On generalized Dyck paths*

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Abstract

We generalize the elegant bijective proof of the Chung Feller theorem from the paper of Young-Ming Chen [*The Chung-Feller theorem revisited*, Disc. Math. **308** (2008), 1328–1329].

1 Introduction

In [1], the *Chung Feller theorem* has been proved by presenting a bijection between *n-Dyck paths* with j *flaws* and *n-Dyck paths* with $j+1$ *flaws* for $j = 0, 1, \ldots, n-1$. The *Chung Feller theorem* states that the number of *n-Dyck paths* with j *flaws* is independent of j and is equal to the *Catalan number* C_n. The bijection consists in switching selected parts of a *Dyck path* in such a way the number of *flaws* increases by one. The author showed how to select the parts to be switched and proved that it is a bijection.

In this paper we present a generalized version of the proof for *Dyck paths* with additional requirements concerning the length and the number of *horizontal steps*. [2] contains a result that covers the main result here, using analytic method. The merit of the current paper is that it offers a simple and elegant bijective proof.

2 Bijection of Dyck paths

We consider a *Dyck path* p as a sequence of n *vertical steps* of the length 1 (meaning one *edge* of a *grid*) and $k \leq n$ *horizontal steps* of the lengths $(l_1, l_2, \ldots, l_k)$ in a grid of $n \times n$ squares such that $l_1 + l_2 + \cdots + l_k = n$. The number of *flaws* is considered as the number of *vertical steps* above the diagonal. Formally, we may define a *Dyck path* as a sequence of positive integers:

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Definition Let $h_n = \{(t_1, l_1), (t_2, l_2), \ldots, (t_m, l_m)\}$ be a set of pairs of ordered positive integers such that $t_1 l_1 + t_2 l_2 + \cdots + t_m l_m = n$ and $l_i \neq l_j$ for $i \neq j$. We define a h_n-*Dyck path* as a sequence c of the elements l_i and the element 0 (representing a *vertical step*), satisfying that every element l_i appears exactly t_i times and the element 0 appears exactly n times in the sequence c.
Let $k = t_1 + t_2 + \cdots + t_m$, then the length of the sequence c is $n + k$.

In the following we write simply a *Dyck path* instead of a h_n-*Dyck path*.

Remark The elements l_i represent the length of a *horizontal step*, whereas the elements t_i represent the number of such *steps*. The order of elements of the sequence corresponds to the order of *steps*.

Remark For $h_n = \{(n, 1)\}$ we get a "customary" *Dyck path*, where all lengths of *horizontal steps* are equal to 1.

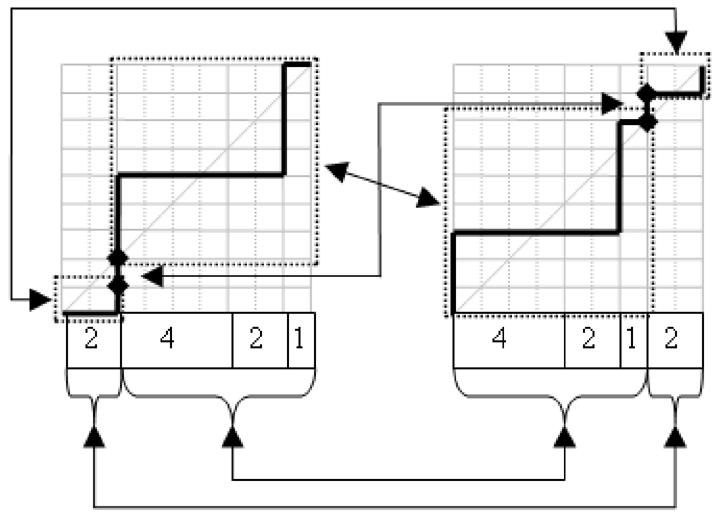
Remark For the set h_n there are $\binom{k}{t_1, t_2, \ldots, t_m}\binom{n+k}{k} = \binom{n+k}{n, t_1, t_2, \ldots, t_m}$ different *Dyck paths*. Just consider that in a sequence of the length $n+k$ we choose k elements to be nonzero (*horizontal steps*) and in these k elements there are $\binom{k}{t_1, t_2, \ldots, t_m}$ permutations.

Next we explain a way how to graphically express the *Dyck paths*. Given the set h_n let us have $\binom{k}{t_1, t_2, \ldots, t_m}$ grids of $n \times n$ squares. Every such grid is assigned to one permutation of *horizontal steps* and for a given grid the *vertical steps* are allowed only in selected vertical lines of the grid accordingly to the given permutation.
Now we can step to the main theorem of the paper:

Theorem 2.1 *Given the set of* Dyck paths *defined by the set* $h_n = \{(t_1, l_1), (t_2, l_2), \ldots, (t_m, l_m)\}$. *There are* $\frac{1}{n+1}\binom{n+k}{n, t_1, t_2, \ldots, t_m}$ subdiagonal Dyck paths *(with* 0 flaws*)*.

Proof Let M_e denote the set of *Dyck paths* with e *flaws*, where $e \in \{0, 1, \ldots, n\}$. We claim that there is a bijection between M_j and M_{j+1}, where $j \in \{0, 1, \ldots, n-1\}$. Let $p = c_1 c_2 \ldots c_{n+k} \in M_j$. The order of nonzero elements of p determines the grid assigned to p. Let c_i be a zero element corresponding to the first *vertical step* below the diagonal that touches the diagonal. Such c_i must exist since we claimed the number of *flaws* $j < n$. Then $q = c_{i+1} c_{i+2} \ldots c_{n+k} c_i c_1 c_2 \ldots c_{i-1} \in M_{j+1}$ and the order of nonzero elements in the sequence q determines the grid assigned to q. The number of *flaws* increased by 1 because the subsequence $c_{i+1} c_{i+2} \ldots c_{n+k}$ is just moved at the beginning, and hence c_i is the only *vertical step* moved above the diagonal.

The figure below shows the bijection for $p = 2000004200001$ and $q = 0004200001020$:



Given p then q is uniquely determined, and also the inverse function exists for $q = c_{i+1}c_{i+2}\dots c_{n+k}c_i c_1 c_2 \dots c_{i-1} \in M_{j+1}$, which yields p: find the last *vertical step* that is above the diagonal and that touches the diagonal. The subsequence $c_1c_2\dots c_{i-1}$ in q contains no such *vertical step*, because we required that c_i in p is the first *vertical step* below the diagonal that touches the diagonal, hence the last *vertical step* in q above the diagonal that touches the diagonal is c_i. Then $p = c_1c_2\dots c_{n+k} \in M_j$.
The bijection between M_j and M_{j+1} proves that the number of *Dyck paths* with j *flaws* does not depend on $j \in \{0, 1, \dots n\}$, so we conclude that there are $\frac{1}{n+1}\binom{n+k}{n, t_1, t_2, \dots, t_m}$ subdiagonal *Dyck paths*.

References

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