since $\{d_x, d_y\} \leq 1 - \lambda$, all these cases yields that $||F|| > (d_x + d_y + \lambda)N - (2\gamma + 13\varepsilon)N$, contrary to (A.6).

Case II. $v(F_i) > 2|P| + 2|Q| - 12\varepsilon N$.

Using (A.7) we have

$$\begin{aligned} ||F|| &> x_0 + y_0 + 2(d_x x_1 + d_y y_1 - 2\varepsilon N) - 12\varepsilon N \\ &= d_x N + d_y N + (x_0 + d_x N - 2d_x x_0) + (y_0 + d_y N - 2d_y y_0) - 16\varepsilon N. \end{aligned}$$

Since $2\gamma > 3\varepsilon$, in order to get a contradiction to (A.6), it suffices to show that

$$(x_0 + d_x N - 2d_x x_0) + (y_0 + d_y N - 2d_y y_0) \geq \lambda N. \tag{A.9}$$

Since $0 \leq d_x, d_y \leq 1$ and $0 \leq x_0, y_0 \leq N$,

$$x_0 - d_x x_0, \quad d_x N - d_x x_0, \quad y_0 - d_y y_0, \quad d_y N - d_y y_0$$

are all non-negative. When $x_0 \geq N/2$, we have $x_0 + d_x N - 2d_x x_0 \geq x_0 - d_x x_0 \geq \lambda N/2$ (by using $d_x \leq 1 - \lambda$). Similarly when $y_0 \geq N/2$, we have $y_0 + d_y N - 2d_y y_0 \geq \lambda N/2$. If both $x_0 \geq N/2$ and $y_0 \geq N/2$, then (A.9) holds. Otherwise assume that $x_0 < N/2$. It is easy to see that $f(d_x) := x_0 + d_x N - 2d_x x_0$ is an increasing function of d_x. Since $d_x \geq \lambda$, this implies that $f(d_x) \geq x_0 + \lambda N - 2\lambda x_0 \geq \lambda N$ (since $\lambda \leq 1/2$). The case when $y_0 < N/2$ is the same.

Now we complete our induction proof by showing one of the cases a), b) or c) holds for i. By induction hypothesis, Case a), b) or c) holds for $i - 1$. Since $X^{i-1} \subseteq X^i$ and $Y^{i-1} \subseteq Y^i$, if either Case b) or Case c) holds for $i - 1$, then it holds for i as well. We may thus assume that Case a) holds for $i - 1$, namely, $||X^{i-1}| - |Y^{i-1}|| < \varepsilon N$. Since $|X| = |Y| = N$, it follows that $||X_1| - |Y_1|| < \varepsilon N$.

Since we embedded F_i by Lemma A.2, one of the cases 1, 2, and 3 in Lemma A.2 must hold. First assume that Case 1 holds. Then $||X_1^*| - |Y_1^*|| \leq \max\{||X_1| - |Y_1||, \varepsilon N\} < \varepsilon N$, where X_1^*, Y_1^* denote the sets of unoccupied vertices in X_1, Y_1 after embedding F_i. Since $X^i = X - X_1^*$ and $Y^i = Y - Y_1^*$, this implies that $||X^i| - |Y^i|| < \varepsilon N$. Hence Case a) holds for i.

Now assume that Lemma A.2 Case 2 holds, namely, $|X^i - X^{i-1}|, |Y^i - Y^{i-1}| \geq |P| - 3\varepsilon N$. By using (A.7), we derive that

$$|X^i| = |X^{i-1}| + |X^i - X^{i-1}| \geq |X^{i-1}| + (d_x|X_1| - \varepsilon N) - 3\varepsilon N \geq d_x N - 4\varepsilon N,$$

and (using $||X^{i-1}| - |Y^{i-1}|| < \varepsilon N$ as well)

$$|Y^i| = |Y^{i-1}| + |Y^i - Y^{i-1}| \geq (|X^{i-1}| - \varepsilon N) + (d_x|X_1| - \varepsilon N) - 3\varepsilon N \geq d_x N - 5\varepsilon N.$$

Thus Case b) holds for F.

Similarly we can derive Case c) for i if Lemma A.2 Case 3 holds. This finally completes the proof of Lemma 5.4 Part 3. □