Counting points of slope varieties over finite fields

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Abstract

The slope variety of a graph is an algebraic set whose points correspond to drawings of that graph. A complement-reducible graph (or cograph) is a graph without an induced four-vertex path. We construct a bijection between the zeroes of the slope variety of the complete graph on n vertices over $\mathbb{F}_2$, and the complement-reducible graphs on n vertices.

1 Introduction

Fix a field $\mathbb{F}$ and a positive integer n. Let $P_1 = (x_1, y_1), \ldots, P_n = (x_n, y_n)$ be points in the plane $\mathbb{F}^2$ such that the x_i are distinct. Let $L_{1,2}, \ldots, L_{n-1,n}$ be the $\binom{n}{2}$ lines in $\mathbb{F}^2$ where $L_{i,j}$ is the line through P_i and P_j. The *slope variety* $S_{\mathbb{F}}(K_n)$ is the set of possible $\binom{n}{2}$-tuples $(m_{1,2}, \ldots, m_{n-1,n})$, where $m_{i,j} = \frac{y_i - y_j}{x_i - x_j}$ denotes the slope of $L_{i,j}$ (if $\mathbb{F}$ is a finite field with q elements then we use the notation $S_q(K_n)$). Over an algebraically closed field, the slope variety is the set of simultaneous solutions of certain polynomials τ_W, called *tree polynomials* [4, 5], indexed by wheel subgraphs of the complete graph K_n. (A k-wheel is a graph formed from a cycle of length k by introducing a new vertex adjacent to all vertices in the cycle.) The ideal generated by all tree polynomials is radical [5, Theorem 1.1]. It is conjectured, and has been verified experimentally for $n \leq 9$, that the ideal of all tree polynomials is in fact generated by the subset $\{\tau_Q\}$ [5], where Q is a 3-wheel (equivalently, a 4-clique) in K_n.

The tree polynomials have integer coefficients, which raises the question of counting their solutions over a finite field. Let $\mathbb{F}_q$ be the field with q elements. In this article, we count the solutions of the tree polynomials over $\mathbb{F}_2$ and give some generalizations for $q > 2$. When the slope variety is considered over $\mathbb{F}_q$, the points correspond to drawings, in $\overline{\mathbb{F}_q}^2$ whose slopes are in $\mathbb{F}_q$. These drawings need not be in $\mathbb{F}_q^2$. If $q = 2$, then there is no way to draw K_n, for $n \geq 3$, so that the vertices have distinct x-coordinates. When considered

over $\mathbb{Z}$, the tree polynomials are in the kernel of the map from $\mathbb{Z}[\{m_{i,j}] \to \mathbb{Q}[\{x_i, y_i, \frac{1}{x_i - x_j}\}]$ defined by $m_{i,j} \mapsto \frac{y_i - y_j}{x_i - x_j}$.

Theorem 1. *Let n be a positive integer and let $\mathbb{F}_2[K_n] := \mathbb{F}_2[m_{1,2}, \ldots, m_{n-1,n}]$. Let I_n denote the ideal of $\mathbb{F}_2[K_n]$ generated by the tree polynomials of wheel subgraphs of K_n, and let J_n denote the ideal generated by the tree polynomials of K_4-subgraphs of K_n (so $J_n \subseteq I_n$).*

Then the following sets are equinumerous:

1. *the zeroes of I_n, i.e., the points in $\mathbb{F}_2^{\binom{n}{2}}$ on which all tree polynomials vanish;*
2. *the zeroes of J_n, i.e., the points in $\mathbb{F}_2^{\binom{n}{2}}$ on which all tree polynomials of 3-wheels vanish;*
3. *complement-reducible graphs (or "cographs") on vertex set $[n] = \{1, 2, \ldots, n\}$, that is, graphs on $[n]$ having no induced subgraph isomorphic to a four-vertex path;*
4. *switching-equivalence classes of graphs on vertex set $[n+1]$ such that no member of the class contains an induced 5-cycle.*

We will explain all these combinatorial interpretations below. The following Theorem appears in [8, Exercise 5.40] and is credited to Cameron [2].

Theorem 2. *The following sets are equinumerous:*

1. *switching-equivalence classes of graphs on vertex set $[n+1]$ such that no member of the class contains an induced 5-cycle;*
2. *series-parallel posets with n labeled vertices;*
3. *series-parallel networks with n labeled edges.*

In this paper, we use the special structure of tree polynomials to prove first the equality of (1), (2) and (3) of Theorem 1 (Proposition 6), and then a bijection between (3) and (4) of Theorem 1 (Proposition 7).

We note that a bijection between *unlabeled* complement-reducible graphs and *unlabeled* series-parallel networks was given by Sloane, see sequence A000084 [6]. We have not found in the literature an explicit bijection for the corresponding labeled objects. The numbers of points in $S_2(K_1)$, $S_2(K_2)$, ..., are

$$1,\ 2,\ 8,\ 52,\ 472,\ 5504,\ 78416,\ \ldots$$

which is sequence A006351 in [6].

2 Background

2.1 Graph theory

We list some necessary notation here; for a general background on graph theory see [1] or [9]. A *graph* G is an ordered pair (V, E) of *vertices* and *edges* i.e., V is a finite set, and E is a set of 2-subsets of V. Two vertices $u, v \in V$ are adjacent if there is an edge $uv \in E$ between them. If V, E are not specified then $V(G)$ is the set of vertices of G and $E(G)$ is the edge set. All graphs in this paper are *simple*, i.e., they have no loops or multiple edges.

For $U \subseteq V$, the *induced subgraph* $G|_U$ of G on U, is the graph with vertex set U and edge set $\{uv \in E(G) \mid u, v \in U\}$. The *intersection* $G \cap H$ of two graphs G and H is the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. The *complement* $\overline{G}$ of G is the graph on the same set of vertices as G whose edges are exactly the non-edges of G.

Let K_n denote the complete graph on n vertices. Let P_n denote the path on n vertices, also called the n-path. A *complement-reducible graph*, or *cograph*, has no induced P_4. An important fact that we will need is that G is complement-reducible if and only if for every induced subgraph $H \subseteq G$, either H or the complement $\overline{H}$ is disconnected; see [3].

The *k-wheel* $W(v_0; v_1, \ldots, v_k)$ is the graph on the vertices $\{v_0, \ldots, v_k\}$ and whose edges are $v_0v_1, \ldots, v_0v_k$, $v_1v_2, \ldots, v_{k-1}v_k, v_kv_1$, $k \geq 3$. Note that the wheel $W(v_0; v_1, \ldots, v_k)$ is invariant up to dihedral permutations of $v_1, \ldots, v_k$. The vertex v_0 is called the *center*; the other vertices are called the *spokes*. The edges incident to the center are called the *radii*, and the other edges are *chords*. Note that a 3-wheel is the complete graph on four vertices.

2.2 Series-parallel networks

A *network* is a graph G with two vertices s_G, t_G designated as the *source* and *sink*, respectively. Two networks G and H can be connected in *series* or *parallel*. The *series connection* $G \oplus H$ is defined by identifying t_G with s_H, and designating s_G as the source and t_H as the sink. The *parallel connection* $G + H$ is defined by identifying s_G with s_H and t_G with t_H.

A *series-parallel network* is a graph obtained from the following rules:

1. a graph with one edge st is a series-parallel network;
2. if G and H are series-parallel networks, then $G \oplus H$ and $G + H$ are series-parallel networks.

One can define series and parallel connections for posets in a similar fashion; see [7, Section 3.2]. Two posets P and Q are connected in series by taking their *ordinal sum* $P \oplus Q$: declaring that all elements of Q are larger than all elements of P (or vice versa) leaving all other relations unchanged. The two posets are connected in parallel by taking the disjoint union. A *series-parallel poset* is a poset built up from single-element posets by series and parallel extensions.

Let $s(n)$ be the number of labeled series-parallel networks on n vertices. The sequence begins

$$s(1) = 1, \quad s(2) = 2, \quad s(3) = 8, \quad s(4) = 52, \quad s(5) = 472, \quad s(6) = 5504, \quad \ldots$$

This is sequence A006351 in the On-Line Encyclopedia of Integer Sequences [6].

2.3 Switching equivalence

Let G be a graph on $[n+1]$ and let $X \subseteq [n]$. The *switch* of G with respect to X is the graph $s_X(G)$ on $[n+1]$ whose edges e satisfy one of two conditions:

1. $e \in E(G)$ and either both vertices of e belong to X or neither do;
2. $e \notin E(G)$ and exactly one vertex of e belongs to X.

This operation is also referred to as graph switching or Seidel switching [10]. Let $\mathscr{G}_{n+1}$ be the set of graphs on $[n+1]$. Then switching defines an action of $\mathbb{Z}_2^n$ on $\mathscr{G}_{n+1}$. For $x = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$, let $X = \{i \mid x_i = 1\} \subset [n]$. Then the group action is $xG = s_X(G)$. This action is free because $s_X(G) = G$ if and only if $X = \emptyset$. The orbits are called *switching classes*, denoted by $[G]$. To see that each orbit contains exactly one graph in which the vertex $n+1$ is isolated, let $G \in \mathscr{G}_{n+1}$ and let $X = N(n+1)$ be the set of neighbors of $n+1$. Then the graph $s_X(G)$ has $n+1$ as an isolated vertex. On the other hand if X is any other subset of $[n]$, then $n+1$ will be adjacent to some vertex of $s_X(G)$. The number of switching classes on $[n+1]$ is $s(n)$, the number of labeled series-parallel networks [8, Exercise 5.40(b)], [2].

2.4 Tree polynomials

We briefly sketch the basics of graph picture spaces; for more details, see [4].

Definition 3. Let $G = (V, E)$ be a graph. For each $e \in E$, let m_e be a variable. For each subset $F \subseteq E$ define

$$m_F = \prod_{f \in F} m_f.$$

We regard the square-free monomial m_F as corresponding to the spanning subgraph (V, F), and we will often ignore the distinction between the monomial and the graph.

A *picture* of a graph $G = (V, E)$ is a collection of labeled points and lines in the plane, corresponding to the vertices and edges of G, respectively, such that the line ℓ_e corresponding to an edge e contains both points corresponding to the endpoints of e. The vertices have distinct locations. The plane in [4] is taken to be $\mathbb{F}^2$ where $\mathbb{F}$ is algebraically closed. Provided that no lines are vertical, each line ℓ_e has a well-defined slope m_e, and so each picture determines a *slope vector* $(m_e)_{e \in E}$. The set of all possible slope vectors is called the *slope variety* $S(G)$.

The slope variety is the set of common zeroes of the set of polynomials called *tree polynomials*, as we now explain. A *(rigidity) pseudocircuit* is a graph H whose edge set can be partitioned into two spanning trees. A *coupled spanning tree* of H is a tree whose complement is also a spanning tree; the set of all coupled spanning trees of H is denoted $Cpl(H)$. For each pseudocircuit $H \subseteq G$, there is a polynomial

$$\tau_H = \sum_{T \in Cpl(H)} \epsilon(H,T) m_T \tag{1}$$

that vanishes on the slope variety of G; where each $\epsilon(H,T) \in \{1,-1\}$. The sign $\epsilon(H,T)$ arises because the tree polynomial is the determinant of a matrix whose entries are either 0 or of the form $m_e - m_f$. For a k-wheel, the sign is determined by the parity of k and the parity of the number of radii in the coupled spanning tree. Call this polynomial the *tree polynomial* of H.

Because the tree polynomials have integer coefficients, it makes sense to consider these polynomials inside the polynomial ring

$$\mathbb{F}_q[G] := \mathbb{F}_q[m_e \mid e \in E(G)].$$

Define the *q-slope variety* $S_q(G)$ to be the zero set of the ideal generated by the tree polynomials of all pseudocircuit subgraphs of G. The main concern of this paper is $S_2(K_n)$, the set of zeroes of the complete graph over $\mathbb{F}_2$.

The most important pseudocircuits are the wheels. The tree polynomial of the wheel $W = W(v_0; v_1, \ldots, v_k)$ has the form

$$\tau_W = \underbrace{\prod_{i=1}^{k} (m_{0,i} - m_{i,i+1})}_{\tau_1} - \underbrace{\prod_{i=1}^{k} (m_{0,i} - m_{i-1,i})}_{\tau_2} \tag{2}$$

where $m_{k,k+1} = m_{1,k}$ [5, eqn. (6)].

Suppose we draw the wheel $W(v_0; v_1, \ldots, v_k)$ with v_0 in the center and the indices of the spokes increasing as we travel clockwise around the perimeter. Each binomial factor in τ_1 is a radius minus the adjacent chord pointing in the clockwise direction, whereas each binomial factor in τ_2 is a radius minus the adjacent chord pointing in the counterclockwise direction. Therefore, if we expand the expression (2) for τ_W, then the claw subgraph (i.e., the graph consisting of three edges that meet at a point) and the cycle of all the chords each occur twice, and with opposite signs. The only remaining terms are coupled spanning trees, which are obtained by picking a nontrivial subset of radii along with all chords pointing clockwise or counterclockwise, but not both. See Figure 1.

The tree polynomials of all the wheels in K_n generate the ideal of tree polynomials of all rigidity pseudocircuits in K_n [5]. Define ideals $I_n, J_n \subseteq \mathbb{F}_2[G]$ as follows:

$$I_n = (\tau_W \mid W \text{ is a wheel in } K_n),$$
$$J_n = (\tau_Q \mid Q \subseteq K_n \text{ is isomorphic to } K_4).$$

It was conjectured in [5] that $I_n = J_n$ when considered as ideals over $\mathbb{C}$. Using the computer algebra system Macaulay this conjecture has been verified for $n \leq 9$ [5].

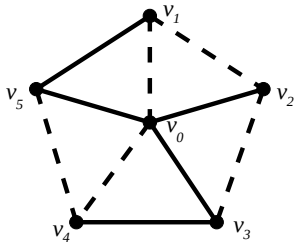

Figure 1: Two complementary coupled spanning trees of a 5-wheel

3 A bijection between slope vectors and complement-reducible graphs

In this section we count the points of $S_2(K_n)$, the slope variety of K_n over $\mathbb{F}_2$. The points of $\mathbb{F}_2^{\binom{n}{2}}$ have their coordinates indexed by the edges of K_n and have value either 0 or 1, which motivates the following notation:

Definition 4. Let $a = (a_{1,2}, a_{1,3}, \ldots, a_{n-1,n}) \in \mathbb{F}_2^{\binom{n}{2}}$. We define the graph G_a to be the graph on $[n]$ with edge set $E(G_a) = \{ij \mid a_{i,j} = 1\}$.

Proposition 5. *Let $W = W(v_0; v_1, \ldots, v_k)$ be a wheel and $a \in \mathbb{F}_2^{\binom{n}{2}}$. Then $\tau_W(a) \neq 0$ if and only if $H_a := G_a \cap W$ is a coupled spanning tree of W.*

Proof. ($\Leftarrow$) Suppose that H_a is a coupled spanning tree of W. When τ_W is written in the form of equation (1), over $\mathbb{F}_2$, it is the sum of all coupled spanning trees of W. Evaluating τ_W at a gives exactly one non-zero term, hence $\tau_W(a) \neq 0$.

($\Rightarrow$) Suppose $\tau_W(a) \neq 0$. First we show that H_a is a spanning tree of W. Over $\mathbb{F}_2$ exactly one of $\tau_1(a)$ or $\tau_2(a)$ has value 1, say $\tau_1(a) = 1$. Each binomial factor of $\tau_1(a)$ must contain exactly one variable with value 1. Therefore, H_a contains exactly k edges, which is the number of edges of a spanning tree of W. In order to show that H_a is a spanning tree it is enough to show that it is acyclic. If H_a contains a cycle C, then either there exist i and j, $1 \leq i < j \leq k$, such that $v_0v_i, v_iv_{i+1}, v_{j-1}v_j, v_0v_j \in E(C)$, or C is the set of chords of W. In the first case, both terms $\tau_1(a)$ and $\tau_2(a)$ have value 0 because $m_{0i} - m_{i(i+1)}$ is a factor of τ_1 and $m_{0j} - m_{(j-1)j}$ is a factor of τ_2. In the second case, formula (2) will be

$$\tau_W(a) = \prod_{i=1}^{k}(a_{0i} - 1) - \prod_{i=1}^{k}(a_{0i} - 1) = 0.$$

Now we show that H_a is in fact a *coupled* spanning tree of W. Define $\overline{a} \in \mathbb{F}_2^{\binom{n}{2}}$ by $\overline{a_{ij}} = 1 - a_{ij}$ for all $1 \leq i < j \leq n$. Therefore, $G_{\overline{a}} = \overline{G_a}$ is the complement of G_a. If $\tau_W(a) \neq 0$ then $\tau_W(\overline{a}) \neq 0$ because each binomial factor of $\tau_i(\overline{a})$ will have the same value

as in $\tau_i(a)$, for $i = 1, 2$. Therefore $H_{\overline{a}} = \overline{H_a} \cap W$ is a spanning tree of W, hence H_a is a coupled spanning tree of W. □

The following proposition gives additional evidence to suggest that I_n and J_n are equal, but whether or not they are equal is still unknown.

Proposition 6. *Let $a \in \mathbb{F}_2^{\binom{n}{2}}$. The following are equivalent:*

1. *a is a zero of I_n;*
2. *a is a zero of J_n;*
3. *G_a is a complement-reducible graph.*

Proof. $(1 \Rightarrow 2)$ This implication follows from the containment $J_n \subseteq I_n$.

$(2 \Rightarrow 3)$ Suppose $a \in \mathbb{F}_2^{\binom{n}{2}}$ is a zero of J_n. By Proposition 5, if $W \subseteq K_n$ is any 3-wheel (and hence isomorphic to K_4), then $G_a \cap W$ is not a coupled spanning tree of W. Since every coupled spanning tree of K_4 is isomorphic to P_4 (the only spanning trees of K_4 are isomorphic to P_4 or the three edge claw), G_a does not contain an induced P_4.

$(3 \Rightarrow 1)$ Let $a \in \mathbb{F}_2^{\binom{n}{2}}$ be such that G_a is a complement-reducible graph. Let $W \subseteq K_n$ be a wheel with $V = V(W)$. Either $G_a|_V$ or $\overline{G_a}|_V$ is disconnected, because G_a is a complement-reducible graph. Therefore either $G_a \cap W$ or $G_{\overline{a}} \cap W$ is disconnected. Since these two graphs are complementary subgraphs of W, neither one is a coupled spanning tree. Therefore, by Proposition 5, $\tau_W(a) = 0$ for every wheel $W \subseteq K_n$. □

4 A bijection between complement-reducible graphs and switching classes

In this section, we establish a bijection (Proposition 7) between the set of graphs on n labeled vertices with an induced P_4, and the switching classes on $n+1$ labeled vertices containing a graph with an induced 5-cycle. Recall from Section 2.3 that each switching class contains exactly one graph in which the vertex $n+1$ is isolated. Therefore the bijection from the set of graphs on $[n]$ to the switching classes on $[n+1]$ is given by sending $G \subseteq K_n$ to $[G]$, the orbit containing G.

Proposition 7. *Let the additive group $\mathbb{Z}_2^n$ act on $\mathscr{G}_{n+1}$ by switching as described in Section 2.3. Then:*

1. *If $G \in \mathscr{G}_{n+1}$ has an induced 5-cycle, then every $H \in [G]$ has an induced 4-path.*
2. *If $G \in \mathscr{G}_n$ has an induced 4-path, then, regarding G as a graph on $[n+1]$ by introducing $n+1$ as an isolated vertex, there is an $H \in \mathscr{G}_{n+1}$ such that $G \in [H]$ and H has an induced 5-cycle.*

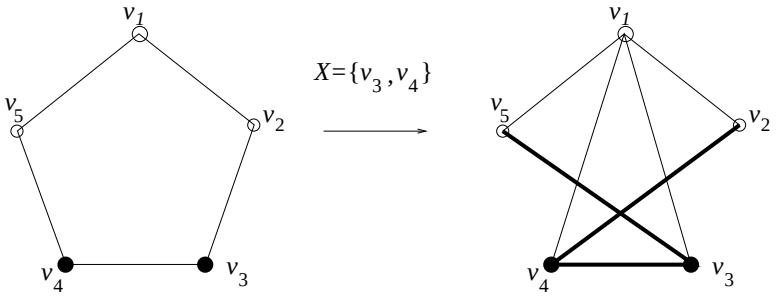

Figure 2: The action with $X = \{v_3, v_4\}$

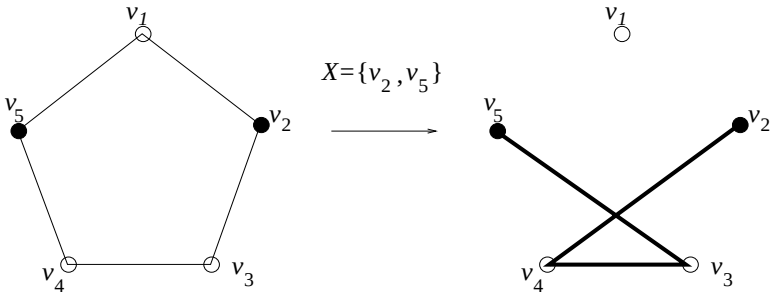

Figure 3: The action with $X = \{v_2, v_5\}$

Proof. (1) Let $G \in \mathscr{G}_{n+1}$ have an induced 5-cycle $C = \{v_1, \ldots, v_5\}$, and let $X \subseteq [n]$. If $|V(C) \cap X| < 2$, then four of the vertices, say $U = \{v_1, v_2, v_3, v_4\}$, are in $[n] \setminus X$. Switching by X does not affect the induced subgraph on U. Similarly, if $|V(C) \cap X| > 3$, then $(s_X(G) \mid_U) \cong P_4$.

Suppose $|V(C) \cap X| = 2$. Without loss of generality we may assume either $X = \{v_2, v_5\}$ or $X = \{v_3, v_4\}$. In both cases $v_5v_3v_4v_2$ is an induced 4-path in $s_X(G)$, as shown in the figure. If $|V(C) \cap X| = 3$, then $|V(C) \cap ([n+1] \setminus X)| = 2$. The same results as above will hold for this case, therefore $s_X(G)$ has an induced P_4.

(2) Suppose $v_5v_3v_4v_2$ is an induced P_4 in $G \subseteq K_n$ and $X = \{v_2, v_5\}$. Then $s_X(G)$ has the induced 5-cycle $C = \{v_1, \ldots, v_5\}$ with $v_1 = n + 1$. □

5 Counting points over other finite fields

It is natural to ask whether these techniques can be extended to enumerate points of the slope variety $S_q(K_n)$ over $\mathbb{F}_q$. This problem appears to be difficult, because the zeroes of a tree polynomial over an arbitrary field do not seem to admit a uniform graph-theoretic description as they do over $\mathbb{F}_2$. In this section, we describe some partial progress in this direction, and explicitly work out the simplest nontrivial case ($n = 4$, $q = 3$) to illustrate the kinds of difficulties involved.

A point in $\mathbb{F}_q^{\binom{n}{2}}$ corresponds to an $\mathbb{F}_q$-*weighted* K_n, that is, a copy of K_n whose edges

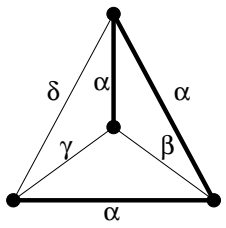

Figure 4: A weight induced P_4

are assigned weights in $\mathbb{F}_q$. For $a \in \mathbb{F}_q^{\binom{n}{2}}$ define G_a to be the $\mathbb{F}_q$-weighted K_n where edge ij is given weight a_{ij}. We say that G_a has a *weight-induced subgraph* H if there is some value $\alpha \in \mathbb{F}_q$ such that

$$E(H) = \{e \in E(K_n) \mid a_e = \alpha\}.$$

One possible approach to generalizing the previous results would be to define a q-analogue to switching. Let the additive group $\mathbb{F}_q^n$ act on $\mathbb{F}_q^{\binom{n}{2}}$ by

$$((x_1, \ldots, x_n) \cdot a))_{ij} = (a_{ij} + x_i + x_j).$$

If $q = 2$, then this is exactly the switching action described in Section 2.3. Note that this is not the same definition of q-switching given by Zaslavsky [10]. The present definition seems more likely to be relevant in the context of slopes because it does not rely on an orientation of the edges. (Recall that the weight of an edge is the slope of the corresponding line segment in a picture of K_n; the slope does not depend on the direction in which way the edge is traversed.) One would hope to generalize the $q = 2$ case by describing the points of $S_q(K_n)$ in terms of forbidden weight-induced subgraphs. It is not clear how to generalize the definition over an arbitrary field, or what the forbidden weight-induced subgraphs should be. However, some facts do carry over to the setting of an arbitrary finite field.

Proposition 8. *Let $W = W(v_0; v_1, v_2, v_3)$ be a 3-wheel, and let $a \in \mathbb{F}_q^{\binom{4}{2}}$ be a point whose coordinates correspond to assigning weights to the edges of W. Then:*

1. *If G_a has a weight-induced P_4, then a is not a zero of τ_W.*
2. *If G_a has a weight-induced claw (that is, a star with three edges), then a is a zero of τ_W.*
3. *If G_a has a weight-induced cycle, then a is a zero of τ_W.*

Proof. (1) Suppose that G_a has a weight-induced P_4. The induced subgraph on the vertices of that P_4 can be drawn as in Figure 4. where $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$, and α does not equal any of the other values. Then,

$$\tau_W(a) = (\alpha - \alpha)(\beta - \alpha)(\gamma - \delta) - (\alpha - \delta)(\beta - \alpha)(\gamma - \alpha) \neq 0.$$

(2) Suppose that G_a has a weight-induced claw, whose edges have the weight $\alpha \in \mathbb{F}_q$. If we draw W so that the center of the claw is the center of the wheel, then

$$\tau_Q(a) = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) - (\alpha-\gamma)(\alpha-\delta)(\alpha-\beta) = 0,$$

for some $\beta, \gamma, \delta \in \mathbb{F}_q$.

(3) Suppose that G_a has a weight-induced cycle C. The graph W can be drawn so that C contains the vertex v_0. Then, for some $1 \le i < j \le 3$, the edges v_0v_i, v_iv_{i+1}, v_0v_j, v_jv_{j-1} all have the same weight α. (Note that if the cycle is a 3-cycle then $v_{j-1} = v_i$.) Then both $\tau_1(a)$ and $\tau_2(a)$ contain the factor $\alpha - \alpha$, so $\tau_W(a) = 0$. □

Corollary 9. *Let $a \in \mathbb{F}_q^{\binom{n}{2}}$. If G_a contains a weight-induced P_4, then a is not a zero of I_n over $\mathbb{F}_q$. Conversely, if every 4-clique of G_a contains a weight-induced cycle or a weight-induced claw, then a is a zero of J_n over $\mathbb{F}_q$.*

Example 10. Let $W = W(v_0; v_1, v_2, v_3)$ be a 3-wheel. We use Proposition 8 to count the number of zeroes of τ_W over $\mathbb{F}_3$.

If some value occurs at least four times in a, then $\tau_W(a) = 0$ because G_a has a weight-induced cycle. If some value α occurs exactly three times in a, then $\tau_W(a) \neq 0$ if and only if the weight-induced graph on α is a 4-path. The cases where each value of a occurs two times are not covered by Proposition 8, so we must consider them separately. For distinct $\alpha, \beta, \gamma \in \mathbb{F}_3$ there are three possibilities, up to a relabeling of the vertices; see Figure 5.

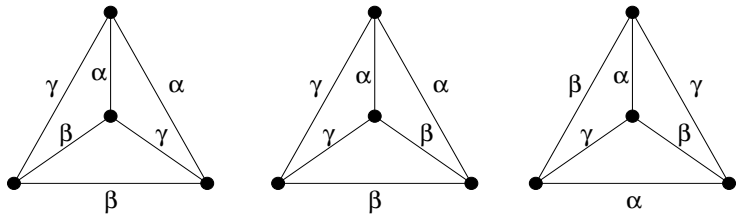

Figure 5: The first weight corresponds to a zero of τ_W, but the second two do not.

Define the *type* of $a \in \mathbb{F}_q^d$ to be the partition whose parts are the numbers of occurrences of each element of $\mathbb{F}_q$ among the entries of a. Some simple counting gives the following table:

Type	Number of zeroes	Number of non-zeroes
(6)	3	0
(5, 1)	36	0
(4, 2)	90	0
(4, 1, 1)	90	0
(3, 3)	24	36
(3, 2, 1)	144	216
(2, 2, 2)	36	54
Total	423	306

If $q > 3$, then there are more cases to check which are not covered by Proposition 8. Using the computer algebra software Maple, one can check that over $\mathbb{F}_3$ the number of zeroes of I_4 and I_5 are 423 and 9243, respectively. Over $\mathbb{F}_5$ the numbers are 4909, 262645, respectively. It is not clear what combinatorial structure (analogous to complement-reducible graphs) might count these points; for instance, these numbers do not appear in the Encyclopedia of Integer Sequences [6].

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