Graph products and new solutions to Oberwolfach problems

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Abstract

We introduce the *circle product*, a method to construct simple graphs starting from known ones. The circle product can be applied in many different situations and when applied to regular graphs and to their decompositions, a new regular graph is obtained together with a new decomposition. In this paper we show how it can be used to construct infinitely many new solutions to the Oberwolfach problem, in both the classic and the equipartite case.

1 Introduction

In this paper we will only deal with undirected simple graphs. For each graph Γ we will denote by $V(\Gamma)$ and $E(\Gamma)$ its vertex–set and edge–set, respectively. By K_v we will denote the *complete graph* on v vertices and by $K_{\{s:r\}}$ the *complete equipartite* graph having r *parts* of *size* s.

The number of edges incident with a vertex a is called the *degree* of a in Γ and is denoted $d_\Gamma(a)$. We will drop the index referring to the underlying graph if the reference is clear. All over the paper we will consider graphs without isolated vertices, i.e., vertices of degree zero. It is well known that a graph in which all vertices have the same degree t is called *t-regular* or simply *regular*.

By $C_n = (a_1, \dots, a_n)$ we will denote a *cycle* of length n, namely a simple graph with vertices $a_1, \dots, a_n$ and edges $[a_i, a_{i+1}]$, where the indices are to be considered modulo n. Also, by $\Gamma_1 \sqcup \Gamma_2$ we will denote the disjoint union of two graphs, namely $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, $V(\Gamma_1 \sqcup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \sqcup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$.

A *decomposition* of a graph K is a set $\mathcal{F} = \{\Gamma_1, \dots, \Gamma_t\}$ of subgraphs of K whose edges partition, altogether, the edge–set of K. If all graphs Γ_i are isomorphic to a given graph

Γ, such a decomposition is generally called a $\Gamma-$*decomposition* of K. If k is a positive integer, a *k-factor* of a graph K is a $k-$regular spanning subgraph and a *k-factorization* of K is a decomposition of K into $k-$factors.

The problem of determining whether a given graph K admits a $\Gamma-$decomposition, for a specified graph Γ, or admits a $k-$factorization with specified properties (for example, on the type of factors or on the automorphism group) can be very difficult to solve. A wide literature exists on these topics, too wide to be mentioned here; therefore, we refer the reader to [14].

Considerable attention has been devoted to the so called *Oberwolfach problem*, both in its classic and generalized formulations.

When v is an *odd* integer, the classic Oberwolfach problem $OP(F)$ asks for a $2-$factorization of the complete graph K_v in which any $2-$factor is isomorphic to the $2-$factor F. This problem was posed by Ringel and first mentioned in [19]. In [21] the authors consider a variant of the Oberwolfach problem asking for a $2-$factorization of $K_v - I$, the complete graph on an *even* set of v vertices minus a $1-$factor I, into isomorphic $2-$factors, and the same notation $OP(F)$ is used. Obviously, in both cases, the $2-$factor F is a disjoint union of cycles. The notation $F(l_1^{s_1}, \ldots, l_r^{s_r})$ will be used to denote a $2-$factor consisting of s_i cycles of length l_i for $i = 1, \ldots, r$ (s_i omitted when equal to 1) and $OP(l_1^{s_1}, \ldots, l_r^{s_r})$ will denote the corresponding Oberwolfach problem. With the same meaning, if $L_1, \ldots, L_h$ are multisets of integers we will set $F(L_1^{s_1}, \ldots, L_h^{s_h})$ and $OP(L_1^{s_1}, \ldots, L_h^{s_h})$. The notation $L_i^{s_i}$ means that all integers in L_i are repeated s_i times. With the notation tL_i the integers in the multiset L_i have to be multiplied by t. We refer to [5] for a survey on known results. In particular, it is well known that $OP(4,5)$, $OP(3,3,5)$, $OP(3,3)$ and $OP(3,3,3,3)$ have no solutions and up to now there is no other known instance with no solution. The Oberwolfach problem $OP(m, m, \ldots, m)$, $m \geq 3$, was completely solved in [1] in the classic case and in [20] for v even. The special case $m = 3$, the famous Kirkman's schoolgirl problem, was solved in [28]. Moreover every instance (except for those mentioned above) has a solution when $v \leq 40$ [15], together with a large number of other special cases for which we refer to [5]. Nevertheless, as v increases, the known results solve only a small fraction of the problem and a general answer seems really hard to find. Recently, complete solutions to the Oberwolfach problem for an infinite set of orders were found in [6]. Moreover it is proved in [4] that when v is even, $OP(F)$ has a solution for any bipartite $2-$factor F. In [22] the author gave a generalization of the problem considering $2-$factorizations of the complete equipartite graph $K_{\{s:r\}}$ into isomorphic 2-factors. Obviously this generalization reduces to the classic Oberwolfach problem when $s = 1$ and to the variant of [21] when $s = 2$. We will use the same notations as before, namely $OP(s : r; l_1^{t_1}, \ldots, l_h^{t_h})$ will denote the Oberwolfach problem for the complete equipartite graph $K_{\{s:r\}}$ in which all $2-$factors are of type $(l_1^{t_1}, \ldots, l_h^{t_h})$. Moreover, in [23], the problem was completely solved in case the $2-$factors are *uniform* of length t, i.e., all cycles have the same length t, $t \geq 3$. The generalized Oberwolfach problem is denoted by $OP(s : r; t)$ in this case and it is proved in [23] that it has a solution if and only if rs is divisible by t, $s(r-1)$ is even, t is even if $r = 2$ and $(r, s, t) \neq (3, 2, 3), (3, 6, 3), (6, 2, 3), (2, 6, 6)$. This result reduces to that found by Piotrowski in [27] when the complete bipartite graph is considered. Moreover, the complete bipartite graph $K_{\{s:2\}}$ does not contain cycles of odd length; hence, its 2–factors can only have cycles of even length. Also, it admits a 2-factorization only if s is even. In [27], Piotrowski proved the sufficiency of all these conditions when $s \neq 6$, namely he proved that $OP(s:2; 2c_1, \ldots, 2c_t)$ has a solution for each set $\{c_1, \ldots, c_t\}$ with $\sum c_i = s$ and $c_i \geq 2$, except for $OP(6:2; 6, 6)$ which has no solution. This completely solves the problem for the bipartite case.

When speaking of a symmetric solution $\mathcal{F}$ to the Oberwolfach problem we mean that $\mathcal{F}$ admits an automorphism group G whose action on a set of objects (mainly vertices, edges or factors) satisfies some properties. A classification result has been achieved in the case where G acts 2–transitively on the set of vertices, [3]. The case where G acts sharply transitively on the vertex–set has been considered in [9]. Also, sufficient conditions for the existence of sharply vertex–transitive solutions to $OP(k^m)$, km odd, with an additional property are provided in [8, Theorem 8.1]. The assumption that seems to be successful for constructing new symmetric solutions to the classic Oberwolfach problem is that the action of G on the vertex–set is 1–rotational. The concept of a 1–rotational solution to the classic Oberwolfach problem has been formally introduced and studied, for the very first time, in [10]. In general, a $k-$factorization of a complete graph is said to be $1-$rotational under a group G if it admits G as an automorphism group acting sharply transitively on all but one vertex, called ∞, which is fixed by each element of G. As pointed out in [10], if a $1-$rotational $k-$factorization $\mathcal{F}$ of K_v exists under a group G, then the vertices of K_v can be renamed over $G \cup \{\infty\}$ in such a way that G acts on vertices by right translation (with the condition $\infty + g = \infty$ for any $g \in G$) and $\mathcal{F}$ is preserved under the action of G, namely $F + g \in \mathcal{F}$ for any $F \in \mathcal{F}$ and $g \in G$. Of course, the graph $F + g$ is obtained by replacing each vertex of the $k-$factor F, say x, with $x + g$, for any $g \in G$. Moreover, the $k-$factorization $\mathcal{F}$ can be obtained as the $G-$orbit of any of its k–factors and when $k = 2$ it readily follows that all cycles in $\mathcal{F}$ passing through ∞ have the same length.
It is well known that for each odd order group G there exists a $1-$factorization which is $1-$rotational under G, [7]. The same result does not hold for $1-$rotational $2-$factorizations: groups have even order in this case and it was proved in [10] that they must satisfy some prescribed properties. It was also proved in the same paper that each $1-$rotational $2-$factorization is a solution to an Oberwolfach problem. Obviously $1-$rotational solutions should be more rare, nevertheless the group structure can be a useful tool to construct them. In fact, new solutions to Oberwolfach problems were constructed in [10] by working entirely in the group. In particular for each symmetrically sequenceable group G, [16], of order $2n$ a $1-$rotational solution to $OP(2n+1)$ under G can be constructed. For completeness, we recall that each solvable group with exactly one involution, except for the quaternion group Q_8, is symmetrically sequenceable, [2]. A wider class of groups realizing 1-rotational solutions to the classic Oberwolfach Problem can be found in [29]. Necessary conditions for the existence of a cyclic 1–rotational solution to $OP(3, 2l_1, \ldots, 2l_t)$, with a complete characterization when $t = 1$, are given in [11]. Although the concept of a 1–rotational solution to the Oberwolfach problem has been formalized and investigated in [10], it should be pointed out that some earlier results have been achieved via the

1–rotational approach. In [25, 24, 26] the authors provide solutions to the Oberwolfach Problem (with a special attention to the cases with two and three parameters) which are 1–rotational under the cyclic group, even though they simply speak of cyclic solutions. Finally, 1–rotational solutions to $OP(3^{2n+1})$ can be found in [12, 13].

In this paper we introduce a product of graphs, that we call the *circle product* and which can be applied to obtain decompositions starting from known ones. In particular we will apply the circle product to combine known solutions of the Oberwolfach Problem and get infinitely many solutions for greater orders, in both classic and non classic cases. When the circle product is applied to 1−rotational solutions, the new obtained solutions will be 1−rotational as well.

2 The circle product

Let Γ_1 and Γ_2 be undirected simple graphs without isolated vertices and let ∞ be a fixed element which either lies in some $V(\Gamma_i)$, $i \in \{1, 2\}$, or not. If $\infty \in V(\Gamma_i)$, we will set $\Gamma_i^* = \Gamma_i - \{\infty\}$. If $\infty \notin V(\Gamma_i)$ when speaking of Γ_i^* we will mean the same graph Γ_i.

For each pair $(e_r, e_s) \in E(\Gamma_1) \times E(\Gamma_2)$, we define the product $e_r \circ e_s$ to be the graph whose vertex-set and edge-set are described below:

1. If $e_r = [\infty, a],\ e_s = [\infty, b]$, then

$$V(e_r \circ e_s) = \Big\{\infty, (a, b)\Big\}$$
$$E(e_r \circ e_s) = \Big\{[\infty, (a, b)]\Big\}$$

2. If $e_r = [\infty, a],\ e_s = [c, d] \in E(\Gamma_2^*)$, then:

$$V(e_r \circ e_s) = \Big\{(a, d), (a, c)\Big\}$$
$$E(e_r \circ e_s) = \Big\{[(a, d), (a, c)]\Big\}$$

3. If $e_r = [a, b] \in E(\Gamma_1^*),\ e_s = [\infty, c]$, then:

$$V(e_r \circ e_s) = \Big\{(a, c), (b, c)\Big\}$$
$$E(e_r \circ e_s) = \Big\{[(a, c), (b, c)]\Big\}$$

4. If $e_r = [a, b] \in E(\Gamma_1^*),\ e_s = [c, d] \in E(\Gamma_2^*)$, then:

$$V(e_r \circ e_s) = \Big\{(a, c), (a, d), (b, c), (b, d)\Big\}$$
$$E(e_r \circ e_s) = \Big\{[(a, c), (b, d)], [(a, d), (b, c)]\Big\}$$

Following the above notations, we can *compose* the graphs Γ_1 and Γ_2 thus obtaining a new graph which is called the *circle product* of Γ_1 and Γ_2.

Definition 2.1. The *circle product* $\Gamma_1 \circ \Gamma_2$ is the graph obtained as the union of all graphs $e_r \circ e_s$ as the pair (e_r, e_s) varies in $E(\Gamma_1) \times E(\Gamma_2)$.

Obviously, the product $e_r \circ e_s$ changes depending on whether e_r or e_s contains the vertex ∞ or not. Besides, if $\infty \notin V(\Gamma_1) \cap V(\Gamma_2)$ then there will not be the products defined in (1), while if $\infty \notin V(\Gamma_1) \cup V(\Gamma_2)$ then there will be only the products defined in (4). Observe also that $V(\Gamma_1 \circ \Gamma_2) = V(\Gamma_1^*) \times V(\Gamma_2^*) \cup \{\infty\}$ whenever $\infty \in V(\Gamma_1) \cap V(\Gamma_2)$, while $V(\Gamma_1 \circ \Gamma_2) = V(\Gamma_1^*) \times V(\Gamma_2^*)$ in all the other cases. If $\infty \notin V(\Gamma_1) \cup V(\Gamma_2)$ then $\Gamma_1 \circ \Gamma_2$ coincides with the usual direct product of graphs, (see [18]).

We will employ the following specific notation to denote $\Gamma_1 \circ \Gamma_2$:

- $\Gamma_1 \diamond \Gamma_2$, if $\infty \in V(\Gamma_1) \cap V(\Gamma_2)$,
- $\Gamma_1 \triangleleft \Gamma_2$, if $\infty \in V(\Gamma_1)$ and $\infty \notin V(\Gamma_2)$,
- $\Gamma_1 \triangleright \Gamma_2$, if $\infty \notin V(\Gamma_1)$ and $\infty \in V(\Gamma_2)$,
- $\Gamma_1 \cdot \Gamma_2$, if $\infty \notin V(\Gamma_1) \cup V(\Gamma_2)$.

When it is not necessary to specify whether ∞ lies in some $V(\Gamma_i)$ or not, we will preserve the notation $\Gamma_1 \circ \Gamma_2$.

Obviously, when considering a graph Γ_i, $i \in \{1, 2\}$, we can always label its vertices in such a way that Γ_i either contains a vertex named ∞ or not, moreover, different choices for the vertex named ∞ may give rise to different graphs as a result of the circle product. If this is the case, we will specify which vertex is labeled with ∞.

Finally, it is easy to check that $\Gamma_1 \circ \Gamma_2$ is a simple graph in all cases.

The next proposition shows what happens when we apply the circle product to some standard graphs.

Proposition 2.2. *The following statements hold:*

1. $K_v \diamond K_w \cong K_{(v-1)(w-1)+1}$*;*
2. $K_v \triangleleft K_w \cong K_w \triangleright K_v \cong K_{\{(v-1):w\}}$*;*
3. $\Gamma \diamond K_2 \cong \Gamma \triangleright K_2 \cong K_2 \triangleleft \Gamma \cong \Gamma$ *for any simple graph* Γ*; in particular,* $C_n \diamond K_2 \cong C_n \triangleright K_2 \cong K_2 \triangleleft C_n \cong C_n$*;*
4. $C_n \triangleleft K_2 \cong C_{2n-2}$*;*
5. $C_n \cdot K_2 \cong K_2 \cdot C_n \cong \begin{cases} C_n \sqcup C_n & \textit{if } n \textit{ is even} \\ C_{2n} & \textit{if } n \textit{ is odd} \end{cases}$

Proof. The proof is an easy check. Point (1) is obvious: consider any pair of distinct vertices $x, y \in V(\Gamma_1^*) \times V(\Gamma_2^*) \cup \{\infty\}$. If $x = \infty$ and $y = (a, b)$, then $[x, y] = [\infty, a] \circ [\infty, b]$ (proceed in the same manner when $y = \infty$). If $x = (c, d)$ and $y = (a, b)$ with $a \neq c$ and $b \neq d$, then $[x, y]$ is an edge of $[a, c] \circ [b, d]$, if $a = c$ and $b \neq d$, then $[x, y] = [\infty, a] \circ [b, d]$, while $[x, y] = [a, c] \circ [\infty, b]$ whenever $b = d$ and $a \neq c$. Concerning point (2), we just observe that $K_v \triangleleft K_w$ is the complete equipartite graph $K_{\{(v-1):w\}}$ with w parts, each containing $v - 1$ elements. In particular, let $\{a_1, \ldots, a_{v-1}\} = V(K_v) - \{\infty\}$, for each $x \in V(K_w)$, the vertices $(a_1, x), \ldots, (a_{v-1}, x)$ are pairwise not adjacent in $K_v \triangleleft K_w$ and form a part of $K_{\{(v-1):w\}}$. In the same manner the vertices $(x, a_1), \ldots, (x, a_{v-1})$ are pairwise not adjacent in $K_w \triangleright K_v$ and form a part of $K_{\{(v-1):w\}}$.
Now, let Γ be a simple graph and observe that both $\Gamma \diamond [\infty, b]$ and $\Gamma \triangleright [\infty, b]$ derive from Γ by simply replacing each vertex different from ∞, say a, with (a, b). In the same manner $[\infty, b] \triangleleft \Gamma$ derives from Γ replacing each vertex $a \in V(\Gamma)$ with (b, a). Thus, point (3) follows.
Finally consider a cycle C_n. If $\infty \in V(C_n)$ and $C_n = (\infty, a_2, \ldots, a_n)$ then $C_n \triangleleft [a, b]$ is the $(2n-2)$-cycle whose vertices are obtained by overlapping the pair (a, b) to the sequence: $a_2, a_3, \ldots, a_{n-1}, a_n, a_n, a_{n-1}, \ldots, a_3, a_2$. More precisely: $C_n \triangleleft [a, b] = ((a_2, a), (a_3, b), \ldots, (a_n, b), (a_n, a), \ldots, (a_3, a), (a_2, b))$ or $C_n \triangleleft [a, b] = ((a_2, a), (a_3, b), \ldots, (a_n, a), (a_n, b), \ldots, (a_3, a), (a_2, b))$ according to whether n is odd or even. Furthermore, if $\infty \notin V(C_n)$ and $C_n = (a_1, \ldots, a_n)$, we have either

$$C_n \cdot [x, y] = ((a_1, x), (a_2, y), \ldots, (a_{n-1}, x), (a_n, y)) \quad \sqcup$$
$$((a_1, y), (a_2, x), \ldots, (a_{n-1}, y), (a_n, x))$$

or

$$C_n \cdot [x, y] = ((a_1, x), (a_2, y) \ldots (a_n, x), (a_1, y), (a_2, x) \ldots (a_n, y))$$

according to whether n is even or odd. □

In the following propositions we point out some properties of the circle product.

Proposition 2.3. *Let Γ_1 and Γ_2 be simple graphs and let (a, b) be a vertex of $\Gamma_1 \circ \Gamma_2$, with $a \in V(\Gamma_1^*)$ and $b \in V(\Gamma_2^*)$. It is $d_{\Gamma_1 \circ \Gamma_2}((a, b)) = d_{\Gamma_1}(a) d_{\Gamma_2}(b)$. Moreover, if ∞ is in $\Gamma_1 \circ \Gamma_2$ then $d_{\Gamma_1 \circ \Gamma_2}(\infty) = d_{\Gamma_1}(\infty) d_{\Gamma_2}(\infty)$.*

Proof. Any edge of $\Gamma_1 \circ \Gamma_2$ passing through (a, b) lies in a product of edges, say $e_1 \circ e_2$, where e_1 and e_2 are incident with a and b, respectively. Since the number of these mutually edge–disjoint products is $d_{\Gamma_1}(a) d_{\Gamma_2}(b)$ and any of them provides exactly one edge passing through (a, b), it follows that $d_{\Gamma_1 \circ \Gamma_2}((a, b)) = d_{\Gamma_1}(a) d_{\Gamma_2}(b)$.

One can proceed in the same manner to get $d_{\Gamma_1 \circ \Gamma_2}(\infty) = d_{\Gamma_1}(\infty) d_{\Gamma_2}(\infty)$. □

As an immediate consequence, we can state that the class of regular graphs is closed under the circle product.

Proposition 2.4. *The circle product of two regular graphs of degree k and t, respectively, is a kt–regular graph.* □

Proposition 2.5. *If $\mathcal{F}_1 = \{\Gamma_1, \dots, \Gamma_s\}$ and $\mathcal{F}_2 = \{\Gamma'_1, \dots, \Gamma'_r\}$ are decompositions of the graphs G_1 and G_2, respectively, then $\mathcal{F}_1 \circ \mathcal{F}_2 = \{\Gamma_i \circ \Gamma'_j \mid i = 1, \dots, s, j = 1, \dots, r\}$ is a decomposition of the graph $G_1 \circ G_2$.*

Proof. Let $[x, y] \in E(G_1 \circ G_2)$. If $x = \infty$ and $y = (a, b)$, $a \in V(G_1)$ $b \in V(G_2)$, we necessarily have $[x, y] = [\infty, a] \circ [\infty, b]$. Let Γ_i (respectively Γ'_j) be the unique graph of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) which contains $[\infty, a]$ (resp. $[\infty, b]$), then $\Gamma_i \circ \Gamma'_j$ is the unique graph of $\mathcal{F}_1 \circ \mathcal{F}_2$ containing $[x, y]$. Proceed in the same manner if $y = \infty$. Now suppose $x \neq \infty$ and $y \neq \infty$, with $x = (a, b)$ and $y = (c, d)$. If $a \neq c$ and $b \neq d$, let Γ_i (respectively Γ'_j) be the unique graph of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) which contains $[a, c]$ (resp. $[b, d]$), then $\Gamma_i \circ \Gamma'_j$ is the unique graph of $\mathcal{F}_1 \circ \mathcal{F}_2$ containing $[x, y]$. Finally suppose $a = c$ and $b \neq d$ and let Γ_i (respectively Γ'_j) be the unique graph of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) which contains $[\infty, a]$ (resp. $[b, d]$), then $\Gamma_i \circ \Gamma'_j$ is the unique graph of $\mathcal{F}_1 \circ \mathcal{F}_2$ containing $[x, y]$. In the same manner proceed if $a \neq c$ and $b = d$. □

3 New solutions to the classic Oberwolfach Problem

Our constructions are presented in Theorems 3.4, 4.1 and 4.2 and need some machinery and preliminary lemmas explained below.

Let $S = \{e_1, e_2, \dots, e_w\}$ be a 1–factor of the complete graph K_{2w} and let $F_1, \dots, F_w$ be w (not necessarily distinct or edge-disjoint) 2−factors of the complete graph K_{2n+1}.
For the constructions explained in Lemma 3.1 and in Lemma 3.2, label the vertices of K_{2n+1} in such a way that $\infty \in V(K_{2n+1})$. For each 2−factor F_i denote by λ_i the length of the cycle through ∞ and let L_i and M_i be multisets of even and odd integers, respectively, so that F_i is a $F_i(\lambda_i, L_i, M_i)$ 2−factor. Then we have:

Lemma 3.1. *Label the vertices of K_{2w} in such a way that $\infty \in V(K_{2w})$ and, without loss of generality, suppose ∞ to be a vertex of e_1.*
The graph $T = (e_1 \diamond F_1) \sqcup (e_2 \triangleright F_2) \sqcup \cdots \sqcup (e_w \triangleright F_w)$ is a 2–factor of $K_{2n(2w-1)+1}$ of type $(\lambda_1, L_1, M_1, 2(\lambda_2 - 1), L_2^2, 2M_2, \dots, 2(\lambda_w - 1), L_w^2, 2M_w)$.

Proof. The graph T is the disjoint union of the graphs $e_i \circ F_i$, $i = 1, \dots, w$, and it is a subgraph of $K_{2w} \diamond K_{2n+1} = K_{2n(2w-1)+1}$. Moreover let $e_1 = [\infty, b_1]$ and $e_i = [a_i, b_i]$, $i = 2, \dots, w$. Recalling how the circle product is defined, we have $V(e_1 \diamond F_1) = \{\infty\} \cup \{b_1\} \times V(K^*_{2n+1})$ and $V(e_i \triangleright F_i) = \{a_i, b_i\} \times V(K^*_{2n+1})$, $i = 2, \dots, w$. Therefore $V(T) = V(K_{2w} \diamond K_{2n+1})$. Also, by Proposition 2.4, each graph $e_i \circ F_i$ is 2−regular and then T is a 2−factor of $K_{2w} \diamond K_{2n+1}$. We can determine the type of T by applying Proposition 2.2. More precisely: the cycles of $e_1 \diamond F_1$ have the same length as those in F_1 (see Proposition 2.2, point 3); for each $i = 2, \dots, w$, the cycle of F_i through ∞ gives rise to a cycle in $e_i \triangleright F_i$ of length $2(\lambda_i - 1)$ (see Proposition 2.2, point 4); each other cycle of F_i of odd length gives rise to a cycle with double length and each of even length gives two cycles of the same length (this from point 5). □

Lemma 3.2. *Label the vertices of K_{2w} in such a way that $\infty \notin V(K_{2w})$. The graph $T = (e_1 \triangleright F_1) \sqcup (e_2 \triangleright F_2) \sqcup \cdots \sqcup (e_w \triangleright F_w)$ is a 2–factor of $K_{\{2n:2w\}}$ of type $(2(\lambda_1 - 1), L_1^2, 2M_1, \ldots, 2(\lambda_w - 1), L_w^2, 2M_w)$*

Proof. Proceed as in the proof of Lemma 3.1 and observe that the graph T is the disjoint union of the graphs $e_i \triangleright F_i$, $i = 1, \ldots, w$, and it is a subgraph of $K_{2w} \triangleright K_{2n+1} = K_{\{2n:2w\}}$. Moreover let $e_i = [a_i, b_i]$, $i = 1, \ldots, w$. Recalling how the circle product of edges is defined, we have $V(e_i \triangleright F_i) = \{a_i, b_i\} \times V(K^*_{2n+1})$. Therefore $V(T) = V(K_{2w} \triangleright K_{2n+1})$. Also, by Proposition 2.4, each graph $e_i \circ F_i$ is 2−regular and then T is a 2−factor of $K_{2w} \triangleright K_{2n+1}$. We can determine the type of T applying Proposition 2.2. More precisely: the cycle of F_i through ∞ gives rise to a cycle in $e_i \triangleright F_i$ of length $2(\lambda_i - 1)$ (apply Proposition 2.2, point 3); each other cycle of F_i of odd length gives rise to a cycle with double length and each of even length gives two cycles of the same length (from point 5). □

Now, for the construction of the following Lemma 3.3, label the vertices of K_{2w} in such a way that ∞ is a vertex of K_{2w} which lies in e_1 and label the vertices of K_{2n+1} in such a way that $\infty \notin V(K_{2n+1})$. For each 2−factor F_i, $i = 1, \ldots, w$, let L_i and M_i be multisets of even and odd integers, respectively, so that F_i is a $F_i(L_i, M_i)$ 2−factor. Then we have:

Lemma 3.3. *The graph $T = (e_1 \triangleleft F_1) \sqcup (e_2 \cdot F_2) \sqcup \cdots \sqcup (e_w \cdot F_w)$ is a 2–factor of $K_{\{(2w-1):(2n+1)\}}$ of type $(L_1, M_1, L_2^2, 2M_2, \ldots, L_w^2, 2M_w)$*

Proof. Observe that the graph T is the disjoint union of the graphs $e_1 \triangleleft F_1$ and $e_i \cdot F_i$, $i = 2, \ldots, w$, and it is a subgraph of $K_{2w} \triangleleft K_{2n+1} = K_{\{(2w-1):(2n+1)\}}$. Moreover let $e_1 = [\infty, b_1]$ and $e_i = [a_i, b_i]$, $i = 2, \ldots, w$.
Applying the rules of the circle product, we have $V(e_1 \triangleleft F_1) = \{b_1\} \times V(K_{2n+1})$ and $V(e_i \cdot F_i) = \{a_i, b_i\} \times V(K_{2n+1})$. Therefore $V(T) = V(K_{2w} \triangleleft K_{2n+1}) = K_{\{(2w-1):(2n+1)\}}$. Also, by Proposition 2.4, each graph $e_i \circ F_i$ is 2−regular and then T is a 2−factor of $K_{\{(2w-1):(2n+1)\}}$. As in the previous lemmas, we can determine the type of T in view of Proposition 2.2: the cycles in $e_1 \triangleleft F_1$ are copies of those in F_1, furthermore, if $i \in \{2, \ldots, w\}$, each cycle of F_i of odd length gives rise to a cycle with double length and each of even length gives two cycles of the same length. □

Theorem 3.4. *Let w be an integer and let $\mathcal{F}_1, \ldots, \mathcal{F}_w$ be w (not necessarily distinct) solutions to an Oberwolfach problem of order $2n+1$. More precisely, let $\mathcal{F}_1$ be a solution to $OP(l_1, \ldots, l_t)$ and for each $i = 2, \ldots, w$ suppose the existence of a vertex in K_{2n+1} such that all cycles of $\mathcal{F}_i$ passing through it have the same length λ_i. For $i = 2, \ldots, w$, denote by L_i and M_i multisets of even and odd integers, respectively, in such a way that $\mathcal{F}_i$ is a solution to $OP(\lambda_i, L_i, M_i)$. Then, there exists a solution to*

$$OP(l_1, \ldots, l_t, 2(\lambda_2 - 1), L_2^2, 2M_2, \ldots, 2(\lambda_w - 1), L_w^2, 2M_w) \tag{3.1}$$

Proof. Label as ∞ the vertex of K_{2n+1} with the property that for each $i = 2, \ldots, w$ all cycles of $\mathcal{F}_i$ passing through ∞ have length λ_i, Let $\{F_i^1, \ldots, F_i^n\}$ be the ordered set of 2−factors in $\mathcal{F}_i$. Let $\mathcal{S}$ be a 1−factorization of K_{2w} and denote by S_j, $j = 1, \ldots, 2w - 1$, the 1−factors of $\mathcal{S}$. Label with ∞ a vertex of K_{2w} and label the edges of each S_j

as $E(S_j) = \{e_{1j}, \ldots, e_{wj}\}$ in such a way that each edge e_{1j} contains ∞, for each $j = 1, \ldots, 2w-1$. Now fix $r \in \{1, \ldots, n\}$ and take the 2−factors $F_1^r, \ldots, F_w^r$, where, following the previous notation, the 2−factor F_i^r is the r−th factor of the 2−factorization $\mathcal{F}_i$. Fix $j \in \{1, \ldots, 2w-1\}$ and take the 1−factor $S_j \in \mathcal{S}$. Now apply Lemma 3.1 and observe that the graph $T_{jr} = (e_{1j} \diamond F_1^r) \sqcup (e_{2j} \triangleright F_2^r) \sqcup \cdots \sqcup (e_{wj} \triangleright F_w^r)$ is a 2–factor of $K_{2n(2w-1)+1}$ of type $(l_1, \ldots, l_t, 2(\lambda_2 - 1), L_2^2, 2M_2, \ldots, 2(\lambda_w - 1), L_w^2, 2M_w)$. To be more precise, observe that $e_{1j} \diamond F_1^r \cong F_1^r$ from point 3 of Proposition 2.2. Therefore $e_{1j} \diamond F_1^r$ gives rise to a set of cycles of length $l_1, \ldots, l_t$ respectively, independently from the cycle of F_1^r on which ∞ lies.

The set $\mathcal{T} = \{T_{jr} \mid j = 1, \ldots, 2w-1,\ r = 1, \ldots, n\}$ contains $n(2w-1)$ 2−factors of $K_{2w} \diamond K_{2n+1} = K_{2n(2w-1)+1}$. To prove that it is a 2−factorization it is sufficient to see that each edge $[x, y] \in E(K_{2w} \diamond K_{2n+1})$ appears in exactly one T_{jr}. Suppose $x = \infty$ and $y = (a, b)$ and then necessarily $[x, y] = [\infty, a] \circ [\infty, b]$. Let S_j be the unique 1−factor of $\mathcal{S}$ containing $[\infty, a] = e_{1j}$ and let F_1^r be the unique 2−factor of $\mathcal{F}_1$ containing $[\infty, b]$. By construction, the 2−factor T_{jr} is the unique one containing $[x, y]$. In the same manner proceed whenever $y = \infty$. Now suppose $x \neq \infty$ and $y \neq \infty$, with $x = (a, b)$ and $y = (c, d)$. If $a \neq c$ let S_j be the unique 1−factor of $\mathcal{S}$ containing $[a, c] = e_{tj}$ $(t > 1)$. If $b \neq d$, respectively if $b = d$, let F_t^r be the unique 2−factor of $\mathcal{F}_t$ which contains $[b, d]$, respectively $[\infty, b]$. By construction, the 2−factor T_{jr} is the unique one containing $[x, y]$. Now suppose $a = c$ and $b \neq d$. Let S_j be the unique 1−factor of $\mathcal{S}$ containing $[\infty, a] = e_{1j}$ and let F_1^r be the unique 2−factor of $\mathcal{F}_1$ containing $[b, d]$. By construction, the 2−factor T_{jr} is the unique one containing $[x, y]$.

Now suppose all $\mathcal{F}_i$'s, $i = 1, \ldots, w$, to be 1–rotational under the same group G. It is proved in [10] that whenever $\mathcal{F}_i$ is 1−rotational, then the vertex of K_{2n+1} which is fixed by G has the property that all cycles through it have the same length. This was already requested by our assumption for each $\mathcal{F}_i$, $i = 2, \ldots, w$ now this holds for $\mathcal{F}_1$ as well. Label by ∞ the vertex of $\mathcal{F}_i$ which is fixed by G. Suppose l_1 to be the length of all cycles of $\mathcal{F}_1$ passing through it, while as above, λ_i, $i = 2, \ldots, w$, denotes the length of all cycles of $\mathcal{F}_i$ through ∞.

It follows from the results of [10] that for any involution j of G there exists at least a 2−factor in $\mathcal{F}_i$ which is fixed by j. Moreover, the 2−factorization $\mathcal{F}_i$ is obtained as the orbit of this 2−factor under the action of a right transversal of $\{1_G, j\}$ in G. Fix an involution $j \in G$ and let $T = \{1_G = t_1, \ldots, t_n\}$ be an ordered right transversal of $\{1_G, j\}$ in G. For each $i = 1, \ldots, w$ choose F_i^1 to be a 2−factor of $\mathcal{F}_i$ which is fixed by j and let $F_i^r = F_i^1 + t_r$, $r = 1, \ldots, n$. Let H be a group of odd order $2w-1$. It is well known that a 1−factorization $\mathcal{S}$ of K_{2w} which is 1−rotational under H exists. Furthermore, H acts sharply transitively on the set $\mathcal{S} = \{S_1, \ldots, S_{2w-1}\}$. Let $S_1 = \{e_{11}, \ldots, e_{w1}\}$ with ∞ a vertex of e_{11}. For each $S_j \in \mathcal{S}$ let $h \in H$ be the unique element of H such that $S_j = S_1 + h$ and set $S_j = \{e_{1j}, \ldots, e_{wj}\}$ with $e_{sj} = e_{s1} + h$, $s = 1, \ldots, w$. With these notations we construct the 2−factorization $\mathcal{T} = \{T_{jr} \mid j = 1, \ldots, 2w-1,\ r = 1, \ldots, n\}$ as above. It is of type $(l_1, \ldots, l_t, 2(\lambda_2 - 1), L_2^2, 2M_2, \ldots, 2(\lambda_w - 1), L_w^2, 2M_w)$ and all cycles through ∞ have length l_1. It is 1−rotational under $H \times G$. In fact for each $T_{jr} \in \mathcal{T}$ and for each pair

$(h,g) \in H \times G$ we have

$$T_{jr} + (h,g) = \bigcup_{i=1}^{w} (e_{ij} + h) \circ (F_i^r + g) = \bigcup_{i=1}^{w} (e_{ij} + h) \circ (F_i^1 + t_r + g)$$

and if we let $S_j + h = S_k$ and $t_r + g \in \{j + t_s, t_s\}$ (i.e., $\{F_1^r + g, \ldots, F_w^r + g\} = \{F_1^s, \ldots, F_w^s\}$), then we have

$$T_{jr} + (h,g) = \bigcup_{i=1}^{w} e_{ik} \circ F_i^s = T_{ks} \in \mathcal{T}$$

□

We point out that a weaker form of Theorem 3.4 appeared in [10] and concerns the case where all $\mathcal{F}_i$'s coincides and then have the same type. In what follows we show a simple example of how Theorem 3.4 works.

Example 3.5. *Let $G = \mathbb{Z}_6 = \{0,1,2,3,4,5\}$, let $H = \mathbb{Z}_3 = \{0,1,2\}$ (in the usual additive notation) and let $F_1^1 = \{(\infty,0,1,5,2,4,3)\}$ and $F_2^1 = \{(\infty,0,3),$ $(1,5,4,2)\}$ be 2–factors of K_7, with $V(K_7) = G \cup \{\infty\}$. F_1^1 and F_2^1 are the base factors of a 1–rotational solution to $OP(7)$ and $OP(3,4)$, respectively. Namely $\mathcal{F}_1 = \{F_1^1, F_1^1 + 1, F_1^1 + 2\} = \{F_1^1, F_1^2, F_1^3\}$ and $\mathcal{F}_2 = \{F_2^1, F_2^1 + 1, F_2^1 + 2\} = \{F_2^1, F_2^2, F_2^3\}$. Consider K_4, with $V(K_4) = H \cup \{\infty\}$ and let $S_1 = \{[\infty,0],[1,2]\}$ be a base 1−factor of a 1−rotational 1−factorization $\mathcal{S} = \{S_1, S_1+1, S_1+2\} = \{S_1, S_2, S_3\}$ of K_4.*

We construct the following 2–factor of K_{19}, with $V(K_{19}) = (H \times G) \cup \{\infty\}$:

$$T_{11} = ([\infty,0] \diamond F_1^1) \sqcup ([1,2] \triangleright F_2^1)$$

It consists of the 7–cycle A' and the three 4–cycles B', C', D' below:

$$\begin{aligned}
A' &= (\infty,(0,0),(0,1),(0,5),(0,2),(0,4),(0,3));\\
B' &= ((1,0),(2,0),(1,3),(2,3));\\
C' &= ((1,1),(2,5),(1,4),(2,2));\\
D' &= ((2,1),(1,5),(2,4),(1,2));
\end{aligned}$$

Moreover, $\mathcal{T} = \{T_{11} + (h,g) \mid (h,g) \in H \times G\}$ turns out to be a 1−rotational solution to $OP(7,4,4,4)$.

We can repeat the construction exchanging the role of $\mathcal{F}_1$ and $\mathcal{F}_2$. In this case we have:

$R_{11} = ([\infty,0] \diamond F_2^1) \sqcup ([1,2] \triangleright F_1^1)$ *which consists of a 3–cycle A'', a 4–cycle B'', and a 12–cycle C'', namely:*

$$\begin{aligned}
A'' &= (\infty,(0,0),(0,3));\\
B'' &= ((0,1),(0,5),(0,4),(0,2));\\
C'' &= ((1,0),(2,1),(1,5),(2,2),(1,4),(2,3),\\
&\qquad (1,3),(2,4),(1,2),(2,5),(1,1),(2,0)).
\end{aligned}$$

Moreover, $\mathcal{R} = \{R_{11} + (h, g) \mid (h, g) \in H \times G\}$ turns out to be a 1−rotational solution to $OP(3, 4, 12)$.

If we identify $\mathbb{Z}_6$ with $\mathbb{Z}_3 \times \mathbb{Z}_2$, and consider two copies of a 1−rotational solution to $OP(3)$ under $\mathbb{Z}_2$, whose unique 2−factor is $(\infty, 0, 1)$, we can also identify F_2^1 with the 2−factor $F = ([\infty, 0] \diamond (\infty, 0, 1)) \sqcup ([1, 2] \triangleright (\infty, 0, 1))$. Therefore, $\mathcal{T}$ and $\mathcal{R}$ can be reasonably considered as the result of the recursive application of Theorem 3.4 to a solution of $OP(3)$ and $OP(7)$.

If we take w solutions $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_w$ to the Oberwolfach Problem of a fixed order $2n+1$ with the property that for every $i = 1, \ldots, n$ there exists a vertex in K_{2n+1} such that all cycles of $\mathcal{F}_i$ passing through it have the same length λ_i, then we can apply Theorem 3.4 and solve

$$\begin{aligned} OP(\lambda_{\sigma(1)}, L_{\sigma(1)}, M_{\sigma(1)}, & 2(\lambda_{\sigma(2)} - 1), L^2_{\sigma(2)}, 2M_{\sigma(2)}, \\ & \vdots \\ & 2(\lambda_{\sigma(w)} - 1), L^2_{\sigma(w)}, 2M_{\sigma(w)}) \end{aligned}$$

where σ denotes a permutation of $\{1, 2, \ldots, w\}$. Therefore, in the case where all $\mathcal{F}_i$'s are mutually distinct we can solve w distinct instances of the Oberwolfach Problem as $\sigma(1)$ varies in $\{1, 2, \ldots, w\}$. Also the constraint consisting of composing solutions to the Oberwolfach Problem of a fixed order can be overcome making a recursive use of Theorem 3.4 and infinite families of new solutions can be obtained. In the following Corollaries we point out just few examples, nevertheless many other combinations and recursive constructions are possible.

Corollary 3.6. *If there exists a solution to $OP(l_1, \ldots, l_t)$, with $2n + 1 = \sum_1^t l_i$, then for any positive integer w there exists a solution to*

$$OP(l_1, \ldots, l_t, (4n)^w) \tag{3.2}$$

Moreover, if there exists a 1–rotational solution to $OP(l_1, \ldots, l_t)$ under a symmetrically sequenceable group G, then (3.2) admits a 1–rotational solution under $H \times G$, for any group H of order $2w + 1$.

Proof. Let $\mathcal{F}_1$ be a solution to $OP(l_1, \ldots, l_t)$ and let $\mathcal{F}_2, \ldots, \mathcal{F}_{w+1}$ be w copies of a solution to $OP(2n + 1)$. By applying Theorem 3.4 we get a solution to (3.2).
Now assume that $\mathcal{F}_1$ is 1–rotational under a symmetrically sequenceable group G and recall that $\mathcal{F}_2, \ldots, \mathcal{F}_{w+1}$ can be chosen to be 1–rotational under G, as well. In view of the second part of Theorem 3.4, we get a 1–rotational solution to (3.2). □

Therefore, if we take a symmetrically sequenceable group G of order $2n$ and consider $w + 1$ copies of a 1−rotational solution to $OP(2n + 1)$ under G, then a 1−rotational solution to $OP(2n + 1, (4n)^w)$ under $H \times G$ exists for each group H of odd order $2w + 1$. For example, starting from the 1−rotational solution to $OP(5, 12)$ under the generalized quaternion group Q_{16} presented in [10], we are able to construct a 1−rotational solution to $OP(5, 12, 32^w)$ under $H \times Q_{16}$ for each group H of order $2w + 1$.

Another application of Corollary 3.6 gives the following:

Corollary 3.7. *Let $d_1, d_2, \ldots, d_u$ be odd positive integers. There exists a $1-$rotational solution to*

$$OP(2n+1, (4n)^{(d_1-1)/2}, (4nd_1)^{(d_2-1)/2}, \ldots, (4nd_1 \ldots d_{u-1})^{(d_u-1)/2}) \tag{3.3}$$

In particular, if $d_1 = d_2 = \ldots = d_u = 3$ then

$$OP(2n+1, 4n, \ldots, 3^i 4n, \ldots, 3^{u-1}4n)$$

has a $1-$rotational solution.

Proof. We proceed by induction on u using Corollary 3.6. If $u = 1$, then the existence of a $1-$rotational solution to (3.3) is a consequence of applying Corollary 3.6 starting from a $1-$rotational solution to $OP(2n+1)$ under a symmetrically sequenceable group G of order $2n$ and setting $w = (d_1 - 1)/2$. Now, let $u > 1$, by the inductive hypothesis there exists a $1-$rotational solution $\mathcal{F}$ to

$$OP(2n+1, (4n)^{(d_1-1)/2}, (4nd_1)^{(d_2-1)/2}, \ldots, (4nd_1 \ldots d_{u-2})^{(d_{u-1}-1)/2})$$

under $G \times H_1 \times \cdots \times H_{u-1}$, with groups H_i of order d_i, $i = 1, \ldots, u-1$. Applying again Corollary 3.6 to $\mathcal{F}$ we get a $1-$rotational solution to (3.3) under $G \times H_1 \times \cdots \times H_u$, with groups H_i of order d_i, $i = 1, \ldots, u$. □

For example, the previous corollary ensures the existence of $1-$rotational solutions to $OP(2n{+}1, 4n)$, $OP(2n{+}1, 4n, 12n)$, $OP(2n{+}1, 4n, 12n, 36n)$, $\ldots$, $OP(2n{+}1, 4n, 12n, 12n, 60n)$, $\ldots$, $OP(2n+1, 4n, 4n, 20n)$, $\ldots$, $OP(2n+1, 4n, 4n, 4n, 28n)$ $\ldots$ and so on.

It is worth pointing out that the benefit we get from constructing a 1–rotational solution $\mathcal{F}$ to $OP(\lambda, l_1, \ldots, l_t)$ under the action of some group G, is that a solution to $OP(2 : (n+1); \lambda+1, l_1, \ldots, l_t)$ for the complete equipartite graph $K_{\{2:(n+1)\}}$ can be constructed as well, where $2n+1 = \lambda + \sum_{i=1}^{t} l_i$ and λ denotes the length of the cycles of $\mathcal{F}$ through the vertex ∞ which is fixed by G. In fact, given a 2–factor F of $\mathcal{F}$ and denoted by $(\infty, a_1, \ldots, a_{2u})$ the cycle of F through ∞, it suffices to construct the 2–factor F' from F as follows: delete the edge $[a_u, a_{u+1}]$ and add the edges $[\infty', a_u]$, $[\infty', a_{u+1}]$, where ∞' is a vertex not belonging to $G \cup \{\infty\}$. The set of all F', where F varies in $\mathcal{F}$, turns out to be a solution of $OP(2 : (n+1); \lambda+1, l_1, \ldots, l_t)$. We will further deal with the equipartite variant to the Oberwolfach Problem in the next section.

Here is another application of Theorem 3.4.

Corollary 3.8. *For every quadruple of non negative integers m, r, w_1, w_2 with both m and r odd and $m \geq 3$, there exists a solution to*

1. $OP(rm, (2rm-2)^{w_1}, (2m-2)^{w_2}, (2m)^{w_2(r-1)})$;
2. $OP(m^r, (2rm-2)^{w_1}, (2m-2)^{w_2}, (2m)^{w_2(r-1)})$.

Proof. First denote by $\mathcal{F}'$ and $\mathcal{F}''$ a solution of $OP(rm)$ and $OP(m^r)$, respectively. Now let $\mathcal{F}_2, \ldots, \mathcal{F}_{w_1+1}$be w_1 copies of $\mathcal{F}'$ and let $\mathcal{F}_{w_1+2}, \ldots, \mathcal{F}_{w_1+w_2+1}$ be w_2 copies of $\mathcal{F}''$. Also let $\mathcal{F}_1$ be either $\mathcal{F}'$ or $\mathcal{F}''$. By applying Theorem 3.4 to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{w_1+w_2+1}$ we get a solution to either (1) or (2) according to whether $\mathcal{F}_1 = \mathcal{F}'$ or $\mathcal{F}_1 = \mathcal{F}''$. □

We point out that applications of Theorem 3.4 provide solutions to the Oberwolfach Problem whose cycles of even length are, at least theoretically, the most. In the following theorem we use the circle product to solve new instances of the Oberwolfach Problem. In particular, this Theorem gives also solutions in which all cycles have odd length.

Theorem 3.9. *Let s be a positive integer and let $t_1, \ldots, t_{2s}$, $k_1, \ldots, k_{2s}$ and d be positive integers such that $t_j \geq 3$ and $t_j k_j = 3d$, for every $j = 1, \ldots, 2s$. If each $t_j \neq 3$ whenever $d = 2$ or 6 and if there exists a solution to $OP(l_1, \ldots, l_r)$ with $l_1 + \cdots + l_r = 2d + 1$, then the Oberwolfach Problem $OP(l_1, \ldots, l_r, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$ has a solution.*

Proof. Let $\mathcal{D}$ be a solution to $OP(3^{2s+1})$ and let $\{D_0, \ldots, D_{3s}\}$ be its set of 2−factors. For each $i = 0, \ldots, 3s$, the graph $K_{d+1} \diamond D_i$ is a spanning subgraph of $K_{d+1} \diamond K_{6s+3} = K_{(6s+2)d+1}$ and $\{K_{d+1} \diamond D_0, K_{d+1} \diamond D_1, \ldots, K_{d+1} \diamond D_{3s}\}$ turns out to be a decomposition of $K_{(6s+2)d+1}$ into isomorphic subgraphs. Each component $K_{d+1} \diamond D_i$ can be decomposed into d 2−factors $F_i^1, \ldots, F_i^d$ of type $(l_1, \ldots, l_r, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$. In fact, let $C_i^1, \ldots, C_i^{2s+1}$ be the 3−cycles composing the 2−factor D_i and suppose $\infty \in V(C_i^{2s+1})$. Observe that $K_{d+1} \diamond C_i^{2s+1} = K_{2d+1}$ and, using a solution to $OP(l_1, \ldots, l_r)$, we can decompose $K_{d+1} \diamond C_i^{2s+1}$ into d 2−factors of type $(l_1, \ldots, l_r)$. Also, for each $1 \leq j \leq 2s$, we have $K_{d+1} \triangleleft C_i^j = K_{\{d:3\}}$ and, using a solution to $OP(d : 3; t_j)$ (whose existence is ensured by [23]), a 2−factorization of $K_{\{d:3\}}$ into d 2−factors each containing k_j cycles of length t_j, with $t_j k_j = 3d$, can be constructed. Since all graphs $K_{d+1} \triangleleft C_i^j$ and $K_{d+1} \diamond C_i^{2s+1}$, with i kept fixed, are vertex-disjoint, we can combine their 2−factors to compose d 2−factors $F_i^1, \ldots, F_i^d$ of $K_{d+1} \diamond D_i$, thus obtaining the 2−factorization $\{F_i^1, \ldots, F_i^d\}$ of $K_{d+1} \diamond D_i$. We conclude that the set $\{F_i^1, \ldots, F_i^d, \mid i = 0, \ldots, 3s\}$ is a solution to $OP(l_1, \ldots, l_r, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$. □

The previous Theorem allows to solve many instances of the Oberwolfach problem. Recalling that $OP(2d + 1)$, $OP(3^{(2d+1)/3})$ with $d \equiv 1 \pmod 6$ and $OP(3, 4^{(d-1)/2})$ with d odd always have a solution, we obtain the following:

Corollary 3.10. *Let s be a positive even integer and let $t_1, \ldots, t_{2s}$, $k_1, \ldots, k_{2s}$ be positive integers such that $t_j \geq 3$ and $t_j k_j = 3d$, for every $j = 1, \ldots, 2s$. If each $t_j \neq 3$ whenever $d = 2$ or 6, then there exists a solution to the following instances of the Oberwolfach Problem:*

1. *$OP(2d + 1, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$;*
2. *$OP(3^{(2d+1)/3}, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$, with $d \equiv 1 \pmod 6$;*
3. *$OP(3, 4^{(d-1)/2}, t_1^{k_1}, \ldots, t_{2s}^{k_{2s}})$, with d odd.*

Many other instances of the Oberwolfach problem can be solved. For example starting from the known solutions presented in [5]. The previous corollaries just give a few of them. Nevertheless not all possible instances can be obtained. For example the problem $OP(3, 3, 3, 10)$ cannot be solved using the previous Theorem 3.9. In fact starting from the known solutions to $OP(3)$, $OP(3, 3, 3)$, $OP(3, 10)$, a recursive use of Theorem 3.9 does not lead to a solution of $OP(3, 3, 3, 10)$.

4 New solutions to the equipartite Oberwolfach problem

Some variations in the proof of Theorem 3.4 leads to the following results on the equipartite Oberwolfach Problem. Cause the evident similarities, we will be more concise in the proof.

Theorem 4.1. *Let w be an integer and let $\mathcal{F}_1, \dots, \mathcal{F}_w$ be w (not necessarily distinct) solutions to an Oberwolfach problem of order $2n+1$. For each $i = 1, \dots, w$ suppose the existence of a vertex in K_{2n+1} such that all cycles of $\mathcal{F}_i$ passing through it have the same length λ_i. Denote by L_i and M_i multisets of even and odd integers, respectively, in such a way that $\mathcal{F}_i$ is a solution to $OP(\lambda_i, L_i, M_i)$. Then, there exists a solution to*

$$OP(2n : 2w; 2(\lambda_1 - 1), L_1^2, 2M_1, \dots, 2(\lambda_w - 1), L_w^2, 2M_w) \tag{4.1}$$

Proof. Without loss of generality, label as ∞ the vertex of K_{2n+1} with the property that each cycle of $\mathcal{F}_i$ passing through it has length λ_i, for each $i = 1, \dots, w$. Let $\{F_i^1, \dots, F_i^n\}$ be the ordered set of 2−factors in $\mathcal{F}_i$. Let $\mathcal{S}$ be a 1−factorization of K_{2w}, with $\infty \notin V(K_{2w})$, and denote as S_j, $j = 1, \dots, 2w-1$, the 1−factors of $\mathcal{S}$. Label the edges of each S_j as $\{e_{1j}, \dots, e_{wj}\}$. Fix $r \in \{1, \dots, n\}$ and take the 2−factors $F_1^r, \dots, F_w^r$, where, following the previous notation, the 2−factor F_i^r is the r−th factor of the 2−factorization $\mathcal{F}_i$. Fix $j \in \{1, \dots, 2w-1\}$ and take the 1−factor $S_j \in \mathcal{S}$. Now apply Lemma 3.2 and observe that the graph $T_{jr} = (e_{1j} \triangleright F_1^r) \sqcup (e_{2j} \triangleright F_2^r) \sqcup \dots \sqcup (e_{wj} \triangleright F_w^r)$ is a 2–factor of $K_{\{2n:2w\}}$ of type

$$(2(\lambda_1 - 1), L_1^2, 2M_1, \dots, 2(\lambda_w - 1), L_w^2, 2M_w)$$

The set $\mathcal{T} = \{T_{jr} \mid j = 1, \dots, 2w-1,\ r = 1, \dots, n\}$ contains $n(2w-1)$ 2−factors of $K_{2w} \triangleright K_{2n+1} = K_{\{2n:2w\}}$ and it is a 2−factorization of $K_{\{2n:2w\}}$. □

Theorem 4.2. *Let w be an integer and let $\mathcal{F}_1, \dots, \mathcal{F}_w$ be w (not necessarily distinct) solutions to an Oberwolfach problem of order $2n+1$. Denote by L_i and M_i multisets of even and odd integers, respectively, in such a way that $\mathcal{F}_i$ is a solution to $OP(L_i, M_i)$. Then, there exists a solution to*

$$OP((2w-1) : (2n+1)); L_1, M_1, L_2^2, 2M_2, \dots, L_w^2, 2M_w) \tag{4.2}$$

Proof. Let $\mathcal{S}$ be a 1−factorization of K_{2w}, denote by S_j, $j = 1, \dots, 2w-1$, the 1−factors of $\mathcal{S}$ and label the edges of each S_j as $\{e_{1j}, \dots, e_{wj}\}$. Without loss of generality, label with ∞ a vertex of K_{2w} in such a way that it is a vertex of e_{1j}, for each $j = 1, \dots, 2w-1$. Label the vertices of $V(K_{2n+1})$ in such a way that $\infty \notin V(K_{2n+1})$, let $\mathcal{F}_i$, $i = 1, \dots, w$, be a solution of $OP(L_i, M_i)$ and let $\{F_i^1, \dots, F_i^n\}$ be the ordered set of its 2−factors. Fix $r \in \{1, \dots, n\}$ and take the 2−factors $F_1^r, \dots, F_w^r$, where, following the previous notation, the 2−factor F_i^r is the r−th factor of the 2−factorization $\mathcal{F}_i$. Fix $j \in \{1, \dots, 2w-1\}$ and take the 1−factor $S_j \in \mathcal{S}$. Now apply Lemma 3.3 and observe that the graph $T_{jr} = (e_{1j} \triangleleft F_1^r) \sqcup (e_{2j} \cdot F_2^r) \sqcup \dots \sqcup (e_{wj} \cdot F_w^r)$ is a 2–factor of $K_{(2w-1):(2n+1)}$ of type

$$(L_1, M_1, L_2^2, 2M_2, \dots, L_w^2, 2M_w)$$

The set $\mathcal{T} = \{T_{jr} \mid j = 1, \dots, 2w-1,\ r = 1, \dots, n\}$ contains $n(2w-1)$ 2−factors of $K_{2w} \triangleleft K_{2n+1} = K_{(2w-1):(2n+1)}$ and it is a 2−factorization of $K_{(2w-1):(2n+1)}$. □

Corollary 4.3. *For every quadruple of non negative integers m, r, w_1, w_2 with both m and r odd, $m \geq 3$ and $(w_1, w_2) \neq (0,0)$, there exists a solution to*

$$OP((rm-1) : 2(w_1+w_2); (2rm-2)^{w_1}, (2m-2)^{w_2}, (2m)^{w_2(r-1)}). \tag{4.3}$$

Proof. First denote by $\mathcal{F}'$ and $\mathcal{F}''$ a solution of $OP(rm)$ and $OP(m^r)$, respectively. Now let $\mathcal{F}_1, \dots, \mathcal{F}_{w_1}$ be w_1 copies of $\mathcal{F}'$ and let $\mathcal{F}_{w_1+1}, \dots, \mathcal{F}_{w_1+w_2}$ be w_2 copies of $\mathcal{F}''$. By applying Theorem 4.1 to $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{w_1+w_2}$ we get a solution to (4.3). □

Corollary 4.4. *For every quadruple of non negative integers m, r, w_1, w_2 with both m and r odd and $m \geq 3$, there exists a solution to*

1. *$OP((2w_1+2w_2+1) : rm; rm, (2rm)^{w_1}, (2m)^{w_2 r})$;*
2. *$OP((2w_1+2w_2+1) : rm; m^r, (2rm)^{w_1}, (2m)^{w_2 r})$.*

Proof. First denote by $\mathcal{F}'$ and $\mathcal{F}''$ a solution of $OP(rm)$ and $OP(m^r)$, respectively. Now let $\mathcal{F}_2, \dots, \mathcal{F}_{w_1+1}$ be w_1 copies of $\mathcal{F}'$ and let $\mathcal{F}_{w_1+2}, \dots, \mathcal{F}_{w_1+w_2+1}$ be w_2 copies of $\mathcal{F}''$. Also let $\mathcal{F}_1$ be either $\mathcal{F}'$ or $\mathcal{F}''$. By applying Theorem 4.2 to $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{w_1+w_2+1}$ we get a solution to either (1) or (2) according to whether $\mathcal{F}_1 = \mathcal{F}'$ or $\mathcal{F}_1 = \mathcal{F}''$. □

We have already mentioned that the bipartite Oberwolfach problem was completely solved by Piotrowski in [27]. Nevertheless, its proof is commonly deemed to be pretty involved meaning that it is to be hoped that a new and less involved proof will be provided. As a particular case of Theorem 4.1 we are able to easily solve a wide class of instances of the bipartite Oberwolfach Problem by combining known solutions of the classic one, as stated below.

Corollary 4.5. *Let L and M be multisets of even and odd integers, respectively. If there exists a solution to $OP(\lambda, L, M)$ with a vertex such that all cycles of passing through it have length λ, then there exists a solution to*

$$OP(2n : 2; 2(\lambda-1), L^2, 2M).$$

Proof. Apply Theorem 4.1 when $w = 1$. □

An analogous of Theorem 3.9 can also be proved.

Theorem 4.6. *Let $n = 6s+3$, with s a positive integer, and let $t_1, \dots, t_{2s+1}$, $k_1, \dots, k_{2s+1}$ and d be positive integers such that $t_j \geq 3$ and $t_j k_j = 3d$, for every $j = 1, \dots, 2s+1$. If each $t_j \neq 3$ whenever $d = 2$ or 6, then the Equipartite Oberwolfach Problem $OP(d : n; t_1^{k_1}, \dots, t_{2s+1}^{k_{2s+1}})$ has a solution.*

Proof. Let $\mathcal{D}$ be a solution to $OP(3^{2s+1})$ and let $\{D_0, \dots, D_{3s}\}$ be its set of $2-$factors. For each $i = 0, \dots, 3s$, the graph $K_{d+1} \triangleleft D_i$ is a spanning subgraph of $K_{d+1} \triangleleft K_{6s+3} = K_{\{d:(6s+3)\}}$ and $\{K_{d+1} \triangleleft D_0, K_{d+1} \triangleleft D_1, \dots, K_{d+1} \triangleleft D_{3s}\}$ turns out to be a decomposition of $K_{\{d:(6s+3)\}}$ into isomorphic subgraphs. Each component $K_{d+1} \diamond D_i$ can be decomposed into d $2-$factors $F_i^1, \dots, F_i^d$ of type $(t_1^{k_1}, t_2^{k_2}, \dots, t_{2s+1}^{k_{2s+1}})$. In fact, let $C_i^1, \dots, C_i^{2s+1}$ be the $3-$cycles composing the $2-$factor D_i, for each $1 \leq j \leq 2s+1$, we have $K_{d+1} \triangleleft C_i^j = K_{\{d:3\}}$ and, using a solution to $OP(d:3; t_j)$ (whose existence is ensured by [23]), a $2-$factorization of $K_{\{d:3\}}$ into d $2-$factors each containing k_j cycles of length t_j, with $t_j k_j = 3d$, can be constructed. Since all the graphs $K_{d+1} \triangleleft C_i^j$, with i kept fixed, are vertex-disjoint, we can combine their $2-$factors to compose d $2-$factors $F_i^1, \dots, F_i^d$ of $K_{d+1} \triangleleft D_i$, thus obtaining the $2-$factorization $\{F_i^1, \dots, F_i^d\}$ of $K_{d+1} \triangleleft D_i$. We conclude that the set $\{F_i^1, \dots, F_i^d, \mid i = 0, \dots, 3s\}$ is a solution to $OP(d:(6s+3); t_1^{k_1}, \dots, t_{2s+1}^{k_{2s+1}})$. $\square$

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