Generating functions attached to some infinite matrices

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Abstract

Let V be an infinite matrix with rows and columns indexed by the positive integers, and entries in a field F. Suppose that $v_{i,j}$ only depends on $i-j$ and is 0 for $|i-j|$ large. Then V^n is defined for all n, and one has a "generating function" $G = \sum a_{1,1}(V^n)z^n$. Ira Gessel has shown that G is algebraic over $F(z)$. We extend his result, allowing $v_{i,j}$ for fixed $i-j$ to be eventually periodic in i rather than constant. This result and some variants of it that we prove will have applications to Hilbert-Kunz theory.

1 Introduction

Throughout, Λ is a ring with identity element 1. Suppose that $w_{i,j}$, i and j ranging over the positive integers, are in Λ and that $w_{i,j} = 0$ whenever $i-j$ lies outside a fixed finite set. Then if W is the infinite matrix $|w_{i,j}|$, one may speak of W^n for all $n \geq 0$, and one gets a generating function $G(W) = \sum_0^\infty a_n z^n$ in $\Lambda[[z]]$, where a_n is the (1,1) entry in the matrix W^n. We shall prove:

Theorem I. *Suppose that $w_{i,j} = 0$ if $i-j \notin \{-1,0,1\}$, and that $w_{i+1,j+1} = w_{i,j}$ unless $i = j = 1$. Suppose further that $\Lambda = M_s(F)$, F a field, so that $G(W)$ may be viewed as an s by s matrix with entries in $F[[z]]$. Then these matrix entries are algebraic over $F(z)$.*

Corollary. *Let F be a field and $v_{i,j}$, i and j ranging over the positive integers, be in F. Suppose:*

(a) $v_{i,j} = 0$ whenever $i-j$ lies outside a fixed finite set.

(b) For fixed r in Z, $v_{i,i+r}$ is an eventually periodic function of i.

Then if V is the matrix $|v_{i,j}|$, the generating function $G(V)$ is algebraic over $F(z)$.

Proof. To derive the corollary we choose s so that:

(1) $v_{i,j} = 0$ whenever $i \le s$ and $j > 2s$ or $j \le s$ and $i > 2s$.

(2) $v_{i+s,j+s} = v_{i,j}$ whenever $i + j \ge s + 2$.

We then write the initial $2s$ by $2s$ block in V as $\left|\begin{smallmatrix} D & C \\ A & B \end{smallmatrix}\right|$ with A, B, C, D in $M_s(F)$. Our choice of s tells us that V is built out of s by s blocks, where the blocks along the diagonal are a single D, followed by B's, those just below a diagonal block are A's, those just above a diagonal block are C's, and all other entries are 0. Now let $\Lambda = M_s(F)$ and $W = |w_{i,j}|$ where $w_{i+1,i} = A$, $w_{i,i+1} = C$, $w_{1,1} = D$, $w_{i,i} = B$ for $i > 1$, and all other $w_{i,j}$ are 0. View $G(W)$ as an s by s matrix with entries in $F[[z]]$. One sees easily that $G(V)$ is the (1,1) entry in this matrix, and Theorem I applied to W gives the corollary. $\square$

Remark. *When $v_{i,j}$ only depends on $i - j$, the above corollary is due to Gessel. (When the matrix entries of V are all 0's and 1's the result is contained in Corollary 5.4 of [1]. The restriction on the matrix entries isn't essential in Gessel's proof, as one can use a generating function for walks with weights.)*

Our proof of Theorem I is easier than Gessel's proof of his special case of the corollary. The reason for this is that by working over Λ rather than over F we are able to restrict our study to walks with step-sizes in $\{-1, 0, 1\}$. (A complication, fortunately minor, is that the weights must be taken in the non-commutative ring Λ.) Our proof is well-adapted to finding an explicit polynomial relation between $G(V)$ and z; we'll work out a few examples. This paper would not have been possible without Ira Gessel's input. I thank him for showing me tools of the combinatorial trade.

2 Walks and generating functions

Definition 2.1. *If $l \ge 0$, an ordered $l+1$-tuple $\alpha = (\alpha_0, \ldots, \alpha_l)$ of integers is a (Motzkin) walk of length $l = l(\alpha)$ if each of $\alpha_1 - \alpha_0, \ldots, \alpha_l - \alpha_{l-1}$ is in $\{-1, 0, 1\}$.*

We say that the start of the walk is α_0, the finish is α_l, and that α is a walk from α_0 to α_l.

Definition 2.2. *If α and β are walks of lengths l and m, the concatenation $\alpha\beta$ of α and β is the walk $(\alpha_0, \ldots, \alpha_l, \alpha_l + (\beta_1 - \beta_0), \ldots, \alpha_l + (\beta_m - \beta_0))$ of length $l + m$.*

Now let Λ be a ring with identity element 1, and A, B, C, D lie in Λ. To each walk α we attach weights $w(\alpha)$ and $w^*(\alpha)$ in Λ:

Definition 2.3. *If $l(\alpha) = 0$, $w(\alpha) = w^*(\alpha) = 1$. If $l(\alpha) > 0$, $w(\alpha) = U_1 \cdot \ldots \cdot U_l$ where $U_i = A$, B or C according as $\alpha_i - \alpha_{i-1}$ is -1, 0, or 1. The definition of $w^*(\alpha)$ is the same with one change: if $\alpha_i = \alpha_{i-1} = 0$ then $U_i = D$ rather than B.*

Evidently $w(\alpha\beta) = w(\alpha)w(\beta)$. Furthermore $w^*(\alpha\beta) = w^*(\alpha)w^*(\beta)$ whenever α and β are walks from 0 to 0.

Definition 2.4. *α is "standard" if each $\alpha_i \geq \alpha_l$. Note that a walk from 0 to 0 is standard if and only if each $\alpha_i \geq 0$.*

Definition 2.5. *α is "primitive" if $l(\alpha) > 0$, $\alpha_0 = \alpha_l$ and no α_i with $0 < i < l$ is α_0. Note that a standard walk from 0 to 0 is primitive if and only if $l(\alpha) > 0$ and each α_i, $0 < i < l$, is > 0.*

Definition 2.6.

(1) $G(w) = \sum w(\alpha) z^{l(\alpha)}$, the sum extending over all standard walks from 0 to 0. $H(w)$ is the sum extending over all primitive standard walks from 0 to 0.

(2) $G(w^*)$ and $H(w^*)$ are defined similarly, using $w^*(\alpha)$ in place of $w(\alpha)$.

Lemma 2.7. *Let $G = G(w)$, $H = H(w)$. Then, in $\Lambda[[z]]$:*

(1) $G = 1 + H + H^2 + \cdots$

(2) $H = Bz + CGAz^2$

Proof. Every standard walk from 0 to 0 of length > 0 is either primitive or uniquely a concatenation of two or more primitive standard walks from 0 to 0. The multiplicative property of w now gives (1). To prove (2) note that the primitive standard walk $(0,0)$ has $w = B$. And a primitive standard walk from 0 to 0 of length $l > 1$ is a concatenation of $(0,1)$, a standard walk, β, from 0 to 0 of length $l-2$ and $(0,-1)$. Then $w(\alpha) = Cw(\beta)A$. Since $\alpha \to \beta$ gives a 1–1 correspondence between primitive standard walks of length l from 0 to 0 and standard walks of length $l-2$ from 0 to 0, we get the result. □

Corollary 2.8. *If $G = G(w)$, then $G - 1 - (BG)z - (CGAG)z^2 = 0$ in $\Lambda[[z]]$.*

Proof. By (1) of Lemma 2.7, $(1-H)\cdot G = 1$. Substituting $H = Bz + CGAz^2$ gives the result. □

Theorem 2.9. *Suppose that $\Lambda = M_s(F)$, F a field, so that $G(w)$ may be viewed as an s by s matrix with entries in $F[[z]]$. Then these matrix entries, $u_{i,j}$, are algebraic over $F(z)$.*

Proof. Let $U = |U_{i,j}|$ be an s by s matrix of indeterminates over F, and $p_{i,j}$ be the (i,j) entry in $U - I_s - (BU)z - (CUAU)z^2$. The $p_{i,j}$ are degree 2 polynomials in $U_{1,1}, \ldots, U_{s,s}$ with coefficients in $F[z]$. By Corollary 2.8, $p_{i,j}(u_{1,1}, \ldots, u_{s,s}) = 0$. Now $p_{i,j} = U_{i,j} - \delta_{i,j} - zf_{i,j}(U_{1,1}, \ldots, U_{s,s}, z)$ where the $f_{i,j}$ are polynomials with coefficients in F. It follows that the Jacobian matrix of the $p_{i,j}$ with respect to the $U_{i,j}$, evaluated at $(u_{1,1}, \ldots, u_{s,s})$, is congruent to $I_{s^2} \mod z$ in the s^2 by s^2 matrix ring over $F[[z]]$, and so is invertible. Thus $(u_{1,1}, \ldots, u_{s,s})$ is an isolated component of the intersection of the hypersurfaces $p_{i,j}(U_{1,1}, \ldots, U_{s,s}) = 0$, and so its co-ordinates, $u_{1,1}, \ldots, u_{s,s}$, are algebraic over $F(z)$. □

Remark. *We sketch a proof, based on the Nullstellensatz and Nakayama's Lemma, of the result from algebraic geometry used in the last sentence above. Suppose then that $K \subset L$ are fields, that $f_1, \ldots, f_n$ are in $K[x_1, \ldots, x_n]$, and that $a_1, \ldots, a_n$ are in L. Suppose further that each $f_i(a_1, \ldots, a_n) = 0$, and that $J(a_1, \ldots, a_n) \neq 0$, where J is the Jacobian determinant of the f_i with respect to the x_j. We shall show that each a_i is algebraic over K. We may assume that K is algebraically closed. The kernel of evaluation at $(a_1, \ldots, a_n)$ is a prime ideal, P, of $K[x_1, \ldots, x_n]$. Each f_i is in P and J is not in P. By the Nullstellensatz, $P \subset$ some $m = (x_1 - b_1, \ldots, x_n - b_n)$ with $J(b_1, \ldots, b_n) \neq 0$. Each f_i is in m. Writing f_i as a polynomial in $x_1 - b_1, \ldots, x_n - b_n$, and using the fact that $J(b_1, \ldots, b_n) \neq 0$, we find that $(P, m^2) = m$. Now P is prime, and it follows from Nakayama's Lemma that $P = m$. So $a_i = b_i$, and is in K.*

Lemma 2.10. $G(w^*)^{-1} - G(w)^{-1} = (B - D)z$.

Proof. The proof of Lemma 2.7 (1) shows that $G(w^*)^{-1} = 1 - H(w^*)$ with $H(w^*)$ as in Definition 2.6. So it suffices to show that $H(w) - H(w^*) = (B - D)z$. Now for a primitive walk α of length > 1 from 0 to 0 one cannot have $\alpha_{i-1} = \alpha_i = 0$, and so $w(\alpha) = w^*(\alpha)$. On the other hand, for the primitive walk $(0, 0)$, $w = B$ and $w^* = D$. This gives the lemma. □

Combining Lemma 2.10 with Theorem 2.9 we get:

Theorem 2.11. *If $\Lambda = M_s(F)$ the matrix entries of the s by s matrix $G(w^*)$ are algebraic over $F(z)$.*

Now let $W = |w_{i,j}|$ where $w_{i+1,i} = A$, $w_{i,i+1} = C$, $w_{1,1} = D$, $w_{i,i} = B$ for $i > 1$, and all the other $w_{i,j} = 0$. In view of Theorem 2.11 the proof of Theorem I will be complete once we show that $G(W) = G(w^*)$ where w^* is the weight function of Definition 2.3. The key to this is:

Lemma 2.12. *For $k \geq 1$ let $u_k^{(n)}$ be $\sum w^*(\alpha)$, the sum extending over all standard walks of length n from $k - 1$ to 0. Then:*

(1) $u_k^{(0)} = 1$ or 0 according as $k = 1$ or $k > 1$.

(2) $u_1^{(n+1)} = Du_1^{(n)} + Cu_2^{(n)}$.

(3) If $k > 1$, $u_k^{(n+1)} = Au_{k-1}^{(n)} + Bu_k^{(n)} + Cu_{k+1}^{(n)}$.

Lemma 2.12 has the following immediate corollaries, with the first proved by induction on n.

Corollary 2.13. *The first column vector in W^n is $(u_1^{(n)}, u_2^{(n)}, \ldots$.*

Corollary 2.14. *The $(1, 1)$ coefficient of W^n is $\sum w^*(\alpha)$, the sum extending over all standard walks of length n from 0 to 0. So $G(W) = G(w^*)$.*

It remains to prove Lemma 2.12. (1) is evident. Let α be a standard walk of length n from 0 or 1 to 0. Then $\beta = (0, \alpha_0, \dots, \alpha_n)$ is a standard walk of length $n+1$ from 0 to 0, and $w^*(\beta)$ is $Dw^*(\alpha)$ in the first case and $Cw^*(\alpha)$ in the second. Also each standard walk β of length $n+1$ from 0 to 0 arises in this way from some α; explicitly $\alpha = (\beta_1, \dots, \beta_n)$. Summing over β we get (2). Similarly, suppose that $k > 1$ and that α is a standard walk of length n from $k-2$, $k-1$ or k to 0. Then $\beta = (k-1, \alpha_0, \dots, \alpha_n)$ is a standard walk of length $n+1$ from $k-1$ to 0 and $w^*(\beta) = Aw^*(\alpha)$ in the first case, $Bw^*(\alpha)$ in the second, and $Cw^*(\alpha)$ in the third. Also, each standard walk β of length $n+1$ arises from such an α; explicitly $\alpha = (\beta_1, \dots, \beta_n)$. Summing over β we get (3), completing the proof.

Remark 2.15. *To calculate the matrix entries of $G(W)$ explicitly as algebraic functions of z by the method of Theorem 2.9 involves solving a system of s^2 quadratic equations in s^2 variables. This isn't practical when $s > 2$; in the next section we give a different proof of Theorem 2.9 that is often better adapted to explicit calculations.*

3 A partial fraction proof of Theorem 2.9

Theorem 3.1. *$\sum w(\alpha)x^{\alpha_0}$, the sum extending over all length n walks (not necessarily standard) with finish 0, is the element $(Ax + B + Cx^{-1})^n$ of $\Lambda[x, x^{-1}]$.*

Proof. Denote the sum by f_n. Since $f_0 = 1$ it's enough to show that $f_{n+1} = (Ax + B + Cx^{-1})f_n$. Let $v_k^{(n)}$ be the coefficient of x^k in f_n. Then $v_k^{(n)} = \sum w(\alpha)$, the sum extending over all length n walks from k to 0. The proof of (3) of Lemma 2.12, using all walks rather than all standard walks, shows that $v_k^{(n+1)} = Av_{k-1}^{(n)} + Bv_k^{(n)} + Cv_{k+1}^{(n)}$ for all k in Z, giving the result. ☐

Definition 3.2.

$M_0(w) = \sum w(\alpha)z^{l(\alpha)}$*, the sum extending over all* 0 *to* 0 *walks.*

$M_{-1}(w)$ *is the sum extending over all* -1 *to* 0 *(or* 0 *to* 1*) walks.*

M_1 *is the sum extending over all* 1 *to* 0 *(or* 0 *to* -1*) walks.*

We'll generally omit the w and just write M_0, M_{-1} or M_1.

Corollary 3.3. *Suppose that $i = 0$, -1 or 1. Then M_i is the coefficient of x^i in the element $\sum_0^\infty (Ax + B + Cx^{-1})^n z^n$ of $\Lambda[x, x^{-1}][[z]]$.*

Definition 3.4. *$J_0 = J_0(w)$ is $\sum w(\alpha)z^{l(\alpha)}$, the sum extending over all primitive* 0 *to* 0 *walks.*

Theorem 3.5.

(1) $M_0 = 1 + J_0 + J_0^2 + \cdots$.

(2) $G(w) = M_0 - M_1 M_0^{-1} M_{-1}$.

Proof. (1) follows from the multiplicative property of w, as in the proof of Lemma 2.7. So $M_0^{-1} = 1 - J_0$, and (2) asserts that $G(w) = M_0 + M_1 J_0 M_{-1} - M_1 M_{-1}$. If α is a walk from 0 to 0 let $r(\alpha)$ be the number of ways of writing α as a concatenation of a walk from 0 to -1 and a walk from -1 to 0. Also let $r_1(\alpha)$ be the number of ways of writing α as a concatenation of a walk from 0 to -1, a primitive walk from -1 to -1 and a walk from -1 to 0. The multiplicative property of w shows that $M_0 + M_1 J_0 M_{-1} - M_1 M_{-1} = \sum w(\alpha)(1 + r_1(\alpha) - r(\alpha)) z^{l(\alpha)}$, the sum extending over all walks from 0 to 0. If α is standard, $r_1(\alpha) = r(\alpha) = 0$. If α is not standard there is an i with $\alpha_i = -1$. Let $i_1 < i_2 < \cdots < i_r$ be those i with $\alpha_i = -1$. One sees immediately that $r(\alpha) = r$ and that $r_1(\alpha) = r - 1$. So $M_0 + M_1 J_0 M_{-1} - M_1 M_{-1}$ is the sum over the standard walks from 0 to 0 of $w(\alpha) z^{l(\alpha)}$, and this is precisely $G(w)$. ☐

Suppose now that $\Lambda = M_s(F)$, F a field, so that M_0, M_1 and M_{-1} may be viewed as s by s matrices with entries in $F[[z]]$. Theorem 3.5, (2), will give a new proof of Theorem 2.9 once we show that these matrix entries are algebraic over $F(z)$. The facts about the matrix entries of M_0, M_1 and M_{-1} follow from a standard partial fraction decomposition argument—we'll give our own version.

The algebraic closure of the field of fractions of $F[[z]]$ is a valued field with value group Q. Let Ω be the completion of this field and $\mathrm{ord} : \Omega \to Q \cup \{\infty\}$ be the ord function in Ω. Let Ω' consist of formal power series $\sum_{-\infty}^{\infty} a_i x^i$ with $a_i \in \Omega$ and $\mathrm{ord}\, a_i \to \infty$ as $|i| \to \infty$. Ω' has an obvious multiplication and is an overring of $F[x, x^{-1}][[z]]$. l_0, l_1 and l_{-1} are the Ω-linear maps $\Omega' \to \Omega$ taking $\sum a_i x^i$ to a_0, a_1 and a_{-1}. Note that $\overline{F(z)}$, the algebraic closure of $F(z)$, imbeds in Ω.

Lemma 3.6. *Suppose $\lambda \in \overline{F(z)}$ with $\mathrm{ord}\,\lambda \neq 0$. Then the element $x - \lambda$ of Ω' is invertible, and for all $k \geq 1$, $(x - \lambda)^{-k} = \sum_{-\infty}^{\infty} a_i x^i$ in Ω' with the a_i in $\overline{F(z)}$. In particular, l_0, l_1 and l_{-1} take each $(x - \lambda)^{-k}$ to an element of $\overline{F(z)}$.*

Proof. If $\mathrm{ord}\,\lambda > 0$, $x - \lambda = x(1 - \lambda x^{-1})$ has inverse $x^{-1}(1 + \lambda x^{-1} + \lambda^2 x^{-2} + \cdots)$, while if $\mathrm{ord}\,\lambda < 0$, $x - \lambda = -\lambda(1 - \lambda^{-1} x)$ has inverse $-\lambda^{-1}(1 + \lambda^{-1} x + \lambda^{-2} x^2 + \cdots)$. ☐

Lemma 3.7. *Let U_1 and U_2 be elements of $F[z, x]$. Suppose that $U_2 \equiv x^s \mod z$ for some s. Then U_2 has an inverse in $F[x, x^{-1}][[z]]$ and the coefficients of x^0, x^1 and x^{-1} in the element $U_1 U_2^{-1}$ of $F[x, x^{-1}][[z]]$ all lie in $\overline{F(z)}$.*

Proof. Write U_2 as $x^s(1 - zp)$ with p in $F[x, x^{-1}, z]$. Then $x^{-s}(1 + zp + z^2 p^2 + \cdots)$ is the desired inverse of U_2. If λ in Ω has ord 0 then $1 - zp(\lambda, \lambda^{-1}, z)$ has ord 0 and cannot be 0. So when we factor U_2 in $\overline{F(z)}[x]$ as $q \cdot \Pi(x - \lambda_i)^{c_i}$ with q in $F[z]$ and λ_i in $\overline{F(z)}$, no ord (λ_i) can be 0. View $U_1 U_2^{-1}$ as an element of $\overline{F(z)}(x)$. As such it is an $\overline{F(z)}$ linear combination of powers of x and powers of the $(x - \lambda_i)^{-1}$. Since l_0, l_1 and l_{-1} are Ω-linear they are $\overline{F(z)}$-linear. Lemma 3.6 then tells us that $U_1 U_2^{-1}$, viewed as an element of Ω', is mapped by each of l_0, l_1 and l_{-1} to an element of $\overline{F(z)}$. This completes the proof. ☐

Lemma 3.8. *Let A, B and C be in $M_s(F)$ and $u \in F[x, x^{-1}][[z]]$ be an entry in the matrix $(I_s - z(Ax + B + Cx^{-1}))^{-1}$. Then the coefficients of x^0, x^1 and x^{-1} in u all lie in $\overline{F(z)}$.*

Proof. u may be written as U_1/U_2 where U_1 and U_2 are in $F[z,x]$ and $U_2 = \det(xI_s - z(Ax^2+Bx+C))$. Then $U_2 \equiv x^s \mod z$, and we apply Lemma 3.7. □

Corollary 3.9. *If $\Lambda = M_s(F)$, F a field, then the matrix entries of M_0, M_1 and M_{-1} are algebraic over $F(z)$. (So by Theorem 3.5 the same is true of the matrix entries of $G(w)$.)*

Proof. $(I_s - z(Ax+B+Cx^{-1}))^{-1} = \sum_0^\infty (Ax+B+Cx^{-1})^n z^n$, and we combine Lemma 3.8 with Corollary 3.3. □

4 Examples

Example 4.1. *For i, j positive integers define $v_{i,j}$ by:*

(1) $v_{i,j} = 1$ if $i-j \in \{-1,0,1\}$.

(2) $v_{i,j} = 1$ if $j = i+3$ and i is odd.

(3) All other $v_{i,j}$ are 0.

We calculate $G(V)$ where $V = |v_{i,j}|$. If we take $s = 2$, (1) and (2) in the corollary to Theorem I are satisfied, and $D = B = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Let $G = G(w) = G(w^*)$. G is a 2 by 2 matrix $\left(\begin{smallmatrix}g_1&g_2\\g_3&g_4\end{smallmatrix}\right)$ with entries in $F[[z]]$, and $g_1 = G(V)$. By Corollary 2.8, $CGAGz^2 + BGz - G + I_2 = 0$. Two of the four equations this gives are:

$$\begin{aligned} z^2 g_1 g_3 + z(g_1+g_3) - g_3 &= 0 \\ z^2 g_3^2 + z(g_1+g_3) - g_1 + 1 &= 0 \end{aligned}$$

Solving the first equation for g_3 and substituting in the second we find that $G(V) = g_1$ is a root of:

$$(z^5 - z^4)x^3 + (3z^4 - 4z^3 + 2z^2)x^2 + (2z^3 - 4z^2 + 3z - 1)x + (z^2 - 2z + 1) = 0.$$

Example 4.2. *For i, j positive integers define $v_{i,j}$ by:*

(1) $v_{i,j} = 1$ if $i-j \in \{-1,0,1\}$.

(2) $v_{i,j} = 1$ if $j = i+3$ and i is even.

(3) All other $v_{i,j}$ are 0.

We calculate $G(V)$ where $V = |v_{i,j}|$. Since $v_{2,5} = 1$, condition (1) of the corollary to Theorem I is not met when $s = 2$, and we instead take $s = 4$.

Now

$$D = B = \begin{pmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{pmatrix} \qquad A = \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\1&0&1&0\end{pmatrix}.$$

Let the entries in the first column of the 4 by 4 matrix $G = G(w)$ be a, b, c and d. Examining the entries in the first column of the matrix equation $G = BGz + CGAGz^2 + I_4$ we see:

$$\begin{aligned} a &= (a+b)z + 1 \\ b &= (a+b+c)z + bdz^2 \\ c &= (b+c+d)z \\ d &= (c+d)z + d(a+c)z^2 \end{aligned}$$

Using Maple to eliminate b, c, and d from this system we find that $a = G(V^*)$ is a root of:

$$\begin{aligned} &(z^2)\cdot(z-1)^3\cdot(3z^2+3z-2)\cdot x^3 \\ &+(z-1)^2\cdot(9z^4+6z^3-11z^2+5z-1)\cdot x^2 \\ &+(2z-1)\cdot(5z^4-13z^2+9z-2)\cdot x \\ &+(2z-1)^2\cdot(z^2+2z-1) = 0. \end{aligned}$$

Example 4.3. *For i, j positive integers define $v_{i,j}$ by:*

(1) $v_{i,j} = 1$ if $i - j \in \{-1, 1\}$.

(2) $v_{i,j} = 1$ if $i - j \in \{-3, 3\}$ and $i \equiv 2 \pmod 3$.

(3) All other $v_{i,j}$ are 0.

We calculate $G(V)$ where $V = |v_{i,j}|$. Take $s = 3$. Then:

$$A = \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 0&0&0 \end{pmatrix} \qquad B = D = \begin{pmatrix} 0&1&0 \\ 1&0&1 \\ 0&1&0 \end{pmatrix} \qquad C = \begin{pmatrix} 0&0&0 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}.$$

The determinant of the matrix $xI_3 - z(Ax^2 + Bx + C)$ is $-x^2(zx^2 + (3z^2 - 1)x + z)$. The splitting field of this polynomial over $F(z)$ is the extension of $F(z)$ generated by $\sqrt{1 - 10z^2 + 9z^4}$. The arguments of section 3 show that M_0, M_1 and M_{-1} have entries in this extension field. It's not hard to write down these matrices explicitly using the partial-fraction decomposition argument. Theorem 3.5 and a Maple calculation then show that the $(1,1)$ entry in $G(w)$ is $4/(3 + z^2 + \sqrt{1 - 10z^2 + 9z^4})$. Since $D = B$, $G(w^*) = G(w)$, and this $(1,1)$ entry is the desired $G(V)$.

5 More algebraic generating functions

Definition 5.1. *Suppose that $\Lambda = M_s(F)$, F a field, and that A, B, C, D are in Λ. Then $\mathcal{L} \subset$ the field of fractions of $F[[z]]$ is the extension field of $F(z)$ generated by the matrix entries of the M_0, M_1 and M_{-1} of Definition 3.2.*

Remark 5.2. *As we've seen $\mathcal{L}$ contains the matrix entries of $G(w)$ and $G(w^*)$ and is finite over $F(z)$. Indeed the proofs of Lemmas 3.7, 3.8 and Corollary 3.9 show that $\mathcal{L} \subset$ a splitting field over $F(z)$ of the polynomial* $\det|xI_s - z(Ax^2 + Bx + C)|$. *One can say a bit more. The above polynomial splits into linear factors in $\Omega[x]$, and one may view its splitting field as a subfield of the valued field Ω. By examining the partial-fraction decomposition one finds that $\mathcal{L}$ is fixed elementwise by each automorphism of the splitting field that is the identity on $F(z)$ and permutes the roots that have positive* ord *among themselves.*

The goal of this section is to show that some generating functions related to $G(w)$ also have their matrix entries in $\mathcal{L}$. These results are used in [3] to show the algebraicity (under a conjecture) of certain Hilbert-Kunz series and Hilbert-Kunz multiplicities; see Theorems 3.1 and 3.4 of that note.

Now let $u_k^{(n)}$ be as in Lemma 2.12 where k is a positive integer. By definition, $G^*(w) = \sum u_1^{(n)} z^n$.

Lemma 5.3. $\sum_n u_{k+1}^{(n)} z^n = G(w)(Az) \sum_n u_k^{(n)} z^n$.

Proof. A standard walk from k to 0 can be written in just one way as the concatenation of a standard walk from k to k, the walk $(k, k-1)$ and a standard walk from $k-1$ to 0. □

Corollary 5.4. *Fix $k \geq 1$. The generating function arising from the $(k, 1)$ entries of the matrices W^n has its matrix entries in $\mathcal{L}$.*

Proof. Corollary 2.13 shows that this generating function is $\sum_n u_k^{(n)} z^n$, and we use Lemma 5.3 and induction. □

Definition 5.5. $G_r^* = \sum \binom{\alpha_0}{r} w^*(\alpha) z^{l(\alpha)}$, *the sum extending over all standard walks finishing at 0.*

Evidently $G_0^* = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} u_{k+1}^{(n)} z^n$. By Lemma 5.3, this is

$$\left(1 + G(w)Az + (G(w)Az)^2 + \cdots\right) G(w^*).$$

So:

Lemma 5.6. $(1 - G(w)Az)G_0^* = G(w^*)$.

A variant of this is:

Lemma 5.7. $(1 - G(w)Az)G_{r+1}^* = G(w)(Az)G_r^*$.

Proof. We introduce new weight functions $w|t$ and $w^*|t$ as follows. Replace Λ, A and C by $\Lambda[[t]]$, $A(1+t)$ and $C(1+t)^{-1}$, and let $w|t$ and $w^*|t$ be the new w and w^* that arise. If $\alpha = (\alpha_0, \ldots, \alpha_l)$ is a walk from k to 0 then there are $k = \alpha_0$ more steps of size -1 in the walk than there are steps of size 1. It follows that $w|t(\alpha)$ and $w^*|t(\alpha)$ are $(1+t)^{\alpha_0} w(\alpha)$ and

$(1+t)^{\alpha_0} w^*(\alpha)$. In particular, $G(w|t) = G(w)$ and $G(w^*|t) = G(w^*)$. Applying Lemma 5.6 in this new situation we find:

$$((1 - G(w)Az) - G(w)Azt) \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (1+t)^k u_{k+1}^{(n)} z^n \right) = G(w^*).$$

In particular, the coefficient of t^{r+1} in the left-hand side of the above equation is 0. Evaluating this coefficient we get the lemma. □

Theorem 5.8. *Let $a_1, a_2, \dots$ be elements of F. Suppose there is a polynomial function whose value at j is a_j for sufficiently large j. Let $R_n = \sum_1^\infty a_k u_k^{(n)}$. Then all the matrix entries of $\sum R_n z^n$ lie in $\mathcal{L}$.*

Proof. Corollary 5.4 shows that the generating function arising from any single $(j,1)$ entry has matrix entries in $\mathcal{L}$. So we may assume that $j \to a_j$ is a polynomial function. Since any polynomial function is an F-linear combination of the functions $j \to \binom{j-1}{r}$, $r = 0, 1, 2, \dots$ we may assume $a_j = \binom{j-1}{r}$. But then $\sum R_n z^n$ is G_r^*, and we use Lemmas 5.6, 5.7 and induction. □

Corollary 5.9. *Suppose $V = |v_{i,j}|$, $i, j \geq 1$ is a matrix with entries in F satisfying:*

(1) $v_{i,j} = 0$ whenever $i \leq s$ and $j > 2s$ or $j \leq s$ and $i > 2s$.

(2) $v_{i+s,j+s} = v_{i,j}$ whenever $i + j \geq s + 2$.

(3) The initial $2s$ by $2s$ block in V is $\left(\begin{smallmatrix} D & C \\ A & B \end{smallmatrix}\right)$.

Suppose further that $a_1, a_2, \dots$ are in F and that for each i, $1 \leq i \leq s$, there is a polynomial function agreeing with $k \to a_{i+sk}$ for large k. Let $v_i^{(n)}$ be the $(i,1)$ entry in V^n. Then $\sum_{i,n} v_i^{(n)} a_i z^n$ is in $\mathcal{L}$.

Proof. Construct W as in the proof of the corollary to Theorem I. As the first column of W^n is $u_1^{(n)}, u_2^{(n)}, \dots$ it follows that $v_{i+sk}^{(n)}$ is just the $(i,1)$ entry in the s by s matrix $u_{k+1}^{(n)}$. Theorem 5.8 shows that for each i with $1 \leq i \leq s$, $\sum_{k,n} v_{i+sk}^{(n)} a_{i+sk} z^n$ is in $\mathcal{L}$. Summing over i we get the result. □

The following results may seem artificial but they're what we need for the applications to Hilbert-Kunz theory in [3].

Lemma 5.10. *Let Y be a finite dimensional vector space over F, $T : Y \to Y$ and $l : Y \to F$ linear maps and $y_1, y_2, \dots$ a sequence in Y. Let V and s be as in Corollary 5.9. Suppose that for each i, $1 \leq i \leq s$, each co-ordinate of y_{i+sk} with respect to a fixed basis of Y is an eventually polynomial function of k. Define $y^{(n)}$ inductively by $y^{(0)} = 0$, $y^{(n+1)} = Ty^{(n)} + \sum v_i^{(n)} y_i$—see Corollary 5.9 for the definition of $v_i^{(n)}$. Then $\sum l\left(y^{(n)}\right) z^n$ is in $\mathcal{L}$.*

Proof. $(I - zT)\sum y^{(n)}z^n = \sum_{i,n} v_i^{(n)} y_i z^{n+1}$. By Corollary 5.9, all the co-ordinates of $(I - zT)\sum y^{(n)}z^n$ with respect to a fixed basis of Y lie in $\mathcal{L}$. Since $\det|I - zT|$ is a non-zero element of $F(z) \subset \mathcal{L}$, the same is true of the co-ordinates of $\sum y^{(n)}z^n$, giving the lemma. □

Theorem 5.11. *Suppose X is a vector space over F, Y is a finite dimensional subspace, $T : X \to X$ is linear with $T(Y) \subset Y$, and $E_1, E_2, \ldots$ lie in X. Suppose further that $T(E_j) = \sum v_{i,j}E_i + y_j$, where $V = |v_{i,j}|$ is as in Lemma 5.10 and $y_1, y_2, \ldots$ is a sequence in Y satisfying the condition of Lemma 5.10. Then if $l : X \to F$ is linear with each $l(E_i) = 0$, the power series $\sum_0^\infty l\left(T^n(E_1)\right) z^n$ is in $\mathcal{L}$.*

Proof. Define $y^{(n)}$ as in Lemma 5.10. Using the identity $\sum_j v_{i,j}v_j^{(n)} = v_i^{(n+1)}$ and induction we find that $T^n(E_1) = \sum v_i^{(n)}E_i + y^{(n)}$. So $l\left(T^n(E_1)\right) = l(y^{(n)})$ and we apply Lemma 5.10. □

The following example is closely related to our calculations in [2]. We explain how this and similar examples relate to Hilbert-Kunz theory in [3].

Example 5.12. *Suppose δ_1 and δ_2 are a basis of Y, that $y_1 = 6\delta_1$ and that $y_k = (8k - 2)\delta_1 + \delta_2$, $k > 1$. Suppose further that $T(\delta_1) = 16\delta_1$, $T(\delta_2) = 4\delta_1 + 4\delta_2$, $T(E_1) = E_1 + E_2 + y_1$, and that $T(E_k) = E_{k-1} + E_{k+1} + y_k$ for $k > 1$. Suppose $l : X \to F$ takes δ_1 to 1, and δ_2 and each E_k to 0. We shall calculate the power series $S = \sum l\left(T^n(E_1)\right) z^n$ explicitly. (Theorem 2.4 of [3] and the observation following it arise from our formula for S.)*

In the above situation, $v_{1,1} = v_{i,i+1} = v_{i+1,i} = 1$ and all other $v_{i,j}$ are 0. So we can take $s = 1$, $A = C = D = 1$ and $B = 0$. Since $s = 1$, $v_k^{(n)} = u_k^{(n)}$. It follows from this and the definition of the y_k that $\sum_{k,n} v_k^{(n)} y_k z^{n+1} = z(8G_1^* + 6G_0^*)\delta_1 + z(G_0^* - G(w^*))\delta_2$.

Now the matrix of $T : Y \to Y$ on the basis (δ_1, δ_2) is $\left(\begin{smallmatrix} 16 & 4 \\ 0 & 4 \end{smallmatrix}\right)$. It follows that the matrix of $I - zT$ is $\left(\begin{smallmatrix} 1-16z & -4z \\ 0 & 1-4z \end{smallmatrix}\right)$ with inverse $\frac{1}{(1-16z)(1-4z)}\left(\begin{smallmatrix} 1-4z & 4z \\ 0 & 1-16z \end{smallmatrix}\right)$. Since S is the coefficient of δ_1 in $\sum l(y^{(n)})z^n = (I - zT)^{-1} \cdot \sum_{k,n} v_k^{(n)} y_k z^{n+1}$, the last paragraph shows that $(1 - 16z)(1 - 4z)S = (z - 4z^2)(8G_1^* + 6G_0^*) + 4z^2(G_0^* - G(w^*))$. It only remains to calculate $G(w^*)$, G_0^* and G_1^*.

Lemma 2.7 and Corollary 2.8 show that $H(w) = z^2G(w)$, and $z^2G(w)^2 - G(w) + 1 = 0$. So $G(w)$ and $H(w)$ are $\frac{1-\sqrt{1-4z^2}}{2z^2}$ and $\frac{1-\sqrt{1-4z^2}}{2}$. Lemma 2.10 then shows $G(w^*) = \frac{1}{2z(1-2z)}(-1 + 2z + \sqrt{1-4z^2})$. Making use of Lemmas 5.6 and 5.7 we find that G_0^* and G_1^* are $\frac{1}{1-2z}$ and $\frac{1}{2(1-2z)^2}(-1 + 2z + \sqrt{1-4z^2})$. A brief calculation then gives the explicit formula:

$$(1 - 16z)(1 - 4z)(1 - 2z)^2 S = 4z(1 - 2z)^2 + (2z - 12z^2)\sqrt{1 - 4z^2}.$$

References

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