# The disjoint $m$-flower intersection problem for latin squares

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**Abstract**

An $m$-flower in a latin square is a set of $m$ entries which share either a common row, a common column, or a common symbol, but which are otherwise distinct. Two $m$-flowers are disjoint if they share no common row, column or entry. In this paper we give a solution of the intersection problem for disjoint $m$-flowers in latin squares; that is, we determine precisely for which triples $(n, m, x)$ there exists a pair of latin squares of order $n$ whose intersection consists exactly of $x$ disjoint $m$-flowers.

## 1 Introduction

Intersection problems for latin squares were first considered by Fu [10]. Since then the area has been extensively investigated, see [6] for a survey of results up until 1990. Subsequent results can be found in [7], [8], [1], [3] and [9].

Intersection problems between pairs of Steiner triple systems were first considered by Lindner and Rosa [12]. Subsequently, the intersection problem, between pairs of Steiner triple systems, $(V, \mathcal{V}_1)$ and $(V, \mathcal{V}_2)$, in which the intersection of $\mathcal{V}_1$ and $\mathcal{V}_2$ is composed of a number of isomorphic copies of some specified partial triple system have also been considered. Mullin, Poplove and Zhu [15] considered the case where the partial triple system in question was a *triangle*. Furthermore, Lindner and Hoffman [11] considered pairs of Steiner triple systems of order $v$ intersecting in a $(\frac{v-1}{2})$-flower and some other

(possibly empty) set of triples; Chang and Lo Faro [4] considered the same problem for Kirkman triple systems.

In [5], Chee investigated the intersection problem for Steiner triple systems in which the intersection was composed of pairwise disjoint triples. An independent and elegant solution to this problem was given by Srinivasan [16]. This result can be considered as pairs of Steiner triple systems whose intersection is composed precisely of disjoint 1-flowers.

A natural progression of the above problems is the intersection problem for pairs of latin squares or Steiner triple systems in which the intersection is composed of a number of disjoint configurations.

In this paper the intersection problem for disjoint $m$-flowers in latin squares is solved. The solution to the corresponding problem for 2-flowers in Steiner triple systems can be found in [14].

Examples labelled A.x for some integer x refer to the appendix, which is available as a separate document from

http://www.combinatorics.org/Volume_18/Abstracts/v18i1p42.html

# 2 Preliminaries

Let $N = \{i \mid 0 \leq i \leq n-1\} \subset \mathbb{N} \cup \{0\}$. Let $N^2$ and $N^3$ denote, respectively, the Cartesian products $N \times N$ and $N \times N \times N$. Let $P \subset N^3$ such that for any pair $n_1, n_2 \in N$, $P$ contains at most one triple of the form $(n_1, n_2, n_3)$, at most one triple of the form $(n_1, n_3, n_2)$ ($P$ is *row latin*), and at most one triple of the form $(n_3, n_1, n_2)$ ($P$ is *column latin*), for some $n_3 \in N$. Then the set $P$ is a *partial latin square*. The number of triples contained in $P$ is known as the *size* of $P$.

For ease of understanding the ordered triple $(n_1, n_2, n_3)$ may be regarded as referring to the occurrence of *symbol* $n_3$ in *cell* $(n_1, n_2)$ of an $n \times n$ array; this cell occurs in *row* $n_1$ and *column* $n_2$. If a cell contains no symbol it is called *empty*. Conversely, if a cell contains a symbol it is said to be *filled*. For a partial latin square $P$, its *shape* is the set of filled cells of $P$. If in a partial latin square, $P$, there are no empty cells then $P$ is called a *latin square* of order $n$.

Let $L$ be a latin square of order $n$; the set of cells $\{(i, i) \mid 0 \leq i \leq n-1\}$ is denoted as the *main diagonal* of $L$.

A pair of partial latin squares, $(P_1, P_2)$, is called a *latin biswap* if the pair satisfies the following: $P_1$ and $P_2$ have the same shape; and the corresponding rows (columns) of $P_1$ and $P_2$ contain the same entries.

Note that if $P_1$ is contained in a latin square $L_1$ then $P_2$ is contained in the latin square $L_2 = (L_1 \setminus P_1) \cup P_2$; moreover, $L_1 \setminus P_1 = L_2 \setminus P_2$.

A latin biswap, $(P_1, P_2)$, is called a *latin bitrade* if it satisfies the additional property that $P_1 \cap P_2 = \emptyset$.

A *transversal*, $T$, in a latin square $L$ of order $n$, (that is $T \subset L$) is a partial latin square which contains $n$ triples such that each element in $N$ occurs precisely once in a coordinate of a triple in $T$.

Let $L$ be a latin square of order $n$ that contains two transversals $S$ and $T$. If the shape of $S$ has no cells in common with the shape of $T$, then $S$ and $T$ are said to be *disjoint*.

In the following a *configuration*, $P$, is an isomorphic copy of some specified partial latin square.

In a latin square, $L$, an *m-flower* is a configuration containing $m$ triples and which is of the form

$F = \{(x_i, y_i, z_i) \mid 1 \leq i \leq m\} \subseteq L$ such that precisely one of the following holds :

for all distinct $i$ and $j$ : $x_i = x_j$, $y_i \neq y_j$ and $z_i \neq z_j$ (a *row-m-flower*); or

for all distinct $i$ and $j$ : $x_i \neq x_j$, $y_i = y_j$ and $z_i \neq z_j$ (a *column-m-flower*); or

for all distinct $i$ and $j$ : $x_i \neq x_j$, $y_i \neq y_j$ and $z_i = z_j$ (a *symbol-m-flower*).

If the intersection between the row coordinates, the intersection between the column coordinates and the intersection between the symbol coordinates of two $m$-flowers are all empty then the two $m$-flowers are said to be pairwise disjoint. If a set of $k$ $m$-flowers satisfy the property that any two are pairwise disjoint then it is said to be a set of $k$ disjoint $m$-flowers.

Consider a set of $k$ disjoint $m$-flowers. A triple in which the row, column and entry coordinates are not equal to, respectively, the row, column or entry of any of the triples in the $k$ disjoint $m$-flowers, is said to be a *disjoint triple* (to the $m$-flowers).

# 3 Constructions for latin squares

For each map $f$ defined in this paper, the image of all the triples in a subset $P \subset N^3$ under $f$ will be denoted by $fP$.

Throughout this paper the well known technique of *prolongation* is extensively used. This section begins by briefly discussing this technique.

Consider a latin square, $L$, of order $n$, and assume that $L$ contains a transversal $T$; then construct a latin square, $L(+1)$, of order $n+1$, as follows: $L(+1) =$

$$(L \setminus T) \cup \{(x, y, n), (n, y, z), (x, n, z) \mid (x, y, z) \in T\} \cup \{(n, n, n)\}.$$

If the latin square $L$ contains $k$ disjoint transversals, $T_i$, where $1 \leq i \leq k$, this idea can be generalised to a $k$-prolongation, yielding a latin square $L(+k)$ of order $n+k$.

Let $\zeta_r$ and $\zeta_c$ be elements of the symmetric group, $S_k$, acting on the set $\{i \mid 1 \leq i \leq k\}$.

Let $1 \leq k, n$ and $\mathcal{P}$ be a partial latin square of order $n+k$ in which the cells in the set $\{(i, j) \mid n \leq i, j \leq n+k-1\}$ are filled with symbols from the set $\{i \mid n \leq i \leq n+k-1\}$ and all other cells are empty; such a partial latin square is denoted as a *completing square* (note that such a partial latin square exists for all orders as it corresponds to a latin square of order $k$ in which each triple $(a, b, c)$ is replaced with $(a+n, b+n, c+n)$).

Let $n', n \in \mathbb{N}$ such that $n' \leq n$, $N' = \{i \mid 0 \leq i \leq n'-1\}$ and $N = \{i \mid 0 \leq i \leq n\}$. Define the following maps from $N'^3$ to $N^3$.

$\gamma_r^n : (u, v, w) \mapsto (n, v, w);$ $\gamma_c^n : (u, v, w) \mapsto (u, n, w);$ and $\gamma_s^n : (u, v, w) \mapsto (u, v, n).$

Then $L(+k) =$

$$\left(L \setminus \bigcup_{1 \leq i \leq k} T_i\right) \cup \bigcup_{1 \leq i \leq k} \left(\gamma_r^{\zeta_r(i)+n-1} T_i \cup \gamma_c^{\zeta_c(i)+n-1} T_i \cup \gamma_s^{i+n-1} T_i\right) \cup \mathcal{P}$$

is a latin square of order $n + k$. See Example A.1. This latin square is referred to as a $(\zeta_r, \zeta_c)$-$k$-*prolongation* of $L$. If $\zeta_r = \zeta_c = id$, the identity permutation, then $L(+k)$ is simply referred to as a $k$-*prolongation* of $L$.

Let $j \in \mathbb{N}$ and $J = \{i \mid 0 \leq i \leq j - 1\}$. Define the following maps from $J^3$ to $J^3$ for $0 \leq i, l \leq j - 1$.

$$id : (u, v, w) \mapsto (u, v, w);$$

$$\sigma_r^j : (u, v, w) \mapsto ((u + 1 \,(\text{mod } j)), v, w);$$

$$\sigma_c^j : (u, v, w) \mapsto (u, (v + 1 \,(\text{mod } j)), w);$$

$$\sigma_s^j : (u, v, w) \mapsto (u, v, (w + 1 \,(\text{mod } j)));$$

$$\sigma_s^{i,j} : (u, v, w) \mapsto (u, v, (w - i + 1 \,(\text{mod } j - i)) + i);$$

$$\upsilon_s^j : (u, v, w) \mapsto \begin{cases} (u, v, (w + 1 \,(\text{mod } j - 1))), & 0 \leq w \leq j - 2 \\ (u, v, w), & w = j - 1 \end{cases};$$

$$\epsilon_s^j : (u, v, w) \mapsto \begin{cases} (u, v, (w + 1 \,(\text{mod } j - 3))), & 0 \leq w \leq j - 4 \\ (u, v, w), & w = j - 2 \\ (u, v, j - 3), & w = j - 1 \\ (u, v, j - 1), & w = j - 3 \end{cases};$$

$$\phi_s^j : (u, v, w) \mapsto (u, v, w - 2 \,(\text{mod } j));$$

$$\delta_i^j : (u, v, w) \mapsto (i + u \,(\text{mod } j), i + w \,(\text{mod } j), i + v \,(\text{mod } j));$$

$$\rho_i^l : (u, v, w) \mapsto \begin{cases} (u, i, w) & \text{if } v = l \\ (u, l, w) & \text{if } v = i \\ (u, v, w) & \text{otherwise} \end{cases} ; \text{ and}$$

$$\psi_i^j : (u, v, w) \mapsto \begin{cases} (u, v, w + 1 \,(\text{mod } i)), & \text{if } 2 \leq w \leq i - 1 \\ (u, v, 2), & \text{if } w = 0 \\ (u, v, w), & \text{otherwise} \end{cases} .$$

Let $A$ be a partial latin square of order $a$ and let $B$ be a partial latin square of order $b$. Let $f_\alpha \in \{id, \sigma_r^b, \sigma_c^b, \sigma_s^b, \phi_s^b\}$ where $\alpha = (u, v, w) \in A$. Define the product of the singleton $\{\alpha\}$ and $B$ as follows; $\{\alpha\} \times f_\alpha B = \{(ub + x, vb + y, wb + z) \mid (x, y, z) \in f_\alpha B\}$. Now define the direct product of $A$ and $B$ as follows; $A \times f_\alpha B = \bigcup_{\alpha \in A}(\{\alpha\} \times f_\alpha B)$. If for all $\alpha \in A$, $f_\alpha = id$, simply write $A \times B$.

Throughout this paper use will be made of the following technical lemma.

**Lemma 3.1.** *Let $P$ and $P'$ be two partial latin squares, both of order $p$; let $\alpha = (u, v, w) \in P$ and $\beta = (u', v', w') \in P'$. Let $\Sigma \in \{\sigma, \phi\}$; let $j \in \{r, c, s\}$, with $j = s$ if $\Sigma = \phi$. Let $T$ be some transversal of order $t > 2$. Finally, let $k \in \mathbb{N}$ such that $k \geq pt$.*
*Then*

$$(\gamma_r^k(\{\alpha\} \times T)) \cap (\gamma_r^k(\{\beta\} \times \Sigma_j^t T)) = \begin{cases} \gamma_r^k(\{\alpha\} \times T), & j = r, \ v = v' \ and \ w = w' \\ \emptyset, & otherwise \end{cases} ;$$

$$(\gamma_c^k(\{\alpha\} \times T)) \cap (\gamma_c^k(\{\beta\} \times \Sigma_j^t T)) = \begin{cases} \gamma_c^k(\{\alpha\} \times T), & j = c, \ u = u' \ and \ w = w' \\ \emptyset, & otherwise \end{cases} ;$$

$$(\gamma_s^k(\{\alpha\} \times T)) \cap (\gamma_s^k(\{\beta\} \times \Sigma_j^t T)) = \begin{cases} \gamma_s^k(\{\alpha\} \times T), & j = s, \ u = u' \ and \ v = v' \\ \emptyset, & otherwise \end{cases} .$$

*Proof.* The third statement will be proved for $\Sigma = \sigma$, the other cases follow similarly.

First let $j = s$.

Then $\quad \{\beta\} \times \sigma_s^t T \quad = \quad \{(u't + x, v't + y, w't + (z + 1 \,(\mathrm{mod}\ \mathrm{t}))) \mid (x, y, z) \in T\},$

hence, $\quad \gamma_s^k(\{\beta\} \times \sigma_s^t T) \quad = \quad \{(u't + x, v't + y, k) \mid (x, y, z) \in T\}.$

Also, $\quad \{\alpha\} \times T \quad = \quad \{(ut + x, vt + y, wt + z) \mid (x, y, z) \in T\},$

so, $\quad \gamma_s^k(\{\alpha\} \times T) \quad = \quad \{(ut + x, vt + y, k) \mid (x, y, z) \in T\}.$

Thus,

$$\gamma_s^k(\{\alpha\} \times T) = \gamma_s^k(\{\beta\} \times \sigma_s^t T)$$

if and only if $u = u'$ and $v = v'$; otherwise the intersection of these two sets is empty.

Now consider the case where $j = c$.

Then $\quad \{\beta\} \times \sigma_c^t T \quad = \quad \{(u't + x, v't + (y + 1 \,(\mathrm{mod}\ \mathrm{t})), w't + z) \mid (x, y, z) \in T\},$

hence, $\quad \gamma_s^k(\{\beta\} \times \sigma_c^t T) \quad = \quad \{(u't + x, v't + (y + 1 \,(\mathrm{mod}\ \mathrm{t})), k) \mid (x, y, z) \in T\}.$

Thus,

$$\gamma_s^k(\{\alpha\} \times T) \cap \gamma_s^k(\{\beta\} \times \sigma_c^t T) = \emptyset$$

regardless of whether or not $u = u'$ or $v = v'$.

The subcase where $j = r$ follows similarly. $\qquad\square$

More often than not, when Lemma 3.1 is applied, $P = P'$ and $\alpha = \beta$.

The following is a well known result [2].

**Lemma 3.2.** *(Bose, Shrikhande & Parker, [2]) For all $3 \leq n$, $n \neq 6$ there exists a latin square which is composed of $n$ disjoint transversals. For $n = 6$ there exists a latin square that contains 4 disjoint transversals.*

Extensive use will be made of the following result.

**Lemma 3.3.** *Let $A$ be a partial latin square of order $a$ that contains a transversal $U$. Let $B$ be a partial latin square of order $b$ that contains a transversal, $T$. Let $f_\alpha \in \{id, \sigma_r^b, \sigma_c^b, \sigma_s^b, \phi_s^b\}$, where $\alpha \in A$. Then $U \times f_\alpha T$ is a transversal in $A \times f_\alpha B$.*

*Proof.* As the rows, columns or symbols can be reordered, without loss of generality $U = \{(j,j,j) \mid 0 \leq j \leq a-1\}$.

Consider $U \times f_\alpha T$, note that $|U \times f_\alpha T| = ab$.

Now, $U \times f_\alpha T = \bigcup_{0 \leq j \leq a-1} \{(jb+x, jb+y, jb+z) \mid (x,y,z) \in f_{(j,j,j)}T\}$.

Note that $f_{(j,j,j)}T$ is a transversal in $f_{(j,j,j)}B$. Hence,

$$
\bigcup_{0 \leq j \leq a-1} \{(jb+x) \mid (x,y,z) \in f_{(j,j,j)}T\} = \bigcup_{0 \leq j \leq a-1} \{(jb+y) \mid (x,y,z) \in f_{(j,j,j)}T\}
$$
$$
= \bigcup_{0 \leq j \leq a-1} \{(jb+z) \mid (x,y,z) \in f_{(j,j,j)}T\}
$$
$$
= \{jb+h \mid 0 \leq h \leq b-1, \ 0 \leq j \leq a-1\}
$$
$$
= \{i \mid 0 \leq i \leq ab-1\}.
$$

Therefore, $U \times f_\alpha T$ is a transversal in $A \times f_\alpha B$. $\qquad\square$

# 4  Solving the intersection problem

The previous two sections have detailed the notation and constructions which will be used to provide a solution to the intersection problem for disjoint $m$-flowers in latin squares. This result is presented in Theorem 1, at the end of this section.

The necessary and sufficient conditions for the proof of Theorem 1 are covered in the following pages. To aid the reader two tables are provided; Table 1 indicates the lemmas that establish necessary conditions whilst Table 2 indicates the lemmas that establish sufficient conditions.

For ease of notation throughout this paper any set of the form $\{i \mid p \leq i \leq p-1\}$ is taken to be the empty set.

Table 1: Necessary Conditions for Theorem 1

| Condition | Lemmas |
|---|---|
| Maximum number of $m$-flowers | 4.1 |
| General exceptions | 4.2, 4.4 |
| Exceptions for pairs of latin squares of small order | 4.5, 4.13 |

**Lemma 4.1.** *Let $l \in \mathbb{N} \cup \{0\}$, $L$ be a latin square of order $n$ and $m \leq n$ then if:*

*$l(2m+1) \leq n < l(2m+1) + m$; $L$ contains a maximum $3l$ disjoint $m$-flowers;*

*$l(2m+1) + m \leq n < l(2m+1) + 2m$; $L$ contains a maximum $3l+1$ disjoint $m$-flowers;*

*$n = l(2m+1) + 2m$; $L$ contains a maximum $3l+2$ disjoint $m$-flowers.*

Table 2: Sufficient Conditions for Theorem 1

| Condition | Lemmas |
|---|---|
| 2-flowers in pairs of latin squares of small order | 4.3 |
| One $m$-flower | 4.11 |
| Two disjoint $m$-flowers | 4.19 |
| Three disjoint $m$-flowers in pairs of latin squares of order $2m+1 \leq n \leq 3m$ | 4.20, 4.23 |
| Three or four disjoint $m(\geq 3)$-flowers in pairs of latin squares of order $3m+1 \leq n \leq 4m$ | 4.24, 4.25 |
| Three, four or five disjoint $m(\geq 3)$-flowers in pairs of latin squares of order $n = 4m+1$ | 4.26, 4.27 |
| Three, four, five or six disjoint $m(\geq 3)$-flowers in pairs of latin squares of order $4m+2 \leq n \leq 6m+2$ | 4.28, 4.29, 4.32, 4.33 |
| Seven disjoint $m(\geq 3)$-flowers in pairs of latin squares of order $5m+2 \leq n \leq 6m+2$ | 4.30 |
| Eight disjoint $m(\geq 3)$-flowers in pairs of latin squares of order $6m+2$ | 4.31 |
| $0 \leq h \leq 3\lfloor \frac{n}{2m+1} \rfloor$ disjoint $m$-flowers in pairs of latin squares of order $n \geq 6m+3$ | 4.34, 4.35 |
| $l \geq 3$; $3l+1$ disjoint $m$-flowers in pairs of latin squares of order $l(2m+1)+m \leq n \leq l(2m+1)+2m$ | 4.36, 4.37, 4.38 |
| $l \geq 3$; $3l+2$ disjoint $m$-flowers in pairs of latin squares of order $n = l(2m+1)+2m$ | 4.39, 4.40 |

*Proof.* Assume that there are $k = k_1 + k_2 + k_3$ disjoint $m$-flowers in a latin square, $L$, of order $n$, such that there are $k_1$ row-$m$-flowers; $k_2$ column-$m$-flowers; and $k_3$ symbol-$m$-flowers.

Thus, in $L$ the $k$ disjoint $m$-flowers contain, $k_1 + mk_2 + mk_3$ distinct rows, $mk_1 + k_2 + mk_3$ distinct columns, and $mk_1 + mk_2 + k_3$ distinct symbols.

Hence, $n \geq k_1 + m(k_2 + k_3) = k_1 + m(k - k_1)$. Without loss of generality, let $k_1 \leq k_2 \leq k_3$; this implies that $k_1 \leq \lfloor \frac{k}{3} \rfloor$. Thus, $n \geq \lfloor \frac{k}{3} \rfloor + m\left(k - \lfloor \frac{k}{3} \rfloor\right)$, and the result follows. $\square$

**Lemma 4.2.** *There does not exist a pair of latin squares of order $2m+1$ whose intersection is precisely three disjoint $m$-flowers.*

*Proof.* In order for a latin square of order $2m + 1$ to contain three disjoint $m$-flowers, one $m$-flower is required to be a symbol-$m$-flower, one to be a row-$m$-flower and one to be a column-$m$-flower.

Consider a latin square $L$ of order $2m + 1$ that contains a row-$m$-flower in row $i$ that is disjoint to a column-$m$-flower in column $j$. Both of these $m$-flowers contain $m$ symbols and all these $2m$ symbols must be distinct. Hence, there is only one choice for the symbol

contained in cell $(i, j)$. $\quad\square$

In [13] latin squares of small orders are provided that establish the following result.

**Lemma 4.3.** *(McCourt, [13]) There exist pairs of latin squares of order $n$ that intersect in $x$ 2-flowers, where:*

> *$n = 5$ and $x = 2$; or $n = 6$ and $2 \leq x \leq 3$; or $n = 7$ and $2 \leq x \leq 4$; or*
> *$n = 8$ and $2 \leq x \leq 4$; or $n = 9$ and $2 \leq x \leq 5$; or $n = 10$ and $2 \leq x \leq 6$; or*
> *$n = 11$ and $2 \leq x \leq 6$; or $n = 12$ and $2 \leq x \leq 7$; or $n = 13$ and $2 \leq x \leq 7$; or*
> *$n = 14$ and $2 \leq x \leq 8$.*

## 4.1 One $m$-flower

In this section pairs of latin squares of order $n$ that intersect precisely in one $m$-flower, where $m \leq n$, and no other triples will be constructed. Without loss of generality, the $m$-flower can be considered to be a symbol-$m$-flower. By permuting the rows, columns or symbols the symbol-$m$-flower can be placed along $m$ cells of the main diagonal, and the common symbol may be chosen to be zero.

**Lemma 4.4.** *No pair of latin squares of order $n$ can intersect in an $(n-1)$-flower.*

*Proof.* Consider a partial latin square, $P$, of order $n$ that contains the triples in the set $\{(i, i, 0) \mid 0 \leq i \leq n-2\}$ and the triple $(n-1, n-1, x)$, where $x \neq 0$. For $P$ to be completed the symbol 0 must occur once more in the latin square, however there is no cell in which it can be placed without invalidating the row or column latin property. $\quad\square$

Thus, the set of possible values of $m$ such that there exists a pair of latin squares of order $n$ that intersect precisely in one $m$-flower is the set $JS(n) = \{0, 1, 2, \ldots, n-2, n\}$. The set of achievable values of $m$ such that there exists a pair of latin squares of order $n$ that intersect precisely in one $m$-flower will be denoted by $IS(n)$.

Let $L$ be a latin square of order $n$. Then $\sigma_r^n L$ is a latin square of order $n$ such that $L$ and $\sigma_r^n L$ have no triples in common. Hence, $0 \in IS(n)$. Also, Fu in [10] showed that for all $n \geq 4$ there exists a pair of latin squares that intersect precisely in one triple. Furthermore, in [10] Fu showed that two latin squares of order three can not intersect precisely in one triple. Hence, the following result has been established.

**Lemma 4.5.** *(Fu, [10]) For all $4 \leq n$, $0, 1 \in IS(n)$. Furthermore $1 \notin IS(3)$.*

Now, pairs of latin squares of order $n$ to establish the contents of $IS(n)$ will be constructed. The construction used is recursive and [13] provides the necessary "ingredient" latin squares, of orders less than 8, required for the recursion to take effect.

By inspection no pair of latin squares of order two intersect in precisely one 2-flower. This coupled with Lemma 4.5 and the intersections between latin squares given in [13] yields the following result.

**Lemma 4.6.** *For pairs of latin squares of order $i$, where $2 \leq i \leq 7$;*

$IS(2) = \{0\}$; $IS(3) = \{0,3\}$; $IS(4) = JS(4)$; $IS(5) = JS(5)$; $IS(6) = JS(6)$; and
$IS(7) = JS(7)$.

The construction used for latin squares of order greater than or equal to eight splits
into two cases.

### 4.1.1 Case A: $n = 2k$

For pairs of latin squares of order $n = 2k \geq 8$ a simple doubling construction is used.

Consider the latin square $A = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$ of order two. Let
$m_1, m_2 \in \{0,1,2,\ldots,k-2,k\}$. Now, assume there exists a pair of latin squares, $\{U_1, U_2\}$,
of order $k$ whose intersection is the set of triples $\{(i,i,0) \mid 0 \leq i \leq m_1 - 1\}$ (a symbol-$m_1$-
flower). Similarly, assume there exists a pair of latin squares, $\{V_1, V_2\}$, of order $k$ whose
intersection is the set of triples $\{(i,i,0) \mid 0 \leq i \leq m_2 - 1\}$ (a symbol-$m_2$-flower).

A pair of latin squares, $\{L_1, L_2\}$, of order $2k = n$ that intersect precisely in one
$(m_1 + m_2)$-flower is now constructed. First, let

$$L_1 = \{(0,0,0)\} \times U_1 \ \cup \ \{(0,1,1)\} \times U_1 \ \cup \ \{(1,0,1)\} \times U_1 \ \cup \ \{(1,1,0)\} \times V_1.$$

Now, let

$$L_2 = \{(0,0,0)\} \times U_2 \ \cup \ \{(0,1,1)\} \times \sigma_s^k U_1 \ \cup \ \{(1,0,1)\} \times \sigma_s^k U_1 \ \cup \ \{(1,1,0)\} \times V_2.$$

The intersection of $L_1$ and $L_2$ is the set of triples $\{(i,i,0),(k+j,k+j,0) \mid 0 \leq i \leq
m_1 - 1, 0 \leq j \leq m_2 - 1\}$, an $(m_1 + m_2)$-flower. Note that the rows or columns of $L_1$
and $L_2$ can be permuted so that the intersection of $L_1$ and $L_2$ (the $(m_1 + m_2)$-flower) is
composed of the set of triples $\{(i,i,0) \mid 0 \leq i \leq m_1 + m_2 - 1\}$. Thus, the following result
has been established.

**Lemma 4.7.** *Assume that $4 \leq k$ and $IS(k) = JS(k)$. Let $n = 2k$, then $IS(n) = JS(n)$.*

### 4.1.2 Case B: $n = 2k + 1$

**One $m$-flower:** $m \in \{i \mid 0 \leq i \leq k-2\} \cup \{k\}$

A pair of latin squares, $\{L_1, L_2\}$, of order $2k + 1 \geq 9$ that intersect precisely in one
$m$-flower, where $m \in \{i \mid 0 \leq i \leq k-2\} \cup \{k\}$ will now be constructed.

Let $m \in \{i \mid 0 \leq i \leq k-2\} \cup \{k\}$. Also, assume there exists a pair of latin
squares, $\{U_1, U_2\}$, of order $k \geq 4$ whose intersection is composed precisely of the triples
$\{(i,i,0) \mid 0 \leq i \leq m-1\}$ (a symbol-$m$-flower).

Consider the following partial latin squares.

$$A(2k+1) = \{(i,j,(i+j \,(\text{mod } k+1)) + k),(j,i,(i+j \,(\text{mod } k+1)) + k),$$
$$(j,j,(j-1 \,(\text{mod } k+1)) + k) \mid 0 \leq i \leq k-1, \, k \leq j \leq 2k\}.$$

$$B(2k+1) = \{(i,j,i-j \,(\text{mod } k)) \mid k \leq i < j \leq 2k\} \cup \{(i,j,i-j-1 \,(\text{mod } k)) \mid k \leq j <$$
$$i \leq 2k\}.$$

**Example 4.1.**

$A(9):$

| | | | | 8 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| | | | | 4 | 5 | 6 | 7 | 8 |
| | | | | 5 | 6 | 7 | 8 | 4 |
| | | | | 6 | 7 | 8 | 4 | 5 |
| 8 | 4 | 5 | 6 | 7 | | | | |
| 4 | 5 | 6 | 7 | | 8 | | | |
| 5 | 6 | 7 | 8 | | | 4 | | |
| 6 | 7 | 8 | 4 | | | | 5 | |
| 7 | 8 | 4 | 5 | | | | | 6 |

$B(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | 3 | 2 | 1 | 0 |
| | | | | 0 | | 3 | 2 | 1 |
| | | | | 1 | 0 | | 3 | 2 |
| | | | | 2 | 1 | 0 | | 3 |
| | | | | 3 | 2 | 1 | 0 | |

Construct the latin square $L_1 = U_1 \cup A(2k+1) \cup B(2k+1)$ of order $2k+1$. Now, construct the latin square $L_2 = U_2 \cup \sigma_s^{k,2k+1} A(2k+1) \cup \sigma_s^k B(2k+1)$ of order $2k+1$.

The intersection of $L_1$ and $L_2$ is the set of triples $\{(i,i,0) \mid 0 \le i \le m\}$, an $m$-flower. Hence, the following result has been established.

**Lemma 4.8.** *Assume that $4 \le k$ and $IS(k) = JS(k)$. Let $n = 2k+1$, then $\{i \mid 0 \le i \le k-2\} \cup \{k\} \subseteq IS(n)$.*

**One $(k-1)$-flower**

Now, a pair of latin squares, $\{L_1, L_3\}$, of order $2k+1 \ge 9$ that intersect precisely in one $(k-1)$-flower will be constructed. Again, assume there exists a pair of latin squares of order $k$ whose intersection is precisely one $k$-flower. Using the above construction, a pair of latin squares, $\{L_1, L_2\}$, is constructed, that intersect precisely in one $k$-flower. Now construct the latin square $L_3 = \rho_{k-2}^{2k} L_2$ (the mapping $\rho_{k-2}^{2k}$ simply swaps column $k-2$ with column $2k$ in $L_2$).

This yields a pair of latin squares, $\{L_1, L_3\}$, of order $2k+1$, whose intersection is precisely one $(k-1)$-flower. Hence, the following result has been established.

**Lemma 4.9.** *Assume that $4 \le k$ and $k \in IS(k)$. Let $n = 2k+1$, then $k-1 \in IS(n)$.*

**One $m$-flower:** $m \in \{i \mid k+1 \le i \le 2k-1\} \cup \{2k+1\}$

Next, a pair of latin squares, $\{L_1, L_2\}$, of order $2k+1$ that intersect precisely in one $m$-flower, where $m \in \{i \mid k+1 \le i \le 2k-1\} \cup \{2k+1\}$ will be constructed.

Let $m_1 \in \{j \mid 0 \le j \le k-2\} \cup \{k\}$. Assume there exists a pair of latin squares, $\{U_1, U_2\}$, of order $k$ whose intersection is composed precisely of the triples $\{(i,i,0) \mid 0 \le i \le m_1 - 1\}$ (a symbol-$m_1$-flower).

Consider the following partial latin squares.

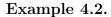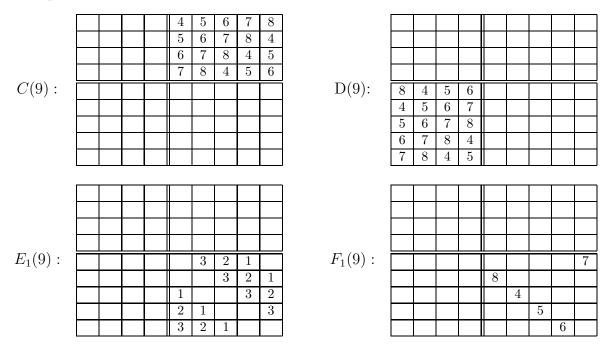$$C(2k+1) = \{(i,j,(i+j+1 \,(\text{mod } k+1))+k) \mid 0 \le i \le k-1,\ k \le j \le 2k\}.$$
$$D(2k+1) = \{(i,j,(i+j \,(\text{mod } k+1))+k) \mid k \le i \le 2k,\ 0 \le j \le k-1\}.$$
$$E_1(2k+1) = \{(i,j,i-j-1 \,(\text{mod } k+1)) \mid k \le i,j \le 2k,\ i \ne j,\ (i \,(\text{mod } k+1))+k \ne j\}.$$
$$F_1(2k+1) = \{(i,(i \,(\text{mod } k+1))+k,(i-1 \,(\text{mod } k+1))+k) \mid k \le i \le 2k\}.$$
$$G(2k+1) = \{(i,i,0) \mid k \le i \le 2k\}.$$

**Example 4.2.**

$C(9):$

| | | | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|---|---|
| | | | 5 | 6 | 7 | 8 | 4 | |
| | | | 6 | 7 | 8 | 4 | 5 | |
| | | | 7 | 8 | 4 | 5 | 6 | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |

$D(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| 8 | 4 | 5 | 6 | | | | | |
| 4 | 5 | 6 | 7 | | | | | |
| 5 | 6 | 7 | 8 | | | | | |
| 6 | 7 | 8 | 4 | | | | | |
| 7 | 8 | 4 | 5 | | | | | |

$E_1(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | 3 | 2 | 1 | | |
| | | | | | 3 | 2 | 1 | |
| | | | 1 | | | 3 | 2 | |
| | | | 2 | 1 | | | 3 | |
| | | | 3 | 2 | 1 | | | |

$F_1(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | 7 |
| | | | | | 8 | | | |
| | | | | | | 4 | | |
| | | | | | | | 5 | |
| | | | | | | 6 | | |

Construct the following latin square of order $2k+1$;

$$L_1 = U_1 \cup C(2k+1) \cup D(2k+1) \cup E_1(2k+1) \cup F_1(2k+1) \cup G(2k+1).$$

Consider the following partial latin squares.

$$E_2(2k+1) = \{(i, j, i - j \,(\mathrm{mod}\ k+1)) \mid k \le i, j \le 2k,\ i \ne j,\ (i + 2 \,(\mathrm{mod}\ k+1)) + k \ne j\}\}.$$

$$F_2(2k+1) = \{(i, (i + 2 \,(\mathrm{mod}\ k+1)) + k, i) \mid k \le i \le 2k\}.$$

**Example 4.3.**

$E_2(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | 3 | 2 | 1 | |
| | | | | 1 | | 3 | 2 | |
| | | | | 2 | 1 | | 3 | |
| | | | | 3 | 2 | 1 | | |
| | | | | | 3 | 2 | 1 | |

$F_2(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | 4 | | | |
| | | | | | | 5 | | |
| | | | | | | | 6 | |
| | | | | | | | | 7 |
| | | | | 8 | | | | |

Now construct the following latin square of order $2k+1$;

$$L_2 = U_2 \cup \sigma_r^k C(2k+1) \cup \sigma_s^{k,2k+1}\sigma_s^{k,2k+1}D(2k+1) \cup E_2(2k+1) \cup F_2(2k+1) \cup G(2k+1).$$

The intersection of $L_1$ and $L_2$ is the set of triples $\{(i, i, 0) \mid 0 \le i \le m_1 - 1\} \cup \{(j, j, 0) \mid k \le j \le 2k\}$, an $(m_1 + k + 1)$-flower. Hence, the following result has been established.

**Lemma 4.10.** *Assume that $4 \leq k$ and $IS(k) = JS(k)$. Let $n = 2k+1$, then $(\{i \mid k+1 \leq i \leq 2k-1\} \cup \{2k+1\}) \subset IS(n)$.*

Combining Lemmas 4.6, 4.7, 4.8, 4.9 and 4.10 yields the following result.

**Lemma 4.11.** *There exist pairs of latin squares of order $n$ yielding the following.*

$$IS(2) = \{0\}; \ IS(3) = \{0, 3\}; \ and \ IS(n) = JS(n) \ for \ 4 \leq n.$$

## 4.2 Two disjoint $m$-flowers

Let $2 \leq m$. In this section pairs of latin squares of order $n$, where $2m \leq n$ that intersect precisely in two disjoint $m$-flowers and no other triples will be constructed. Note that in these constructions both $m$-flowers will be symbol-$m$-flowers.

**Lemma 4.12.** *There exists a pair of latin squares of order five that intersect precisely in two disjoint $2$-flowers.*

*Proof.* The intersection (shown in bold) of the two latin squares shown below is the set of triples $A \cap B = \{(0, 0, 0), (1, 1, 0)\} \cup \{(2, 2, 1), (3, 3, 1)\}$, two disjoint 2-flowers.

$$A: \begin{array}{|c|c|c|c|c|} \hline \mathbf{0} & 1 & 2 & 3 & 4 \\ \hline 2 & \mathbf{0} & 3 & 4 & 1 \\ \hline 4 & 3 & \mathbf{1} & 0 & 2 \\ \hline 3 & 2 & 4 & \mathbf{1} & 0 \\ \hline 1 & 4 & 0 & 2 & 3 \\ \hline \end{array} \qquad B: \begin{array}{|c|c|c|c|c|} \hline \mathbf{0} & 3 & 4 & 2 & 1 \\ \hline 1 & \mathbf{0} & 2 & 3 & 4 \\ \hline 3 & 2 & \mathbf{1} & 4 & 0 \\ \hline 2 & 4 & 0 & \mathbf{1} & 3 \\ \hline 4 & 1 & 3 & 0 & 2 \\ \hline \end{array}$$

□

Once again, the construction splits into two cases.

### 4.2.1 Case A: $n = 2k$

First note the following exception, which follows from inspection of the latin squares of order four.

**Lemma 4.13.** *There does not exist a pair of latin squares of order four that intersect precisely in two disjoint $2$-flowers.*

**Two disjoint $m$-flowers:** $m \in \{i \mid 0 \leq i \leq k-2\} \cup \{k\}$

A pair of latin squares, $\{L_1, L_2\}$, of order $2k \geq 6$ that intersect precisely in two disjoint $m$-flowers, where $m \in \{i \mid 2 \leq i \leq k-2\} \cup \{k\}$, will now be constructed.

Consider the latin square $A = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of order two. Let $m \in \{i \mid 2 \leq i \leq k-2\} \cup \{k\}$. Consider the pair of latin squares (constructed in Section 4.1), $\{U_1, U_2\}$, of order $k \geq 3$ whose intersection is the set of triples $\{(i, i, 0) \mid 0 \leq i \leq m\}$ (a symbol-$m$-flower).

A pair of latin squares, $\{L_1, L_2\}$, of order $2k$ that intersect precisely in two disjoint $m$-flowers will now be constructed. First, construct

$$L_1 = \{(0,0,0), (1,0,1), (0,1,1)\} \times U_1 \ \cup \ \{(1,1,0)\} \times \sigma_s^k U_1.$$

Then, construct

$$L_2 = \{(0,0,0)\} \times U_2 \ \cup \ \{(1,0,1), (0,1,1)\} \times \sigma_s^k U_1 \ \cup \ \{(1,1,0)\} \times \sigma_s^k U_2.$$

The intersection of $L_1$ and $L_2$ is the set of triples $\{(i,i,0) \mid 0 \le i \le m-1\} \cup \{(j,j,1) \mid k \le j \le k+m-1\}$, two disjoint $m$-flowers. Hence, the following result has been established.

**Lemma 4.14.** *Assume $6 \le n = 2k$, then there exists a pair of latin squares of order $n$ that intersect precisely in two disjoint $m$-flowers, where $m \in \{i \mid 2 \le i \le k-2\} \cup \{k\}$.*

**Two disjoint $(k-1)$-flowers**

A pair of latin squares, $\{L_1, L_3\}$, of order $2k \ge 6$ that intersect precisely in two disjoint $(k-1)$-flowers will now be constructed.

Using the above, construct a pair of latin squares, $\{L_1, L_2\}$, that intersect precisely in two disjoint $k$-flowers. Then, let $L_3 = \rho_{2k-1}^{k-1} L_2$ (the map $\rho_{2k-1}^{k-1}$ simply interchanges column $k-1$ with column $2k-1$ in $L_2$).

This yields a pair of latin squares, of order $2k$, whose intersection is precisely composed of two disjoint $(k-1)$-flowers. Hence, the following result has been established.

**Lemma 4.15.** *Assume $3 \le k$, then there exists a pair of latin squares of order $n = 2k$ that intersect in two disjoint $(k-1)$-flowers.*

### 4.2.2 Case B: $n = 2k+1$

**Two disjoint $k$-flowers**

First, a pair of latin squares, $\{L_1', L_2\}$, of order $2k+1 \ge 7$ that intersect precisely in one $(k+1)$-flower and one disjoint $k$-flower will be constructed. This pair is then used to construct a pair of latin squares, $\{L_1, L_2\}$, of order $2k+1 \ge 7$ that intersect precisely in two disjoint $k$-flowers.

Consider a pair of latin squares (constructed in Section 4.1), $\{U_1, U_2\}$, of order $k \ge 3$, whose intersection is composed precisely of the triples $\{(i,i,0) \mid 0 \le i \le k-1\}$ (a symbol $k$-flower). The symbols in $U_1$ and $U_2$ can be permuted, hence, without loss of generality $(k-2, k-1, 1) \in U_1$.

First, a latin square $L_1'$ will be constructed. From the previous section, the partial latin squares $C(2k+1)$, $D(2k+1)$, $F_1(2k+1)$ and $G(2k+1)$ will be made use of. In addition the following partial latin square will also be used.

$$H(2k+1) = \{(i,j,(i-j-1 \,(\mathrm{mod}\ k+1))), (i,(i-1\,(\mathrm{mod}\ k+1))+k, 0) \mid k \le i,j \le 2k,\ i \ne j,$$
$$(i-1\,(\mathrm{mod}\ k+1))+k \ne j,\ (i\,(\mathrm{mod}\ k+1))+k \ne j\}.$$

**Example 4.4.**

$H(9):$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | 3 | 2 | 0 | |
| | | | | | | 3 | 2 | 0 |
| | | | | 0 | | | 3 | 2 |
| | | | | 2 | 0 | | | 3 |
| | | | | 3 | 2 | 0 | | |

Construct the following latin square of order $2k + 1$;

$$L'_1 = U_1 \cup C(2k+1) \cup D(2k+1) \cup F_1(2k+1) \cup H(2k+1) \cup \sigma_s^{k+1} G(2k+1).$$

The following set is claimed to be a latin square of order $2k + 1$;

$$L_2 = U_2 \cup \sigma_s^{k,2k+1}(C(2k+1) \cup D(2k+1) \cup F_1(2k+1)) \cup \psi_k^{2k+1} H(2k+1) \cup \sigma_s^{k+1} G(2k+1).$$

*Proof.* The set of symbols in $H(2k+1)$ is the set $\{h \mid 0 \le h \le k-1,\ h \ne 1\}$. Thus, $\psi_k^{2k+1}$ merely permutes the symbols of $H$ such that there are no fixed points. Hence, as $L'_1$ is a latin square of order $2k + 1$ it follows that $L_2$ is a latin square of order $2k + 1$. $\square$

The intersection of $L'_1$ and $L_2$ is the set of triples $\{(i, i, 0) \mid 0 \le i \le k-1\} \cup \{(j, j, 1) \mid k \le j \le 2k\}$, a $k$-flower and a disjoint $k + 1$-flower. Hence, the following result has been established.

**Lemma 4.16.** *Assume $3 \le k$. Let $n = 2k + 1$, then there exists a pair of latin squares of order $n$ that intersect precisely in one $(k+1)$-flower and one disjoint $k$-flower.*

Notice, $S_1 = \{(k-2, k-1, 1), (k-2, 2k, (3k-1 \,(\mathrm{mod}\ k+1)) + k), (2k, k-1, (3k-1 \,(\mathrm{mod}\ k+1)) + k), (2k, 2k, 1)\} \subset L'_1$; forms one mate of the latin bitrade $(S_1, S_2)$ (an intercalate) where $S_2 = \{(k-2, k-1, (3k-1 \,(\mathrm{mod}\ k+1)) + k), (k-2, 2k, 1), (2k, k-1, 1), (2k, 2k, (3k-1 \,(\mathrm{mod}\ k+1)) + k)\}$. Hence, construct the latin square $L_1 = (L'_1 \setminus S_1) \cup S_2$.

The intersection of $L_1$ and $L_2$ is composed precisely of the triples in the set $\{(i, i, 0) \mid 0 \le i \le k-1\} \cup \{(j, j, 1) \mid k \le j \le 2k-1\}$, two disjoint $k$-flowers. Thus, the following result has been established.

**Lemma 4.17.** *Assume $3 \le k$. Let $n = 2k + 1$, then there exists a pair of latin squares of order $n$ that intersect precisely in two disjoint $k$-flowers.*

**Two disjoint $m$-flowers, where:** $m \in \{i \mid 2 \le i \le k-1\}$

In the following, a pair of latin squares, $\{L_3, L_2\}$, that intersect precisely in two disjoint $m$-flowers, where $2 \le m \le k-1$, will be constructed.

Begin by using the above construction to form the pair, $\{L_1, L_2\}$, of latin squares of order $2k + 1$ that intersect in two disjoint $k$-flowers.

Let $2 \leq m < k$. Note that if $L$ is a latin square of order $n$ and $0 \leq i, j \leq n$, then the map $\rho_i^j$ applied to a latin square $L$ simply swaps column $j$ with column $i$. Thus the following set is a latin square of order $2k + 1$;

$$L_3 = \rho_1^k \circ \rho_2^{k+1} \circ \rho_3^{k+2} \circ \ldots \circ \rho_{k-m}^{2k-m-1} L_1.$$

Furthermore, $L_3$ intersects $L_2$ precisely in two disjoint $m$-flowers, specifically $\{(0,0,0), (i,i,0) \mid k - m + 1 \leq i \leq k - 1\}$ and $\{(i,i,1) \mid 2k - m \leq i \leq 2k - 1\}$. Hence, the following result has been established.

**Lemma 4.18.** *Assume $3 \leq k$. Let $n = 2k + 1$, then there exists a pair of latin squares of order $n$ that intersect precisely in two disjoint $m$-flowers, where $m \in \{i \mid 2 \leq i \leq k - 1\}$.*

Combining Lemmas 4.14, 4.15, 4.17 and 4.18 yields the following result.

**Lemma 4.19.** *There exists a pair of latin squares of order $n \geq 6$ whose intersection is composed of two disjoint $m$-flowers, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$.*

## 4.3 $n = 2m + 2k + 1$, where $0 \leq k \leq \frac{m-1}{2}$

In this section pairs of latin squares of order $n = 2m + 2k + 1$, where $k \in \mathbb{N}$ and $1 \leq k \leq \frac{m-1}{2}$, that intersect precisely in three disjoint $m$-flowers, and pairs of latin squares of order $2m + 1$ that intersect precisely in three disjoint $m$-flowers and one other triple (by Lemma 4.2 there does not exist a pair of latin squares of order $2m + 1$ whose intersection is precisely three disjoint $m$-flowers) are constructed.

**Example 4.5.** A pair of latin squares, $\{L_1, L_2\}$, of order five that intersect in three disjoint 2-flowers and one other triple is shown below (the triples in the 2-flowers are shown in bold, while the additional triple is shown in italics).

$$L_1 : \begin{array}{|c|c|c|c|c|} \hline \mathbf{4} & 3 & 1 & 0 & 2 \\ \hline 2 & \mathbf{4} & 0 & 1 & 3 \\ \hline 0 & 2 & 3 & 4 & \mathbf{1} \\ \hline 3 & 1 & 4 & 2 & \mathbf{0} \\ \hline 1 & 0 & \mathbf{2} & \mathbf{3} & \mathit{4} \\ \hline \end{array} \qquad L_2 : \begin{array}{|c|c|c|c|c|} \hline \mathbf{4} & 2 & 0 & 1 & 3 \\ \hline 1 & \mathbf{4} & 3 & 0 & 2 \\ \hline 3 & 0 & 4 & 2 & \mathbf{1} \\ \hline 2 & 3 & 1 & 4 & \mathbf{0} \\ \hline 0 & 1 & \mathbf{2} & \mathbf{3} & \mathit{4} \\ \hline \end{array}$$

Let $3 \leq m$.

First, latin squares, $L_i$, where $i \in \{1, 2\}$, of order $2m$ will be constructed. Then, using a method similar to prolongation pairs of latin squares of the required orders will be obtained.

Let $A = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$.

Let $B$ be a latin square of order $m$ that contains at least $\lfloor \frac{m}{2} \rfloor + 1$ transversals labelled $T^i$, where $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i,i,i) \mid 0 \leq i \leq m - 1\}$.

First a latin square, $L_1$, of order $2m$ is constructed; $L_1 = A \times B$.

Recall that $k \leq \frac{m-1}{2}$; construct the following latin square of order $2m + 2k + 1$;

$L_1(+2k + 1) =$

$L_1 \setminus [\{(0,0,0)\} \times T^0 \,\cup\, \{(1,0,1)\} \times T^0 \,\cup\, \{(1,1,0)\} \times T^0 \,\cup\, \bigcup_{1 \le i \le k}(A \times T^i)] \,\cup$

$\gamma_c^{2m}(\{(0,0,0)\} \times T^0) \,\cup\, \gamma_s^{2m}(\{(0,0,0)\} \times T^0) \,\cup\, \gamma_r^{2m}(\{(1,0,1)\} \times T^0) \,\cup$
$\gamma_c^{2m}(\{(1,0,1)\} \times T^0) \,\cup\, \gamma_r^{2m}(\{(1,1,0)\} \times T^0) \,\cup\, \gamma_s^{2m}(\{(1,1,0)\} \times T^0) \,\cup$
$\{(m+i,i,i) \mid 0 \le i \le m-1\} \,\cup$

$\bigcup_{1 \le h \le k}$

$[\gamma_r^{2m+2h-1}(\{(0,0,0)\} \times T^h) \,\cup\, \gamma_c^{2m+2h-1}(\{(0,0,0)\} \times T^h) \,\cup\, \gamma_s^{2m+2h-1}(\{(0,0,0)\} \times T^h) \,\cup$

$\gamma_r^{2m+2h}(\{(1,0,1\} \times T^h) \,\cup\, \gamma_c^{2m+2h-1}(\{(1,0,1)\} \times T^h) \,\cup\, \gamma_s^{2m+2h}(\{(1,0,1)\} \times T^h) \,\cup$

$\gamma_r^{2m+2h}(\{(1,1,0)\} \times T^h) \,\cup\, \gamma_c^{2m+2h}(\{(1,1,0)\} \times T^h) \,\cup\, \gamma_s^{2m+2h-1}(\{(1,1,0)\} \times T^h) \,\cup$

$\gamma_r^{2m+2h-1}(\{(0,1,1)\} \times T^h) \,\cup\, \gamma_c^{2m+2h}(\{(0,1,1)\} \times T^h) \,\cup\, \gamma_s^{2m+2h}(\{(0,1,1)\} \times T^h)] \,\cup$

$\{(2m+j, 2m+l, 2m+(j+l \,(\mathrm{mod}\; 2k+1))) \mid 0 \le j,l \le 2k\}.$

See Example A.2.

Next, a second latin square, $L_2$, of order $2m$ is constructed; $L_2 = A \times f_\alpha B$, where $f_{(0,0,0)} = f_{(0,1,1)} = \phi_s^m$, $f_{(1,0,1)} = \sigma_c^m$, and $f_{(1,1,0)} = \sigma_r^m$.

Construct the following latin square of order $2m+2k+1$;

$L_2(+2k+1) =$

$L_2 \setminus [\{(0,0,0)\} \times \phi_s^m T^0 \,\cup\, \{(1,0,1)\} \times \sigma_c^m T^0 \,\cup\, \{(1,1,0)\} \times \sigma_r^m T^0) \,\cup\, \bigcup_{1 \le i \le k}(A \times f_\alpha T^i)] \cup$

$\gamma_c^{2m}(\{(0,0,0)\} \times \phi_s^m T^0) \,\cup\, \gamma_s^{2m}(\{(0,0,0)\} \times \phi_s^m T^0) \,\cup\, \gamma_r^{2m}(\{(1,0,1)\} \times \sigma_c^m T^0) \,\cup$
$\gamma_c^{2m}(\{(1,0,1)\} \times \sigma_c^m T^0) \,\cup\, \gamma_r^{2m}(\{(1,1,0)\} \times \sigma_r^m T^0) \,\cup\, \gamma_s^{2m}(\{(1,1,0)\} \times \sigma_r^m T^0) \,\cup$
$\{(m+i, i+1\,(\mathrm{mod}\; m), i-1\,(\mathrm{mod}\; m)) \mid 0 \le i \le m-1\} \,\cup$

$\bigcup_{1 \le h \le k}$

$[\gamma_r^{2m+2h}(\{(0,0,0)\} \times \phi_s^m T^h) \,\cup\, \gamma_c^{2m+2h}(\{(0,0,0)\} \times \phi_s^m T^h) \,\cup\, \gamma_s^{2m+2h}(\{(0,0,0)\} \times \phi_s^m T^h) \cup$

$\gamma_r^{2m+2h-1}(\{1,0,1\}) \times \sigma_c^m T^h) \,\cup\, \gamma_c^{2m+2h}(\{(1,0,1)\} \times \sigma_c^m T^h) \,\cup\, \gamma_s^{2m+2h-1}(\{(1,0,1)\} \times \sigma_c^m T^h) \cup$

$\gamma_r^{2m+2h-1}(\{(1,1,0)\} \times \sigma_r^m T^h) \,\cup\, \gamma_c^{2m+2h-1}(\{(1,1,0)\} \times \sigma_r^m T^h) \,\cup\, \gamma_s^{2m+2h}(\{(1,1,0)\} \times \sigma_r^m T^h) \cup$

$\gamma_r^{2m+2h}(\{(0,1,1)\} \times \phi_s^m T^h) \,\cup\, \gamma_c^{2m+2h-1}(\{(0,1,1)\} \times \phi_s^m T^h) \,\cup\, \gamma_s^{2m+2h-1}(\{(0,1,1)\} \times \phi_s^m T^h)] \,\cup$

$\{(2m+j, 2m+l, 2m+(j+l+1\,(\mathrm{mod}\; 2k+1))) \mid 0 \le j,l \le 2k\}.$

See Example A.3.

Consider the intersection of $L_1(+2k+1)$ and $L_2(+2k+1)$. Note that

for $k \geq 1$ this intersection is composed precisely of the disjoint $m$-flowers $\{(i, i, 2m) \mid 0 \leq i \leq m - 1\}$, $\{(m + i, 2m, m + i) \mid 0 \leq i \leq m - 1\}$ and $\{(2m, m + i, i) \mid 0 \leq i \leq m - 1\}$;

for $k = 0$ this intersection is composed precisely of the disjoint $m$-flowers $\{(i, i, 2m) \mid 0 \leq i \leq m - 1\}$, $\{(m + i, 2m, m + i) \mid 0 \leq i \leq m - 1\}$ and $\{(2m, m + i, i) \mid 0 \leq i \leq m - 1\}$ and the additional triple $(2m, 2m, 2m)$.

See Example A.4.

The existence of latin squares of order seven that intersect precisely in three disjoint 2-flowers is shown in [13]. Also, Example 4.5 shows a pair of latin squares of order five that intersect precisely in three disjoint 2-flowers and one other triple. Hence, the following result has been established.

**Lemma 4.20.** *Assume* $2 \leq m$ *and* $n = 2m + 2k + 1$, *where* $1 \leq k \leq \frac{m-1}{2}$, *then there exists a pair of latin squares of order $n$ that intersect precisely in three disjoint $m$-flowers. Furthermore, there exists a pair of latin squares of order $2m + 1$ that intersect precisely in three disjoint $m$-flowers and one other triple.*

## 4.4 $n = 2m + 2k$, where $1 \leq k \leq \frac{m}{2}$

**Example 4.6.** A pair of latin squares, $\{L_1, L_2\}$, of order six that intersect in three disjoint 2-flowers is shown below (the triples in the 2-flowers are shown in bold).

$L_1 :$

| **0** | 2 | 4 | 3 | 5 | 1 |
|---|---|---|---|---|---|
| 1 | **0** | 5 | 4 | 2 | 3 |
| 3 | 5 | **1** | **2** | 0 | 4 |
| 2 | 4 | 0 | 1 | **3** | 5 |
| 5 | 1 | 3 | 0 | **4** | 2 |
| 4 | 3 | 2 | 5 | 1 | 0 |

$L_2 :$

| **0** | 1 | 5 | 4 | 2 | 3 |
|---|---|---|---|---|---|
| 5 | **0** | 2 | 3 | 1 | 4 |
| 4 | 3 | **1** | **2** | 5 | 0 |
| 1 | 5 | 4 | 0 | **3** | 2 |
| 3 | 2 | 0 | 5 | **4** | 1 |
| 2 | 4 | 3 | 1 | 0 | 5 |

Let $3 \leq m$. In the remainder of this section pairs of latin squares of order $n = 2m + 2k$, where $3 \leq m$, $k \in \mathbb{N} \cup \{0\}$ and $1 \leq k \leq \frac{m}{2}$, that intersect precisely in three disjoint $m$-flowers will be constructed.

The constructions all start by forming a latin square, $L_i$, where $i \in \{1, 2\}$, of order $2m$ and then increasing this order through a method similar to prolongation.

For the first latin square (of the pair) two separate constructions are used, one for when $k = 1$ and one for when $2 \leq k \leq \frac{m}{2}$. The same construction, for both $k = 1$ and $2 \leq k \leq \frac{m}{2}$, is used for the second latin square.

Let $A = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

Let $B$ be a latin square of order $m$ that contains at least $\lfloor \frac{m}{2} \rfloor + 1$ transversals labelled $T^i$, where $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$ (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i, i, i) \mid 0 \leq i \leq m - 1\}$.

First a latin square, $L_1$, of order $2m$ is constructed; $L_1 = A \times B$.

Now construct the following latin square of order $2m + 2$;

$L_1'(+2) =$

$L_1 \setminus [A \times T^0] \cup$

$\gamma_c^{2m}(\{(0,0,0) \times T^0\}) \ \cup \ \gamma_s^{2m}(\{(0,0,0) \times T^0\}) \ \cup \ \gamma_r^{2m}(\{(0,0,0) \times T^0\}) \cup$
$\gamma_c^{2m}(\{(1,0,1) \times T^0\}) \ \cup \ \gamma_s^{2m+1}(\{(1,0,1) \times T^0\}) \ \cup \ \gamma_r^{2m+1}(\{(1,0,1) \times T^0\}) \cup$
$\gamma_c^{2m+1}(\{(0,1,1) \times T^0\}) \ \cup \ \gamma_s^{2m+1}(\{(0,1,1) \times T^0\}) \ \cup \ \gamma_r^{2m}(\{(0,1,1) \times T^0\}) \cup$
$\gamma_c^{2m+1}(\{(1,1,0) \times T^0\}) \ \cup \ \gamma_s^{2m}(\{(1,1,0) \times T^0\}) \ \cup \ \gamma_r^{2m+1}(\{(1,1,0) \times T^0\}) \cup$

$\{(2m+j, 2m+l, 2m+(j+l \ (\mathrm{mod} \ 2))) \mid 0 \le j,l \le 1\}.$

See Example A.5.

Notice, $(R_1 = \{(2m-1,i,x),(2m+1,i,m+i) \mid 0 \le i \le m-2, (2m-1,i,x) \in L_1'(+2)\}, R_2 = \{(2m-1,i,m+i),(2m+1,i,x) \mid 0 \le i \le m-2, (2m-1,i,x) \in L_1'(+2)\})$ is a latin bitrade where $R_1 \subset L_1'(+2)$.

Similarly, $(S_1 = \{(2m-1,m-1,2m+1),(2m-1,2m,2m-1),(2m+1,m-1,2m-1),(2m+1,2m,2m+1)\}, S_2 = \{(2m-1,m-1,2m-1),(2m-1,2m,2m+1),(2m+1,m-1,2m+1),(2m+1,2m,2m-1)\})$ is a latin bitrade where $S_1 \subset L_1'(+2)$.

Hence, construct the latin square $L_1(+2) = (L_1'(+2) \setminus (R_1 \cup S_1)) \cup (R_2 \cup S_2)$. See Example A.6

Let $2 \le k \le \frac{m}{2}$; consider the latin square of order $2m+2k$; $L_1(+2k) =$

$L_1 \setminus [\bigcup_{0 \le h \le k-1} A \times T^h]$

$\bigcup_{0 \le h \le 1}$

$[\gamma_r^{2m+h}(\{(0,0,0) \times T^h\}) \ \cup \ \gamma_c^{2m+2h}(\{(0,0,0)\} \times T^h) \ \cup \ \gamma_s^{2m+2h}(\{(0,0,0) \times T^h\}) \cup$

$\gamma_r^{2m+h}(\{(0,1,1) \times T^h\}) \ \cup \ \gamma_c^{2m+2h+1}(\{(0,1,1)\} \times T^h) \ \cup \ \gamma_s^{2m+2h+1}(\{(0,1,1) \times T^h\}) \cup$

$\gamma_r^{2m+2+h}(\{(1,0,1) \times T^h\}) \ \cup \ \gamma_c^{2m+2h}(\{(1,0,1)\} \times T^h) \ \cup \ \gamma_s^{2m+2h+1}(\{(1,0,1) \times T^h\}) \cup$

$\gamma_r^{2m+3-h}(\{(1,1,0) \times T^h\}) \ \cup \ \gamma_c^{2m+2h+1}(\{(1,1,0)\} \times T^h) \ \cup \ \gamma_s^{2m+2h}(\{(1,1,0) \times T^h\})] \cup$

$\bigcup_{2 \le h \le k-1}$

$[\gamma_r^{2m+2h}(\{(0,0,0) \times T^h\}) \ \cup \ \gamma_c^{2m+2h}(\{(0,0,0)\} \times T^h) \ \cup \ \gamma_s^{2m+2h}(\{(0,0,0) \times T^h\}) \cup$

$\gamma_r^{2m+2h}(\{(0,1,1) \times T^h\}) \ \cup \ \gamma_c^{2m+2h+1}(\{(0,1,1)\} \times T^h) \ \cup \ \gamma_s^{2m+2h+1}(\{(0,1,1) \times T^h\}) \cup$

$\gamma_r^{2m+2h+1}(\{(1,0,1) \times T^h\}) \ \cup \ \gamma_c^{2m+2h}(\{(1,0,1)\} \times T^h) \ \cup \ \gamma_s^{2m+2h+1}(\{(1,0,1) \times T^h\}) \cup$

$\gamma_r^{2m+2h+1}(\{(1,1,0) \times T^h\}) \ \cup \ \gamma_c^{2m+2h+1}(\{(1,1,0)\} \times T^h) \ \cup \ \gamma_s^{2m+2h}(\{(1,1,0) \times T^h\})] \cup$

$\{(2m+j, 2m+l, 2m+(j+l \ (\mathrm{mod} \ 2k))) \mid 0 \le j,l \le 2k-1\}.$

See Example A.7.

Next, a latin square, $L_2$, of order $2m$ will be constructed; $L_2 = A \times f_\alpha B$, where $f_{(0,0,0)} = \sigma_s^m$, $f_{(1,0,1)} = f_{(0,1,1)} = \sigma_r^m$ and $f_{(1,1,0)} = \sigma_c^m$.

Let $1 \le k \le \frac{m}{2}$. Consider the latin square of order $2m+2k$;
$L_2(+2k) =$

$L_2 \setminus [\bigcup_{0 \le h \le k-1} A \times \{f_\alpha T^h\}_{\alpha \in A}] \cup$

$\gamma_r^{2m}(\{(0,0,0)\} \times \sigma_s^m T^0) \cup \gamma_c^{2m}(\{(0,0,0)\} \times \sigma_s^m T^0) \cup \gamma_s^{2m}(\{(0,0,0)\} \times \sigma_s^m T^0) \cup$

$\gamma_r^{2m}(\{(0,1,1)\} \times \sigma_r^m T^0) \cup \gamma_c^{2m+1}(\{(0,1,1)\} \times \sigma_r^m T^0) \cup \gamma_s^{2m+1}(\{(0,1,1)\} \times \sigma_r^m T^0) \cup$

$\gamma_r^{2m+1}(\{(1,0,1)\} \times \sigma_r^m T^0) \cup \gamma_c^{2m}(\{(1,0,1)\} \times \sigma_r^m T^0) \cup \gamma_s^{2m+1}(\{(1,0,1)\} \times \sigma_r^m T^0) \cup$

$\gamma_r^{2m+1}(\{(1,1,0)\} \times \sigma_c^m T^0) \cup \gamma_c^{2m+1}(\{(1,1,0)\} \times \sigma_c^m T^0) \cup \gamma_s^{2m}(\{(1,1,0)\} \times \sigma_c^m T^0)$

$\bigcup_{1 \le h \le k-1}$

$[\gamma_r^{2m+2h+1}(\{(0,0,0)\} \times \sigma_s^m T^h) \cup \gamma_c^{2m+2h+1}(\{(0,0,0)\} \times \sigma_s^m T^h) \cup \gamma_s^{2m+2h+1}(\{(0,0,0)\} \times \sigma_s^m T^h) \cup$

$\gamma_r^{2m+2h+1}(\{(0,1,1)\} \times \sigma_r^m T^h) \cup \gamma_c^{2m+2h}(\{(0,1,1)\} \times \sigma_r^m T^h) \cup \gamma_s^{2m+2h}(\{(0,1,1)\} \times \sigma_r^m T^h) \cup$

$\gamma_r^{2m+2h}(\{(1,0,1)\} \times \sigma_r^m T^h) \cup \gamma_c^{2m+2h+1}(\{(1,0,1)\} \times \sigma_r^m T^h) \cup \gamma_s^{2m+2h}(\{(1,0,1)\} \times \sigma_r^m T^h) \cup$

$\gamma_r^{2m+2h}(\{(1,1,0)\} \times \sigma_c^m T^h) \cup \gamma_c^{2m+2h}(\{(1,1,0)\} \times \sigma_c^m T^h) \cup \gamma_s^{2m+2h+1}(\{(1,1,0)\} \times \sigma_c^m T^h)] \cup$

$\{(2m+j, 2m+l, 2m+(j+l+1 \pmod{2k})) \mid 0 \le j, l \le 2k-1\}.$

See Example A.8

The intersection of $L_1(+2)$ and $L_2(+2)$ is composed precisely of the disjoint $m$-flowers $\{(i,i,2m) \mid 0 \le i \le m-1\}$, $\{(2m, m+i, m+i) \mid 0 \le i \le m-1\}$ and $\{(m+i, 2m+1, i) \mid 0 \le i \le m-1\}$. See Example A.9.

Recall that, Example 4.6 shows a pair of latin squares of order six that intersect precisely in three disjoint 2-flowers. Thus, the following result has been established.

**Lemma 4.21.** *Assume $2 \le m$ then there exists a pair of latin squares of order $2m+2$ that intersect precisely in three disjoint $m$-flowers.*

Let $4 \le m$. Let $2 \le k \le \frac{m}{2}$. Consider the intersection of $L_1(+2k)$ and $L_2(+2k)$. This intersection is composed precisely of the disjoint $m$-flowers $\{(i,i,2m) \mid 0 \le i \le m-1\}$, $\{(2m, m+i, m+i) \mid 0 \le i \le m-1\}$ and $\{(m+i, 2m+1, i) \mid 0 \le i \le m-1\}$. See Example A.10. Hence, the following result has been established.

**Lemma 4.22.** *Assume $4 \le m$ and $2 \le k \le \frac{m-1}{2}$ then there exists a pair of latin squares of order $2m+2k$ that intersect precisely in three disjoint $m$-flowers.*

Combining Lemmas 4.21 and 4.22 yields the following result.

**Lemma 4.23.** *Assume $2 \le m$ and $1 \le k \le \frac{m-1}{2}$ then there exists a pair of latin squares of order $2m+2k$ that intersect precisely in three disjoint $m$-flowers.*

## 4.5  $3m + 1 \leq n \leq 4m$

Let $3 \leq m$. In this section pairs of latin squares of order $n = 3m + k$, where $k \in \mathbb{N}$ and $1 \leq k \leq m$, that intersect precisely in three or four disjoint $m$-flowers will be constructed.

First, a pair of latin squares, of order $3m + k$, that intersect in three disjoint $m$-flowers is constructed. Then this construction is modified to obtain a pair of latin squares, of order $3m + k$ that intersect in four disjoint $m$-flowers.

Let $A = \{(0,0,0), (0,1,2), (0,2,1), (1,0,2), (1,1,1), (1,2,0), (2,0,1), (2,1,0), (2,2,2)\}$.

Let $B$ be a latin square of order $m$ that contains at least $\lfloor \frac{m}{2} \rfloor + 1$ transversals labelled $T^i$, where $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$ (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i,i,i) \mid 0 \leq i \leq m - 1\}$.

Begin by constructing a latin square, $L_1$, of order $3m$; $L_1 = A \times B$.

Let $k_1 = \lceil \frac{k-1}{2} \rceil$ and $k_2 = \lfloor \frac{k-1}{2} \rfloor$. Also let, $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. By Lemma 3.3 the sets $\{(t,t,t) \mid 0 \leq t \leq 2\} \times T^0$, $\{(0,1,2), (1,2,0), (2,0,1)\} \times T^i$ and $\{(t,t,t) \mid 0 \leq t \leq 2\} \times T^j$ are transversals in $L_1$. So, $k$-prolongate down the following $k$ transversals in $L_1$:

$$T_0 = \{(t,t,t) \mid 0 \leq t \leq 2\} \times T^0;$$

$$T_i = \{(0,1,2), (1,2,0), (2,0,1)\} \times T^i, \text{ where } 1 \leq i \leq k_1; \text{ and}$$

$$T_{k_1+j} = \{(t,t,t) \mid 0 \leq t \leq 2\} \times T^j, \text{ where } 1 \leq j \leq k_2;$$

using the completing square $\{(3m + h, 3m + l, 3m + (h + l \,(\text{mod k}))) \mid 0 \leq h, l \leq k - 1\}$ to form the latin square $L_1(+k)$. See Example A.11.

Now, a latin square, $L_2$, of order $3m$ will be constructed. Let $A'$ be the set $A \setminus \{(1,1,1), (2,2,2)\}$. Then, $L_2 = A \times \{f_\alpha B\}_{\alpha \in A}$, where $f_\alpha = \sigma_s^m$ when $\alpha \in A'$, $f_{(1,1,1)} = \sigma_c^m$, and $f_{(2,2,2)} = \sigma_r^m$.

As before, let $k_1 = \lceil \frac{k-1}{2} \rceil$ and $k_2 = \lfloor \frac{k-1}{2} \rfloor$. Also, as before, let $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. By Lemma 3.3 the sets $\{(t,t,t) \mid 0 \leq t \leq 2\} \times f_\alpha T^0$, $\{(t,t,t) \mid 0 \leq t \leq 2\} \times f_\alpha T^i$ and $\{(0,1,2), (1,2,0), (2,0,1)\} \times f_\alpha T^j$ are transversals in $L_2$. So, $k$-prolongate down the following $k$ transversals

$$T_i = \{(h,h,h) \mid 0 \leq h \leq 2\} \times f_\alpha T^i, \text{ where } 0 \leq i \leq k_1; \text{ and}$$

$$T_{k_1+j} = \{(0,1,2), (1,2,0), (2,0,1)\} \times f_\alpha T^j, \text{ where } 1 \leq j \leq k_2;$$

and use the completing square $\{(3m + h, 3m + l, 3m + (h + l + 1 \,(\text{mod } k))) \mid 0 \leq h, l \leq k - 1\}$ to form the latin square $L_2(+k)$. See Example A.12.

The latin squares $L_1(+k)$ and $L_2(+k)$ intersect precisely in three disjoint $m$-flowers if $2 \leq k$ (namely $\{(i,i,i) \mid 0 \leq i \leq m - 1\}$, $\{(m + i, 3m, m + i) \mid 0 \leq i \leq m - 1\}$ and $\{(3m, 2m + i, 2m + i) \mid 0 \leq i \leq m - 1\}$ (see Example A.13)) and precisely in the same three disjoint $m$-flowers and one other triple (namely $(3m, 3m, 3m)$) if $k = 1$.

Notice,

$$R_1 = \{(0, m, 2m), (0, 2m, m), (0, 3m, 0), (m, 0, 2m), (m, m, 3m), (m, 2m, 0),$$

$$(2m, 0, m), (2m, 2m, 3m), (2m, 3m, 2m), (3m, 0, 0), (3m, m, m), (3m, 3m, 3m)\} \subset L_1(+1);$$

this set is one mate of the latin bitrade $(R_1, R_2)$, where

$$R_2 = \{(0, m, m), (0, 2m, 0), (0, 3m, 2m), (m, 0, 0), (m, m, 2m), (m, 2m, 3m), (2m, 0, 2m)$$

$$, (2m, 2m, m), (2m, 3m, 3m), (3m, 0, m), (3m, m, 3m), (3m, 3m, 0)\}.$$

Construct the latin square $L_3(+1) = (L_1 \setminus R_1) \cup R_2$. See Example A.14.

The latin squares $L_3(+1)$ and $L_2(+1)$ intersect precisely in three disjoint $m$-flowers. Hence, the following result has been established.

**Lemma 4.24.** *Assume $3 \le m$ and $3m + 1 \le n \le 4m$, then there exists a pair of latin squares of order $n$ that intersect precisely in three disjoint $m$-flowers.*

Next a latin square $L_4(+k)$, where $1 \le k \le m$, is constructed in the same manner as $L_2(+k)$ except letting $f_{(2,1,0)} = v_s^m$.

The latin squares $L_1(+k)$ and $L_4(+k)$, where $2 \le k \le m$, and $L_3(+1)$ and $L_4(+1)$ intersect precisely in four disjoint $m$-flowers, namely $\{(i, i, i) \mid 0 \le i \le m - 1\}$, $\{(m + i, 3m, m + i) \mid 0 \le i \le m - 1\}$, $\{(3m, 2m + i, 2m + i) \mid 0 \le i \le m - 1\}$ and $\{(2m + i, m + i, m - 1) \mid (i, j, m - 1) \in B\}$. Thus, the following result has been proved.

**Lemma 4.25.** *Assume $3 \le m$ and $3m + 1 \le n \le 4m$, then there exists a pair of latin squares of order $n$ that intersect precisely in four disjoint $m$-flowers.*

## 4.6   $n = 4m + 1$

In this section pairs of latin squares of order $n = 4m + 1$ that intersect precisely in 3, 4 and 5 disjoint $m$-flowers, are constructed.

Let $A = \{(0, 0, 0), (0, 1, 3), (0, 2, 1), (0, 3, 2), (1, 0, 2), (1, 1, 1), (1, 2, 3), (1, 3, 0), (2, 0, 3), (2, 1, 0), (2, 2, 2), (2, 3, 1), (3, 0, 1), (3, 1, 2), (3, 2, 0), (3, 3, 3)\}$.

Let $B$ be a latin square of order $m$ that contains a transversal, labelled $T^0$ (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i, i, i) \mid 0 \le i \le m - 1\}$.

First the latin square, $L_1$, of order $4m$ is constructed; $L_1 = A \times B$. By Lemma 3.3, the set $\{(i, i, i) \mid 0 \le i \le 3\} \times T^0$ is a transversal in $L_1$. Thus, prolongate down this transversal to form the latin square $L_1(+1)$ of order $4m + 1$. See Example A.15.

Now a latin square, $L_2$, of order $4m$ is constructed. Let $A' = A \backslash \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$. Then, $L_2 = A \times \{f_\alpha B\}_{\alpha \in A}$, where $f_\alpha = \sigma_s^m$ when $\alpha \in A'$, $f_{(1,1,1)} = \sigma_r^m$, $f_{(2,2,2)} = \sigma_c^m$ and $f_{(3,3,3)} = id$.

By Lemma 3.3 the set $\{(i, i, i) \mid 0 \le i \le 3\} \times f_\alpha T^0$ is a transversal in $L_2$. Prolongate down this transversal to form the latin square $L_2'(+1)$.

Note that the triples in the set $R_1 = \{(x, y, z) \mid 3m \le x, y \le 4m, (x, y, z) \in L_1(+1)\}$ occur in both $L_1(+1)$ and $L_2'(+1)$. Also, note that these triples form one mate of a latin bitrade $(R_1, R_2)$, where $R_2 = \{(x, y, (z - 3m + 1 \,(\mathrm{mod}\; m + 1)) + 3m) \mid (x, y, z) \in R_1\}$ ($R_1$ is in fact a latin square where $N = \{i \mid 3m \le i \le 4m\}$). Construct $L_2(+1) = (L_2'(+1) \setminus R_1) \cup R_2$. See Example A.16.

The latin squares $L_1(+1)$ and $L_2(+1)$ intersect precisely in three disjoint $m$-flowers, namely $\{(i,i,i) \mid 0 \le i \le m-1\}$, $\{(4m, m+i, m+i) \mid 0 \le i \le m-1\}$ and $\{(2m+i, 4m, 2m+i) \mid 0 \le i \le m-1\}$. Hence, the following result has been proved.

**Lemma 4.26.** *Assume $3 \le m$ then there exists a pair of latin squares of order $4m+1$ that intersect precisely in three disjoint $m$-flowers.*

Now construct a latin square $L_3(+1)$ in the same manner as $L_2(+1)$ except let $f_{(3,2,0)} = v_s^m$; and construct $L_4(+1)$ in the same manner as $L_2(+1)$ except let $f_{(3,2,0)} = v_s^m$ and $f_{(1,3,0)} = \epsilon_s^m$.

The latin squares $L_1(+1)$ and $L_3(+1)$ intersect precisely in four disjoint $m$-flowers (namely $\{(i,i,i) \mid 0 \le i \le m-1\}$, $\{(4m, m+i, m+i) \mid 0 \le i \le m-1\}$, $\{(2m+i, 4m, 2m+i) \mid 0 \le i \le m-1\}$ and $\{(3m+i, 2m+j, m-1) \mid (i,j,m-1) \in B\}$) and the latin squares $L_1(+1)$ and $L_4(+1)$ intersect precisely in five disjoint $m$-flowers (namely $\{(i,i,i) \mid 0 \le i \le m-1\}$, $\{(4m, m+i, m+i) \mid 0 \le i \le m-1\}$, $\{(2m+i, 4m, 2m+i) \mid 0 \le i \le m-1\}$, $\{(3m+i, 2m+j, m-1) \mid (i,j,m-1) \in B\}$ and $\{(m+i, 3m+j, m-2) \mid (i,j,m-2) \in B\}$). Hence, the following result has been established.

**Lemma 4.27.** *Assume $3 \le m$. Then there exists a pair of latin squares of order $4m+1$ that intersect precisely in four disjoint $m$-flowers and there exists a pair of latin squares of order $4m+1$ that intersect precisely in five disjoint $m$-flowers.*

## 4.7 $\quad 4m+2 \le n \le 6m+2$

Let $3 \le m$. In this section pairs of latin squares of order $n$ are constructed that intersect in: 3, 4, 5 or 6 disjoint $m$-flowers when $4m+2 \le n \le 5m+1$; in 3, 4, 5, 6 or 7 disjoint $m$-flowers when $5m+2 \le n \le 6m+1$; and in 3, 4, 5, 6, 7 or 8 disjoint $m$-flowers when $n = 6m+2$.

First a pair of latin squares of order $n$, where $4m+2 \le n \le 6m+2$, that intersect in 5 disjoint $m$-flowers will be constructed.

Let $B$ be a latin square of order $m$ that contains at least $\lceil \frac{m}{2} \rceil + 1$ transversals labelled $T^i$, where $0 \le i \le \lceil \frac{m}{2} \rceil$ (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i,i,i) \mid 0 \le i \le m-1\}$.

Let $A_1 = \{(0,0,0), (0,1,1), (0,2,2), (0,3,3), (1,0,1), (1,1,0), (1,2,3), (1,3,2), (2,0,2), (2,1,3), (2,2,0), (2,3,1), (3,0,3), (3,1,2), (3,2,1), (3,3,0)\}$. Notice that $A_1 = U_1 \cup U_2 \cup U_3 \cup U_4$, where:

$U_1 = \{(0,0,0), (1,2,3), (2,3,1), (3,1,2)\}$; $U_2 = \{(0,1,1), (1,3,2), (2,2,0), (3,0,3)\}$;
$U_3 = \{(0,2,2), (1,0,1), (2,1,3), (3,3,0)\}$; $U_4 = \{(0,3,3), (1,1,0), (2,0,2), (3,2,1)\}$.

Also, note that each $U_i$, where $1 \le i \le 4$, is a transversal in $A_1$ and that these transversals are disjoint.

$A_1 :$

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0 | 3 | 2 |
| 2 | 3 | 0 | 1 |
| 3 | 2 | 1 | 0 |

Transversals:

| $U_1$ | $U_2$ | $U_3$ | $U_4$ |
|---|---|---|---|
| $U_3$ | $U_4$ | $U_1$ | $U_2$ |
| $U_4$ | $U_3$ | $U_2$ | $U_1$ |
| $U_2$ | $U_1$ | $U_4$ | $U_3$ |

First, a latin square, $L_1$, of order $4m$ is constructed; $L_1 = A_1 \times B$.

Next, let $2 \le k \le 2m + 2$ and $k - 2 = k_1 + k_2 + k_3 + k_4$, where $0 \le k_4 \le k_3 \le k_2 \le k_1 \le \lceil \frac{m}{2} \rceil$. Construct the following latin square of order $4m + k$;
$L_1'(+k) =$

$L_1 \setminus [((A_1 \setminus \{(0,1,1),(0,3,3),(2,1,3),(2,3,1)\}) \times T^0) \cup \bigcup_{1 \le h_1 \le k_1}(U_1 \times T^{h_1}) \cup$
$\bigcup_{1 \le h_2 \le k_2}(U_2 \times T^{h_2}) \cup \bigcup_{1 \le h_3 \le k_3}(U_3 \times T^{h_3}) \cup \bigcup_{1 \le h_4 \le k_4}(U_4 \times T^{h_4})] \cup$

$\gamma_r^{4m}(\{(1,0,1),(1,1,0),(1,2,3),(1,3,2)\} \times T^0) \cup$
$\gamma_r^{4m+1}(\{(3,0,3),(3,1,2),(3,2,1),(3,3,0)\} \times T^0) \cup$
$\gamma_c^{4m}(\{(3,0,3),(2,0,2),(1,0,1),(0,0,0)\} \times T^0) \cup$
$\gamma_c^{4m+1}(\{(3,2,1),(2,2,0),(1,2,3),(0,2,2)\} \times T^0) \cup$
$\gamma_s^{4m}(\{(0,0,0),(1,1,0),(2,2,0),(3,3,0)\} \times T^0) \cup$
$\gamma_s^{4m+1}(\{(0,2,2),(1,3,2),(2,0,2),(3,1,2)\} \times T^0) \cup$

$\{(m+i,i,i),(m+i,2m+i,2m+i),(3m+i,i,2m+i),(3m+i,2m+i,i) \mid 0 \le i \le m-1\} \cup$

$\bigcup_{1 \le h_1 \le k_1}(\gamma_r^{4m+1+h_1}(U_1 \times T^{h_1}) \cup \gamma_c^{4m+1+h_1}(U_1 \times T^{h_1}) \cup \gamma_s^{4m+1+h_1}(U_1 \times T^{h_1})) \cup$

$\bigcup_{1 \le h_2 \le k_2}(\gamma_r^{4m+1+k_1+h_2}(U_2 \times T^{h_2}) \cup \gamma_c^{4m+1+k_1+h_2}(U_2 \times T^{h_2}) \cup \gamma_s^{4m+1+k_1+h_2}(U_2 \times T^{h_2})) \cup$

$\bigcup_{1 \le h_3 \le k_3}(\gamma_r^{4m+1+k_1+k_2+h_3}(U_3 \times T^{h_3}) \cup \gamma_c^{4m+1+k_1+k_2+h_3}(U_3 \times T^{h_3}) \cup \gamma_s^{4m+1+k_1+k_2+h_3}(U_3 \times T^{h_3})) \cup$

$\bigcup_{1 \le h_4 \le k_4}(\gamma_r^{4m+1+k_1+k_2+k_3+h_4}(U_4 \times T_4^h) \cup \gamma_c^{4m+1+k_1+k_2+k_3+h_4}(U_4 \times T^{h_4}) \cup \gamma_s^{4m+1+k_1+k_2+k_3+h_4}(U_4 \times T_4^h)) \cup$

$\{(4m+j,4m+l,4m+(j+l \,(\mathrm{mod}\, k))) \mid 0 \le j, l \le k-1\}.$

See Example A.17.

Notice, $R_1 = \{(m+i,m+i,4m),(m+i,3m+i,4m+1),(3m+i,m+i,4m+1),(3m+i,3m+i,4m) \mid 0 \le i \le m-1\} \subset L_1'(+k)$; this set is one mate of the latin bitrade $(R_1, R_2)$, where $R_2 = \{(m+i,m+i,4m+1),(m+i,3m+i,4m),(3m+i,m+i,4m),(3m+i,3m+i,4m+1) \mid 0 \le i \le m-1\}$.

Hence, construct the latin square $L_1(+k) = (L_1'(+k) \setminus R_1) \cup R_2$.

Let $A_2 = \{(0,0,0),(0,1,1),(0,2,3),(0,3,2),(1,0,1),(1,1,0),(1,2,2),(1,3,3),(2,0,3),(2,1,2),(2,2,0),(2,3,1),(3,0,2),(3,1,3),(3,2,1),(3,3,0)\}$. Notice that $A_2 = U_1' \cup U_2' \cup U_3' \cup U_4'$, where:

$U_1' = \{(0,3,2),(1,1,0),(2,0,3),(3,2,1)\}$; $U_2' = \{(0,2,3),(1,0,1),(2,1,2),(3,3,0)\}$;
$U_3' = \{(0,1,1),(1,3,3),(2,2,0),(3,0,2)\}$; $U_4' = \{(0,0,0),(1,2,2),(2,3,1),(3,1,3)\}$.

Also, note that each $U_i'$, where $1 \le i \le 4$, is a transversal in $A_2$ and that these transversals are disjoint.

$A_2:$

| 0 | 1 | 3 | 2 |
|---|---|---|---|
| 1 | 0 | 2 | 3 |
| 3 | 2 | 0 | 1 |
| 2 | 3 | 1 | 0 |

Transversals:

| $U'_4$ | $U'_3$ | $U'_2$ | $U'_1$ |
|---|---|---|---|
| $U'_2$ | $U'_1$ | $U'_4$ | $U'_3$ |
| $U'_1$ | $U'_2$ | $U'_3$ | $U'_4$ |
| $U'_3$ | $U'_4$ | $U'_1$ | $U'_2$ |

Now, the latin square, $L_2$, of order $4m$ will be constructed. $L_2 = A_2 \times f_\alpha B$, where $f_{(0,3,2)} = f_{(1,2,2)} = f_{(1,3,3)} = f_{(2,0,3)} = f_{(3,1,3)} = id$, $f_{(2,1,2)} = f_{(3,0,2)} = f_{(0,0,0)} = f_{(0,1,1)} = f_{(1,0,1)} = f_{(1,1,0)} = f_{(2,3,1)} = f_{(0,2,3)} = \sigma_s^m$, $f_{(3,3,0)} = \phi_s^m$, $f_{(2,2,0)} = \sigma_c^m$ and $f_{(3,2,1)} = \sigma_r^m$.

| $\sigma_s^m$ | $\sigma_s^m$ | $\sigma_s^m$ | $id$ |
|---|---|---|---|
| $\sigma_s^m$ | $\sigma_s^m$ | $id$ | $id$ |
| $id$ | $\sigma_s^m$ | $\sigma_c^m$ | $\sigma_s^m$ |
| $\sigma_s^m$ | $id$ | $\sigma_r^m$ | $\phi_s^m$ |

Let $2 \le k \le 2m + 2$, and $k_i$, where $1 \le i \le 4$, be the same as in the construction of $L_1'(+k)$. So consider the following latin square of order $4m + k$.

$L_2(+k) =$

$L_2 \setminus [((A \setminus \{(0,1,1),(1,3,3),(2,0,3),(2,3,1)\}) \times f_\alpha T^0) \cup \bigcup_{1 \le h_1 \le k_1}(U'_1 \times f_\alpha T^{h_1}) \cup \bigcup_{1 \le h_2 \le k_2}(U'_2 \times f_\alpha T^{h_2}) \cup \bigcup_{1 \le h_3 \le k_3}(U'_3 \times f_\alpha T^{h_3}) \cup \bigcup_{1 \le h_4 \le k_4}(U'_4 \times f_\alpha T^{h_4})] \cup$

$\gamma_r^{4m}(\{(1,0,1),(1,1,0),(0,2,3),(0,3,2)\} \times f_\alpha T^0) \cup$
$\gamma_r^{4m+1}(\{(3,0,2),(3,1,3),(3,2,1),(3,3,0)\} \times f_\alpha T^0) \cup$
$\gamma_c^{4m}(\{(0,0,0),(1,0,1),(2,1,2),(3,1,3)\} \times f_\alpha T^0) \cup$
$\gamma_c^{4m+1}(\{(0,2,3),(1,2,2),(2,2,0),(3,2,1)\} \times f_\alpha T^0) \cup$
$\gamma_s^{4m}(\{(0,0,0),(1,1,0),(2,2,0),(3,3,0)\} \times f_\alpha T^0) \cup$
$\gamma_s^{4m+1}(\{(0,3,2),(1,2,2),(2,1,2),(3,0,2)\} \times f_\alpha T^0) \cup$

$\{(i, 2m+i, 2m+i), (m+i, i, (i+1 \pmod m)), (3m+i, m+i, 2m+(i+1 \pmod m)), (3m+(i+1 \pmod m), 2m+i, (i-1 \pmod m)) \mid 0 \le i \le m-1\} \cup$

$\bigcup_{1 \le h_1 \le k_1}(\gamma_r^{4m+1+h_1}(U'_1 \times f_\alpha T^{h_1}) \cup \gamma_c^{4m+1+h_1}(U'_1 \times f_\alpha T^{h_1}) \cup \gamma_s^{4m+1+h_1}(U'_1 \times f_\alpha T^{h_1})) \cup$

$\bigcup_{1 \le h_2 \le k_2}(\gamma_r^{4m+1+k_1+h_2}(U'_2 \times f_\alpha T^{h_2}) \cup \gamma_c^{4m+1+k_1+h_2}(U'_2 \times f_\alpha T^{h_2}) \cup \gamma_s^{4m+1+k_1+h_2}(U'_2 \times f_\alpha T^{h_2})) \cup$

$\bigcup_{1 \le h_3 \le k_3}(\gamma_r^{4m+1+k_1+k_2+h_3}(U'_3 \times f_\alpha T^{h_3}) \cup \gamma_c^{4m+1+k_1+k_2+h_3}(U'_3 \times f_\alpha T^{h_3}) \cup \gamma_s^{4m+1+k_1+k_2+h_3}(U'_3 \times f_\alpha T^{h_3})) \cup$

$\bigcup_{1 \le h_4 \le k_4}(\gamma_r^{4m+1+k_1+k_2+k_3+h_4}(U'_4 \times f_\alpha T^{h_4}) \cup \gamma_c^{4m+1+k_1+k_2+k_3+h_4}(U'_4 \times f_\alpha T^{h_4}) \cup \gamma_s^{4m+1+k_1+k_2+k_3+h_4}(U'_4 \times f_\alpha T^{h_4})) \cup$

$\{(4m+j, 4m+l, 4m+(j+l+1 \pmod k)) \mid 0 \le j, l \le k-1\}.$

See Example A.18.

The latin squares $L_1(+k)$ and $L_2(+k)$ intersect precisely in five disjoint $m$-flowers (namely $\{(i,i,4m) \mid 0 \le i \le m-1\}$, $\{(2m+i,4m+1,i) \mid 0 \le i \le m-1\}$, $\{3m+i,4m,3m+i \mid 0 \le i \le m-1\}$, $\{(4m,3m+i,2m+i) \mid 0 \le i \le m-1\}$ and $\{(4m+1,2m+i,m+i) \mid 0 \le i \le m-1\}$) and no other triples. See Example A.19.

Thus, the following result has been established.

**Lemma 4.28.** *Assume $3 \leq m$ and $4m + 2 \leq n \leq 6m + 2$, then there exists a pair of latin squares of order $n$ that intersect precisely in five disjoint $m$-flowers.*

Notice $S_1 = \{(m+i, m+i, 4m), (m+i, 2m+i, 4m+1), (2m+i, m+i, 4m+1), (2m+i, 2m+(i+1 \,(\mathrm{mod}\, m)), 4m) \mid 0 \leq i \leq m-1\} \subset L_2(+k)$; this set of triples is one mate of the latin bitrade $(S_1, S_2)$, where $S_2 = \{(m+i, m+i, 4m+1), (m+i, 2m+i, 4m), (2m+i, m+i, 4m), (2m+i, 2m+(i+1 \,(\mathrm{mod}\, m)), 4m+1) \mid 0 \leq i \leq m-1\}$.

Construct the latin square $L_3(+k) = (L_2(+k) \setminus S_1) \cup S_2$.

The latin squares $L_1(+k)$ and $L_3(+k)$ intersect in precisely in six disjoint $m$-flowers, specifically $\{(i, i, 4m) \mid 0 \leq i \leq m-1\}$, $\{(2m+i, 4m+1, i) \mid 0 \leq i \leq m-1\}$, $\{3m+i, 4m, 3m+i \mid 0 \leq i \leq m-1\}$, $\{(4m, 3m+i, 2m+i) \mid 0 \leq i \leq m-1\}$, $\{(4m+1, 2m+i, m+i) \mid 0 \leq i \leq m-1\}$ and $\{(m+i, m+i, 4m+1) \mid 0 \leq i \leq m-1\}$. Hence, the following result has been established.

**Lemma 4.29.** *Assume $3 \leq m$ and $4m + 2 \leq n \leq 6m + 2$, then there exists a pair of latin squares of order $n$ that intersect precisely in six disjoint $m$-flowers.*

Let $m + 2 \leq k \leq 2m + 2$; let $V_1 = \{(i, j, u) \mid 4m \leq i, j \leq 4m + k - 1, (i, j, u) \in L_1(+k)\}$ and $W_1 = \{(i, j, v) \mid 4m \leq i, j \leq 4m + k - 1, (i, j, v) \in L_3(+k)\}$.

From Lemma 4.11 there exists a pair of latin squares of order $k$, where $m + 2 \leq k \leq 2m + 2$, that intersect precisely in one symbol-$m$-flower. By relabelling the rows, columns and symbols (the same permutations are applied to both squares) the symbol-$m$-flower is the set $\{(i, i, 2) \mid 2 \leq i \leq m + 1\}$. Denote this pair of latin squares by $\{V_2', W_2'\}$.

Let $V_2 = \delta^{4m} V_2'$ and $W_2 = \delta^{4m} W_2'$. Note that, $(V_1, V_2)$ and $(W_1, W_2)$ are latin biswaps. So, construct the latin squares $L_4(+k) = (L_1(+k) \setminus V_1) \cup V_2$ and $L_5(+k) = (L_1(+k) \setminus W_1) \cup W_2$.

Thus, the latin squares $L_4(+k)$ and $L_5(+k)$ intersect precisely in seven disjoint $m$-flowers, specifically $\{(i, i, 4m) \mid 0 \leq i \leq m-1\}$, $\{(2m+i, 4m+1, i) \mid 0 \leq i \leq m-1\}$, $\{3m+i, 4m, 3m+i \mid 0 \leq i \leq m-1\}$, $\{(4m, 3m+i, 2m+i) \mid 0 \leq i \leq m-1\}$, $\{(4m+1, 2m+i, m+i) \mid 0 \leq i \leq m-1\}$, $\{(m+i, m+i, 4m+1) \mid 0 \leq i \leq m-1\}$ and $\{(4m+2+i, 4m+2+i, 4m+2) \mid 0 \leq i \leq m-1\}$. Hence, the following result has been established.

**Lemma 4.30.** *Assume $3 \leq m$ and $5m + 2 \leq n \leq 6m + 2$, then there exists a pair of latin squares of order $n$ that intersect precisely in seven disjoint $m$-flowers.*

Let $k = 2m + 2$; let $X_1 = \{(i, j, u) \mid 4m \leq i, j \leq 4m + k - 1, (i, j, u) \in L_1(+k)\}$ and $Y_1 = \{(i, j, v) \mid 4m \leq i, j \leq 4m + k - 1, (i, j, v) \in L_3(+k)\}$.

From Lemma 4.19 there exists a pair of latin squares of order $k = 2m + 2$ that intersect precisely in two disjoint symbol-$m$-flowers. By relabelling the rows, columns and symbols (the same permutations are applied to both squares) the symbol-$m$-flowers are the sets $\{(i, i, 2) \mid 2 \leq i \leq m + 1\}$ and $\{(i, i, 3) \mid m + 2 \leq i \leq 2m + 1\}$. Denote this pair of latin squares by $\{X_2', Y_2'\}$.

Let $X_2 = \delta^{4m} X_2'$ and $Y_2 = \delta^{4m} Y_2'$. Note that, $(X_1, X_2)$ and $(Y_1, Y_2)$ are latin biswaps. Construct the latin squares $L_6(+k) = (L_1(+k) \setminus X_1) \cup X_2$ and $L_7(+k) = (L_3(+k) \setminus Y_1) \cup Y_2$.

The latin squares $L_6(+k)$ and $L_7(+k)$ intersect precisely in eight disjoint $m$-flowers, specifically $\{(i, i, 4m) \mid 0 \le i \le m - 1\}$, $\{(2m + i, 4m + 1, i) \mid 0 \le i \le m - 1\}$, $\{3m + i, 4m, 3m + i \mid 0 \le i \le m - 1\}$, $\{(4m, 3m + i, 2m + i) \mid 0 \le i \le m - 1\}$, $\{(4m + 1, 2m + i, m + i) \mid 0 \le i \le m - 1\}$, $\{(m + i, m + i, 4m + 1) \mid 0 \le i \le m - 1\}$, $\{(4m + 2 + i, 4m + 2 + i, 4m + 2) \mid 0 \le i \le m - 1\}$ and $\{(5m + 2 + i, 5m + 2 + i, 4m + 3) \mid 0 \le i \le m - 1\}$. Hence, the following result has been established.

**Lemma 4.31.** *Assume $3 \le m$, then there exists a pair of latin squares of order $6m + 2$ that intersect precisely in eight disjoint $m$-flowers.*

Now a pair of latin squares of order $n$, where $4m + 2 \le n \le 6m + 2$, that intersect in 3 disjoint $m$-flowers; and a pair of latin squares of order $n$, where $4m + 2 \le n \le 6m + 2$, that intersect in 4 disjoint $m$-flowers will be constructed.

Let $B$ be a latin square of order $m$ that contains at least $\lceil \frac{m}{2} \rceil + 1$ transversals labelled $T^i$, where $0 \le i \le \lceil \frac{m}{2} \rceil$ (such a latin square exists by Lemma 3.2); without loss of generality, in the latin square $B$, $T^0 = \{(i, i, i) \mid 0 \le i \le m - 1\}$.

Let $A = \{(0, 0, 0), (0, 1, 1), (0, 2, 2), (0, 3, 3), (1, 0, 1), (1, 1, 0), (1, 2, 3), (1, 3, 2), (2, 0, 2), (2, 1, 3), (2, 2, 0), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (3, 3, 0)\}$. Notice that $A = U_1 \cup U_2 \cup U_3 \cup U_4$, where:

$U_1 = \{(0, 0, 0), (1, 2, 3), (2, 3, 1), (3, 1, 2)\}$; $U_2 = \{(0, 1, 1), (1, 3, 2), (2, 2, 0), (3, 0, 3)\}$;
$U_3 = \{(0, 2, 2), (1, 0, 1), (2, 1, 3), (3, 3, 0)\}$; $U_4 = \{(0, 3, 3), (1, 1, 0), (2, 0, 2), (3, 2, 1)\}$.

Also note that each $U_i$, where $1 \le i \le 4$, is a transversal in $A$ and that these transversals are disjoint.

$A:$

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0 | 3 | 2 |
| 2 | 3 | 0 | 1 |
| 3 | 2 | 1 | 0 |

Transversals:

| $U_1$ | $U_2$ | $U_3$ | $U_4$ |
|---|---|---|---|
| $U_3$ | $U_4$ | $U_1$ | $U_2$ |
| $U_4$ | $U_3$ | $U_2$ | $U_1$ |
| $U_2$ | $U_1$ | $U_4$ | $U_3$ |

First, a latin square, $N_1$, of order $4m$ is constructed; $N_1 = A \times B$.

Let $2 \le k \le 2m + 2$ and $k - 2 = k_1 + k_2 + k_3 + k_4$, where $0 \le k_4 \le k_3 \le k_2 \le k_1 \le \lceil \frac{m}{2} \rceil$. Let $0 \le j \le \lceil \frac{m}{2} \rceil$ and $1 \le i \le 4$; by Lemma 3.3 the sets $U_i \times T^j$ are transversals in $N_1$. Consider the transversals

$D_0 = U_1 \times T^0$; $D_1 = U_3 \times T^0$; $D_{1+h_1} = U_1 \times T^{h_1}$, where $1 \le h_1 \le k_1$;
$D_{k_1+1+h_2} = U_2 \times T^{h_2}$, where $1 \le h_2 \le k_2$; $D_{k_1+k_2+1+h_3} = U_1 \times T^{h_3}$, where $1 \le h_3 \le k_3$;
and $D_{k_1+k_2+k_3+1+h_4} = U_1 \times T^{h_4}$, where $1 \le h_4 \le k_4$.

Hence, $k$-prolongate down the $k$ disjoint transversals $D_i$, where $0 \le i \le k - 1$, and use the completing square $\{(4m + i, 4m + j, 4m + (i + j \,(\mathrm{mod}\, k))) \mid 0 \le i, j \le k - 1\}$ to form the latin square $N_1(+k)$. See Example A.20.

Next, the latin square, $N_2$, of order $4m$ will be constructed; $N_2 = A \times f_\alpha B$, where $f_{(0,0,0)} = f_{(0,1,1)} = f_{(0,2,2)} = f_{(0,3,3)} = f_{(1,1,0)} = f_{(1,3,2)} = f_{(2,0,2)} = f_{(2,1,3)} = f_{(2,2,0)} = f_{(2,3,1)} = f_{(3,0,3)} = f_{(3,2,1)} = \sigma_s^m$, $f_{(1,0,1)} = f_{(3,1,2)} = \sigma_c^m$ and $f_{(1,2,3)} = f_{(3,3,0)} = \sigma_r^m$.

| $\sigma_s^m$ | $\sigma_s^m$ | $\sigma_s^m$ | $\sigma_s^m$ |
|---|---|---|---|
| $\sigma_c^m$ | $\sigma_s^m$ | $\sigma_r^m$ | $\sigma_s^m$ |
| $\sigma_s^m$ | $\sigma_s^m$ | $\sigma_s^m$ | $\sigma_s^m$ |
| $\sigma_s^m$ | $\sigma_c^m$ | $\sigma_s^m$ | $\sigma_r^m$ |

By Lemma 3.3 the sets $\{U_i \times f_\alpha T^j\}$ are transversals in $N_2$. Consider the transversals

$$E_0 = U_3 \times f_\alpha T^0; \ E_1 = U_1 \times f_\alpha T^0; \ E_{1+h_1} = U_2 \times f_\alpha T^{h_1}, \text{ where } 1 \leq h_1 \leq k_1;$$
$$E_{k_1+1+h_2} = U_3 \times f_\alpha T^{h_2}, \text{ where } 1 \leq h_2 \leq k_2; \ E_{k_1+k_2+1+h_3} = U_4 \times f_\alpha T^{h_3}, \text{ where}$$
$$1 \leq h_3 \leq k_3; \text{ and } E_{k_1+k_2+k_3+1+h_4} = U_1 \times f_\alpha T^{h_4}, \text{ where } 1 \leq h_4 \leq k_4.$$

Let $\zeta_r = \zeta_v = (1\ 2)$. Now, $(\zeta_r, \zeta_c)$-$k$-prolongate down the $k$ disjoint transversals $E_i$, where $0 \leq i \leq k-1$, and use the completing square $\{(4m+i, 4m+j, 4m+(i+j+1\,(\mathrm{mod}\ k))) \mid 0 \leq i, j \leq k-1\}$ to form the latin square $N_2(+k)$. See Example A.21.

The latin squares $N_1(+k)$ and $N_2(+k)$ intersect precisely in four disjoint $m$-flowers, namely $\{(4m, 2m+i, 3m+i) \mid 0 \leq i \leq m-1\}$, $\{(4m+1, 3m+i, i) \mid 0 \leq i \leq m-1\}$, $\{(3m+i, 4m, 2m+i) \mid 0 \leq i \leq m-1\}$ and $\{(m+i, 4m+1, m+i) \mid 0 \leq i \leq m-1\}$. See Example A.22.

Hence, the following result has been proved.

**Lemma 4.32.** *Assume $3 \leq m$ and $4m+2 \leq n \leq 6m+2$, then there exists a pair of latin squares of order $n$ that intersect precisely in four disjoint $m$-flowers.*

Now a latin square $N_3(+k)$ is constructed. This construction is the same as the construction for $N_2(+k)$ except that $f_{(1,2,3)} = \sigma_s^m$.

The latin squares $N_1(+k)$ and $N_3(+k)$ intersect precisely in three disjoint $m$-flowers, namely $\{(4m+1, 3m+i, i) \mid 0 \leq i \leq m-1\}$, $\{(3m+i, 4m, 2m+i) \mid 0 \leq i \leq m-1\}$ and $\{(m+i, 4m+1, m+i) \mid 0 \leq i \leq m-1\}$. Hence, the following result has been established.

**Lemma 4.33.** *Assume $3 \leq m$ and $4m+2 \leq n \leq 6m+2$, then there exists a pair of latin squares of order $n$ that intersect precisely in three disjoint $m$-flowers.*

## 4.8 $6m+3 \leq n$

In this section constructions for the remaining cases required to prove Theorem 1 will be provided.

Let $2 \leq m$.

Let $A$ be a latin square of order $l \geq 3$ that contains at least three disjoint transversals labelled $U_0$, $U_1$ and $U_2$; without loss of generality, in the latin square $A$, $U_0 = \{(i, i, i) \mid 0 \leq i \leq l-1\}$.

Let $B_1$ be a latin square of order $2m+1$ which contains $2m+1$ disjoint transversals. Without loss of generality $(0, 0, 0) \in B_1$. Label the $2m+1$ disjoint transversals in $B_1$ as $V_1^j$, where $0 \leq j \leq 2m$, such that $(0, 0, 0) \in V_1^0$. Let $B_2 = \sigma_s^{2m+1} B_1$ and label the transversal $\sigma_s^{2m+1} V_1^j$ as $V_2^j$, where $0 \leq j \leq 2m$. Note that $B_1 \cap B_2 = \emptyset$.

The existence of the above latin squares ($A$ and $B_1$) is guaranteed by Lemma 3.2.

Let $C_1$ and $C_2$ be a pair of latin squares of order $2m+1$ that intersect in three disjoint $m$-flowers and one other triple, namely the triple $(0,0,0)$ (note that this triple will not be disjoint from the $m$-flowers). Such a pair exists by Lemma 4.20.

Let $D_1$ and $D_2$ be a pair of latin squares of order $2m+1$ that intersect precisely in two disjoint $m$-flowers. Without loss of generality $(0,0,0) \in D_1$ and $(0,0,0) \notin D_2$. Such a pair exists by Lemma 4.19.

Let $E_1$ and $E_2$ be a pair of latin squares of order $2m+1$ that intersect precisely in one $m$-flower. Without loss of generality $(0,0,0) \in E_1$ and $(0,0,0) \notin E_2$. Such a pair exists by Lemma 4.11.

Recall $3 \leq l$. A pair of latin squares, $\{L_1^h(+k), L_2^h(+k)\}$, of order $l(2m+1)+k$, where $0 \leq k \leq 2m$, that intersect in precisely $h$, where $0 \leq h \leq 3l$, disjoint $m$-flowers will now be constructed.

Determine $b, c, d, e \in \mathbb{N} \cup \{0\}$ such that $3c + 2d + e = h$ and $b + c + d + e = l$ (it should be noted that this is always possible).

First, construct the latin square $L_1^h(+0)'$ of order $l(2m+1)$.

$$L_1^h(+0)' = \{(i,i,i) \mid 0 \leq i \leq c-1\} \times C_1 \ \cup \ \{(i,i,i) \mid c \leq i \leq c+d-1\} \times D_1 \ \cup$$
$$\{(i,i,i) \mid c+d \leq i \leq c+d+e-1\} \times E_1 \ \cup \ \{(i,i,i) \mid c+d+e \leq i \leq l-1\} \times B_1 \ \cup \ (A \backslash U_0) \times B_1.$$

As $(0,0,0) \in B_1 \cap C_1 \cap D_1 \cap E_1$, the set $R_1 = \{(x(2m+1), y(2m+1), z(2m+1)) \mid (x,y,z) \in A\}$ is one mate of the latin bitrade $(R_1, R_2)$, where $R_2 = \{(x(2m+1), y(2m+1), (z+1 \,(\mathrm{mod}\, l))(2m+1)) \mid (x,y,z) \in A\}$. Hence, construct the latin square $L_1^h(+0) = (L_1^h(+0)' \backslash R_1) \cup R_2$.

Let $1 \leq k \leq 2m$.

By Lemma 3.3 the set of triples $U_1 \times V_1^f$, where $0 \leq f \leq 2m-1$, forms a transversal in $L_1^h(+0)$.

The latin square $L_1^h(+k)$ is constructed by $k$-prolongating $L_1^h(+0)$ down the transversals

$$T_1^f = U_1 \times V_1^f, \text{ where } 1 \leq f \leq k,$$

and using the completing square

$$\{(i, j, (i+j-2l(2m+1) \,(\mathrm{mod}\, k)) + l(2m+1)) \mid l(2m+1) \leq i, j \leq l(2m+1) + k - 1\}.$$

See Example A.23.

A second latin square, $L_2^h(+0)$, of order $l(2m+1)$ will now be constructed.

$$L_2^h(+0) = \{(i,i,i) \mid 0 \leq i \leq c-1\} \times C_2 \ \cup \ \{(i,i,i) \mid c \leq i \leq c+d-1\} \times D_2 \ \cup$$
$$\{(i,i,i) \mid c+d \leq i \leq c+d+e-1\} \times E_2 \ \cup \ \{(i,i,i) \mid c+d+e \leq i \leq l-1\} \times B_2 \ \cup \ (A \backslash U_0) \times B_2.$$

By Lemma 3.3 the set of triples $U_2 \times V_2^f$, where $0 \leq f \leq 2m-1$, forms a transversal in $L_2^h(+0)$.

Let $1 \leq k \leq 2m$. Construct the latin square $L_2^h(+k)$ by $k$-prolongating $L_2^h(+0)$ down the transversals

$$T_2^f = U_2 \times V_2^f, \text{ where } 1 \leq f \leq k,$$

and using the completing square

$$\{(i, j, (i+j-2l(2m+1)+1 \,(\mathrm{mod}\ k))+l(2m+1)) \mid l(2m+1) \le i, j \le l(2m+1)+k-1\}.$$

See Example A.24.

Let $0 \le k \le 2m$; when $k \ne 1$, $L_1^h(+k)$ intersects $L_2^h(+k)$ precisely in $h$ disjoint $m$-flowers and, when $k = 1$, $L_1^h(+1)$ intersects $L_2^h(+1)$ precisely in $h$ disjoint $m$-flowers and one other disjoint triple. See Example A.25.

Hence, the following result has been established.

**Lemma 4.34.** *Assume $2 \le m$, $3 \le l$, $6m + 3 \le n$ and $0 \le h \le 3\lfloor \frac{n}{2m+1} \rfloor$. If $n \ne l(2m+1)+1$ then there exists a pair of latin squares of order $n$ that intersect precisely in $h$ disjoint $m$-flowers. If $n = l(2m+1)+1$ then there exists a pair of latin squares of order $n$ that intersect in precisely $h$ disjoint $m$-flowers and one additional disjoint triple.*

Let $k = 1$; as above prolongate $L_1^h(+0)$ down the transversal $T_1^1 = U^1 \times V_1^1$ to achieve the latin square $L_1^h(+1)$.

Note that the set $S_1 = \{(l(2m + 1), l(2m + 1), l(2m + 1)), (l(2m + 1) - 1, l(2m + 1), z), (l(2m + 1) - 1, y, l(2m + 1)), (l(2m + 1), y, z) \mid (l(2m + 1) - 1, y, z) \in T_1^1\}$, forms one mate of the latin bitrade $(S_1, S_2)$, where $S_2 = \{(l(2m+1), l(2m+1), z), (l(2m+1) - 1, l(2m+1), l(2m+1)), (l(2m+1)-1, y, z), (l(2m+1), y, l(2m+1)) \mid (l(2m+1)-1, y, z) \in T_1^1\}$. Construct the latin square $L_3^h(+1) = (L_1^h(+1) \setminus S_1) \cup S_2$.

The latin squares $L_3^h(+1)$ and $L_2^h(+1)$ intersect precisely in $h$ disjoint $m$-flowers. Thus, the following result has been established.

**Lemma 4.35.** *Assume $2 \le m$, $3 \le l$, $n = l(2m+1)+1$ and $0 \le h \le 3l$, then there exists a pair of latin squares of order $n$ that intersect in precisely $h$ disjoint $m$-flowers.*

Next, a pair of latin squares of order $l(2m + 1) + m \le n < (l + 1)(2m + 1)$, $n \ne l(2m + 1) + m + 1$ and $(n, m) \ne (5l + 2, 2)$ that intersect precisely in $3l + 1$ disjoint $m$-flowers will be constructed.

Let $m \le k \le 2m$, $k \ne m + 1$ and $(k, m) \ne (2, 2)$.

Let $X_1 = \{(i, j, (i + j - 2l(2m + 1) \,(\mathrm{mod}\ k)) + l(2m + 1)) \mid l(2m + 1) \le i, j \le l(2m + 1) + k - 1\}$ and $Y_1 = \{(i, j, (i + j - 2l(2m + 1) + 1 \,(\mathrm{mod}\ k)) + l(2m + 1)) \mid l(2m + 1) \le i, j \le l(2m + 1) + k - 1\}$. Note that $X_1 \subset L_1^{3l}(+k)$ and $Y_1 \subset L_2^{3l}(+k)$ (they are the completing squares used in the $k$-prolongations).

By Lemma 4.11, as $k \ne m + 1$ and $(k, m) \ne (2, 2)$, there exists a pair of latin squares of order $k$ that intersect precisely in one $m$-flower. Denote this pair of latin squares by $\{X_2', Y_2'\}$. Let $X_2 = \delta^{l(2m+1)} X_2'$ and $Y_2 = \delta^{l(2m+1)} Y_2'$.

Hence, $(X_1, X_2)$ and $(Y_1, Y_2)$ are latin biswaps. Construct the latin squares

$$L_4^{3l+1}(+k) = (L_1^{3l}(+k) \setminus X_1) \cup X_2 \text{ and } L_5^{3l+1}(+k) = (L_2^{3l}(+k) \setminus Y_1) \cup Y_2.$$

The latin squares $L_4^{3l+1}(+k)$ and $L_5^{3l+1}(+k)$ intersect precisely in $3l + 1$ disjoint $m$-flowers. Hence, the following result has been established.

**Lemma 4.36.** *Assume $2 \leq m$, $3 \leq l$, $l(2m+1) + m \leq n \leq l(2m+1) + 2m$, $n \neq l(2m+1) + m + 1$ and $(n,m) \neq (5l+2, 2)$; then there exists a pair of latin squares of order $n$ that intersect precisely in $3l+1$ disjoint $m$-flowers.*

A pair of latin squares of order $l(2m+1) + m + 1$ that intersect precisely in $3l+1$ disjoint $m$-flowers will now be constructed.

Similarly to before, let $X_1 = \{(i, j, (i+j-2l(2m+1) \,(\mathrm{mod}\, m+1)) + l(2m+1)) \mid l(2m+1) \leq i, j \leq l(2m+1) + m\}$ and $Y_1 = \{(i, j, (i+j-2(l(2m+1)) + 1 \,(\mathrm{mod}\, m+1)) + l(2m+1)) \mid l(2m+1) \leq i, j \leq l(2m+1) + m\}$. Again, note that $X_1 \subset L_1^{3l}(+m+1)$ and $Y_1 \subset L_2^{3l}(+m+1)$.

By Lemma 4.11 there exists a pair of latin squares of order $m+1$ that intersect precisely in one $(m+1)$-flower. Without loss of generality the $(m+1)$-flower is comprised by the set of cells $\{(i, i, i) \mid 0 \leq i \leq m\}$. Denote this pair of latin squares by $\{X_2', Y_2'\}$.

Let $X_2 = \delta^{l(2m+1)} X_2'$ and $Y_2 = \delta^{l(2m+1)} Y_2'$.

Hence, $(X_1, X_2)$ and $(Y_1, Y_2)$ are latin biswaps. Construct the latin squares

$$L_6^{3l+1}(+m+1) = (L_1^{3l}(+m+1) \setminus X_1) \cup X_2 \text{ and } L_7^{3l+1}(+m+1) = (L_2^{3l}(+m+1) \setminus Y_1) \cup Y_2.$$

Note that the set of triples $S_1 = \{(l(2m+1), l(2m+1), l(2m+1)), (l(2m+1)-1, l(2m+1), z), (l(2m+1)-1, y, l(2m+1)), (l(2m+1), y, z) \mid (l(2m+1)-1, y, z) \in T_1^1\}$, forms one mate of the latin bitrade $(S_1, S_2)$, where $S_2 = \{(l(2m+1), l(2m+1), z), (l(2m+1)-1, l(2m+1), l(2m+1)), (l(2m+1)-1, y, z), (l(2m+1), y, l(2m+1)) \mid (l(2m+1)-1, y, z) \in T_1^1\}$.

Construct the latin square $L_8^{3l+1}(+m+1) = (L_6^{3l+1}(+m+1) \setminus S_1) \cup S_2$.

The latin squares $L_8^{3l+1}(+m+1)$ and $L_7^{3l+1}(+m+1)$ intersect precisely in $3l+1$ disjoint $m$-flowers. Hence, the following result has been established.

**Lemma 4.37.** *Assume $2 \leq m$, $3 \leq l$ and $n = l(2m+1) + m + 1$, then there exists a pair of latin squares of order $n$ that intersect precisely in $3l+1$ disjoint $m$-flowers.*

A pair of latin squares of order $5l+2$ that intersect precisely in $3l+1$ disjoint 2-flowers will now be constructed. Recall that there does not exist a pair of latin squares of order two that intersect precisely in one 2-flower.

Consider the latin square of order two, $G_1 = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$.

Let $m = 2$. Consider the permutation $\zeta = (1\ 2)$. Construct the latin squares $L_1^{3l}(+2)$ as above using the transversals $T_1^1$ and $T_1^2$ and the completing square $\delta^{5l} G_1$.

Note, the set of triples $S_1 = \{(0, y, 5l), (0, 5l, z), (5l, y, z), (5l, 5l, 5l) \mid (0, y, z) \in T_1^1\} \subset L_1^{3l}(+2)$, forms one mate of the latin bitrade $(S_1, S_2)$, where $S_2 = \{(0, y, z), (0, 5l, 5l), (5l, y, 5l), (5l, 5l, z) \mid (0, y, z) \in T_1^1\}$. Hence, construct the latin square $L_9^{3l+1}(+2) = (L_1^{3l}(+2) \setminus S_1) \cup S_2$.

Next, construct the latin square $L_{10}^{3l+1}(+2)$ by $(\zeta, \zeta)$-2-prolongating the latin square $L_2^{3l}(+0)$ down the transversals $T_2^1$ and $T_2^2$ and using the completing square $\delta^{5l} G_1$.

Note that the set of triples $S_3 = \{(0, p, 5l), (0, 5l+1, q), (5l+1, p, q), (5l+1, 5l+1, 5l) \mid (0, p, q) \in T_2^1\} \subset L_{10}^{3l+1}(+2)$, forms one mate of the latin bitrade $(S_3, S_4)$, where

$S_4 = \{(0, p, q), (0, 5l+1, 5l), (5l+1, p, 5l), (5l+1, 5l+1, q) \mid (0, p, q) \in T_2^1\}$. Construct the latin square $L_{11}^{3l+1}(+2) = (L_{10}^{3l+1}(+2) \setminus S_3) \cup S_4$.

The latin squares $L_9^{3l+1}(+2)$ and $L_{11}^{3l+1}(+2)$ intersect precisely in $3l+1$ disjoint 2-flowers. Hence, the following result has been established.

**Lemma 4.38.** *Assume $3 \le l$ and $n = 5l+2$, then there exists a pair of latin squares of order $n$ that intersect precisely in $3l+1$ disjoint 2-flowers.*

Let $3 \le m$. Next, a pair of latin squares of order $l(2m+1) + 2m$, where $3 \le l$ that intersect precisely in $3l+2$ disjoint $m$-flowers will be constructed.

Let $X_1 = \{(i, j, (i + j - 2l(2m+1) \,(\mathrm{mod}\, 2m)) + l(2m+1)) \mid l(2m+1) \le i, j \le l(2m+1) + 2m - 1\}$ and $Y_1 = \{(i, j, (i + j - l(2m+1) + 1 \,(\mathrm{mod}\, 2m)) + l(2m+1)) \mid l(2m+1) \le i, j \le l(2m+1) + 2m - 1\}$. Once again, note that $X_1 \subset L_1^{3l}(+2m)$ and $Y_1 \subset L_2^{3l}(+2m)$.

By Lemma 4.19 there exists a pair of latin squares of order $2m$ that intersect precisely in two disjoint $m$-flowers. Denote this pair of latin squares by $\{X_2', Y_2'\}$. Let $X_2 = \delta^{l(2m+1)} X_2'$ and $Y_2 = \delta^{l(2m+1)} Y_2'$.

Hence, $(X_1, X_2)$ and $(Y_1, Y_2)$ are latin biswaps. Construct the latin squares

$$L_{12}^{3l+2}(+2m) = (L_1^{3l}(+2m) \setminus X_1) \cup X_2 \text{ and } L_{13}^{3l+2}(+2m) = (L_2^{3l}(+2m) \setminus Y_1) \cup Y_2.$$

The latin squares $L_{12}^{3l+2}(+2m)$ and $L_{13}^{3l+2}(+2m)$ intersect precisely in $3l+2$ disjoint $m$-flowers. Hence, the following result has been established.

**Lemma 4.39.** *Assume $3 \le m$, $3 \le l$ and $n = l(2m+1) + 2m$, then there exists a pair of latin squares of order $n$ that intersect precisely in $3l+2$ disjoint $m$-flowers.*

Finally, a pair of latin squares of order $5l+4$ that intersect in $3l+2$ disjoint 2-flowers will be constructed. This construction uses similar ideas to those underlying the construction used to prove Lemma 4.38.

Consider the following pair of latin squares of order four, $G_2' = \{(0, 0, 0), (0, 1, 1), (0, 2, 2), (0, 3, 3), (1, 0, 1), (1, 1, 0), (1, 2, 3), (1, 3, 2), (2, 0, 2), (2, 1, 3), (2, 2, 1), (2, 3, 0), (3, 0, 3), (3, 1, 2), (3, 2, 0), (3, 3, 1)\}$ and $G_3' = \{(0, 0, 0), (0, 1, 1), (0, 2, 3), (0, 3, 2), (1, 0, 1), (1, 1, 0), (1, 2, 2), (1, 3, 3), (2, 0, 3), (2, 1, 2), (2, 2, 1), (2, 3, 0), (3, 0, 2), (3, 1, 3), (3, 2, 0), (3, 3, 1)\}$.

Let $m = 2$. Consider the permutations $\zeta_a = (1\ 2\ 3)$ and $\zeta_b = (1\ 4)$. Construct the latin square $L_{14}^{3l+2}(+4)$ by $(\zeta_a, \zeta_b)$-4-prolongating the latin square $L_1^{3l}(+0)$ down the transversals $T_1^i$, where $1 \le i \le 4$ and using the completing square $G_2 = \delta^{5l} G_2'$.

Note that the set of triples $S_1 = \{(0, y, 5l), (0, 5l+3, z), (5l+2, y, z), (5l+2, 5l+3, 5l), (0, v, 5l+1), (0, 5l+1, w), (5l, v, w), (5l, 5l+1, 5l+1) \mid (0, y, z) \in T_1^1, (0, v, w) \in T_1^2\} \subset L_{14}^{3l+2}(+4)$, forms one mate of the latin bitrade $(S_1, S_2)$, where $S_2 = \{(0, y, z), (0, 5l+3, 5l), (5l+2, y, 5l), (5l+2, 5l+3, z), (0, v, w), (0, 5l+1, 5l+1), (5l, v, 5l+1), (5l, 5l+1, w) \mid (0, y, z) \in T_1^1, (0, v, w) \in T_1^2\}$. Construct the latin square $L_{16}^{3l+2}(+4) = (L_9^{3l+2}(+4) \setminus S_1) \cup S_2$.

Now, construct the latin square $L_{15}^{3l+2}(+4)$ by $(\zeta_b, \zeta_a)$-4-prolongating the latin square $L_2^{3l}(+0)$ down the transversals $T_2^i$, where $1 \le i \le 4$, and using the completing square $G_3 = \delta^{5l} G_3'$.

Note that the set of triples $S_3 = \{(0, p, 5l), (0, 5l+2, q), (5l+3, p, q), (5l+3, 5l+2, 5l),$ $(0, s, 5l+1), (0, 5l, t), (5l+1, s, t), (5l+1, 5l, 5l+1) \mid (0, p, q) \in T_2^1, (0, s, t) \in T_2^2\} \subset$ $L_{15}^{3l+2}(+4)$, forms one mate of the latin bitrade $(S_3, S_4)$, where $S_4 = \{(0, p, q), (0, 5l+2, 5l), (5l+3, p, 5l), (5l+3, 5l+2, q), (0, s, t), (0, 5l, 5l+1), (5l+1, s, 5l+1), (5l+1, 5l, t) \mid$ $(0, p, q) \in T_2^1, (0, s, t) \in T_2^2\}$. Hence, construct the latin square $L_{17}^{3l+2}(+4) = (L_{15}^{3l+2}(+4) \setminus S_3) \cup S_4$.

The latin squares $L_{16}^{3l+2}(+4)$ and $L_{17}^{3l+2}(+4)$ intersect precisely in $3l + 2$ disjoint 2-flowers. Hence, the following result has been proved.

**Lemma 4.40.** *Assume $m = 2$, $3 \le l$ and $n = 5l + 4$; then there exists a pair of latin squares of order $n$ that intersect precisely in $3l + 2$ disjoint 2-flowers.*

## 4.9 Main theorem

At the beginning of this paper necessary conditions for the disjoint $m$-flower intersection problem in latin squares were established. Through the rest of the paper pairs of latin squares that prove that these conditions are in fact sufficient have been constructed.

**Theorem 1.** *There exists a pair of latin squares of order $n$ whose intersection is composed precisely of $x$ disjoint $m$-flowers, where $2 \le m \le n$, $0 \le x \le i$ and*

$$
\begin{array}{lll}
i = 3l & for & l(2m+1) \le n < l(2m+1) + m \text{ and } n \ne 2m+1, \\
i = 3l + 1 & for & l(2m+1) + m \le n < l(2m+1) + 2m \text{ and} \\
& & (m, n) \notin \{(n-1, n), (2, 4)\}, \\
i = 3l + 2 & for & n = l(2m+1) + 2m, \\
i = 0 & for & m = n - 1, \\
i = 2 & for & n = 2m + 1, \text{ and} \\
i = 1 & for & m = 2 \text{ and } n = 4.
\end{array}
$$

*Furthermore there does not exist a pair of latin squares of order $n$ that intersect in precisely $x > i$ disjoint $m$-flowers.*

*Proof.* The necessary conditions for this result follow from Lemmas 4.1, 4.2, 4.4, 4.5 and 4.13. The sufficient conditions follow from Lemmas 4.3, 4.11, 4.19, 4.20, 4.23, 4.24, 4.25, 4.26, 4.27, 4.28, 4.29, 4.30, 4.31, 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39 and 4.40. □

## Acknowledgments

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