Orthogonal arrays with parameters $OA(s^3, s^2{+}s{+}1, s, 2)$ and 3-dimensional projective geometries

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Abstract

There are many nonisomorphic orthogonal arrays with parameters $OA(s^3, s^2 + s + 1, s, 2)$ although the existence of the arrays yields many restrictions. We denote this by $OA(3, s)$ for simplicity. V. D. Tonchev showed that for even the case of $s = 3$, there are at least 68 nonisomorphic orthogonal arrays. The arrays that are constructed by the $n-$dimensional finite spaces have parameters $OA(s^n, (s^n - 1)/(s-1), s, 2)$. They are called Rao-Hamming type. In this paper we characterize the $OA(3, s)$ of 3-dimensional Rao-Hamming type. We prove several results for a special type of $OA(3, s)$ that satisfies the following condition:
For any three rows in the orthogonal array, there exists at least one column, in which the entries of the three rows equal to each other.

We call this property α-type.

We prove the following.

(1) An $OA(3, s)$ of α-type exists if and only if s is a prime power.
(2) $OA(3, s)$s of α-type are isomorphic to each other as orthogonal arrays.
(3) An $OA(3, s)$ of α-type yields $PG(3, s)$.
(4) The 3-dimensional Rao-Hamming is an $OA(3, s)$ of α-type.
(5) A linear $OA(3, s)$ is of α-type.

Keywords: *orthogonal array*; *projective space*; *projective geometry*

1 Introduction

An $N \times k$ array A with entries from a set S that contains s symbols is said to be an *orthogonal array with s levels, strength t and index* λ if every $N \times t$ subarray of A contains

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each $t-$tuple based on S exactly λ times as a row. We denote the array A by $OA(N, k, s, t)$. Orthogonal arrays with parameters $OA(s^n, (s^n-1)/(s-1), s, 2)$ are known for any prime power s and any integer $n \geq 2$. For example, orthogonal arrays of Rao-Hamming type have such parameters. We are interested in whether orthogonal arrays with above parameters exist or not when s is not a prime power, but do not know the existence of arrays with such parameters. In this paper we prove that s is prime power when $n = 3$, under an additional assumption. Throughout this paper, let s be a positive integer with $s \geq 2$.

Notation 1.1 Let S be a set of s symbols, A an orthogonal array $OA(s^3, s^2+s+1, s, 2)$. Then we use the following notations.
(1) $OA(s^3, s^2+s+1, s, 2)$ is denoted by $OA(3, s)$ for simplicity.
(2) $\Omega(A)$ is the set of rows of A.
(3) $\Gamma(A)$ is the set of columns of A.
(4) $u = (u(C))_{C\in\Gamma(A)}$ for $u \in \Omega(A)$.
(5) Set $k(s) = s^2 + s + 1$.

Definition 1.2 Let A be an $OA(3, s)$ and set $\Omega = \Omega(A),\ \Gamma = \Gamma(A),\ k = k(s)$.
(1) For $u, v \in \Omega$ and $C \in \Gamma$, let

$$K(u, v, C) = \begin{cases} 1 & if\ u(C) = v(C), \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let $[u_1, u_2, \ldots, u_r] = |\{C \in \Gamma | u_1(C) = u_2(C) = \cdots = u_r(C)\}|$.
Especially, we have $[u_1, u_2] = \sum_{C\in\Gamma} K(u_1, u_2, C)$.

Lemma 1.3 *Let A be an $OA(3, s)$ and set $\Omega = \Omega(A),\ \Gamma = \Gamma(A),\ k = k(s)$. Then the following statements hold.*
(1) $K(u, u, C) = 1$ *and* $(K(u, v, C))^2 = K(u, v, C)$ *for* $u, v \in \Omega$ *and* $C \in \Gamma$.
(2) $[u, u] = k$ *for* $u \in \Omega$.
(3) $\sum_{v\in\Omega} K(u, v, C) = s^2$ *and* $\sum_{v\in\Omega, v\neq u} K(u, v, C) = s^2 - 1$ *for* $u \in \Omega$ *and* $C \in \Gamma$, *and so* $\sum_{v\in\Omega, v\neq u} [u, v] = (s^2 + s + 1)(s^2 - 1)$.
(4) $\sum_{v\in\Omega} K(u, v, C_1)K(u, v, C_2) = s$ *and* $\sum_{v\in\Omega, v\neq u} K(u, v, C_1)K(u, v, C_2) = s - 1$ *for* $u \in \Omega$ *and distinct* $C_1, C_2 \in \Gamma$.

PROOF. The lemma is clear from the definition of an orthogonal array. □

Lemma 1.4 *Let A be an $OA(3, s)$ and set $\Omega = \Omega(A), \Gamma = \Gamma(A)$. Then* $[u, v] = s + 1$ *for distinct* $u, v \in \Omega$.

Proof. Let $u \in \Omega$.

$$\sum_{v\in\Omega, v\neq u}([u,v])^2 = \sum_{v\in\Omega, v\neq u}\{\sum_{C\in\Gamma}(K(u,v,C))^2 + \sum_{C_1\in\Gamma}(\sum_{C_2\in\Gamma, C_2\neq C_1} K(u,v,C_1)K(u,v,C_2))\}$$
$$= \sum_{C\in\Gamma}(\sum_{v\in\Omega, v\neq u}(K(u,v,C))^2) + \sum_{C_1\in\Gamma}(\sum_{C_2\in\Gamma, C_2\neq C_1}(\sum_{v\in\Omega, v\neq u} K(u,v,C_1)K(u,v,C_2)))$$
$$= \sum_{C\in\Gamma}(s^2-1) + \sum_{C_1\in\Gamma}(\sum_{C_2\in\Gamma, C_2\neq C_1}(s-1))$$
$$= (s^2+s+1)(s^2-1) + (s^2+s+1)(s^2+s)(s-1)$$
$$= (s^2+s+1)(s+1)^2(s-1).$$

Hence,

$$\sum_{v\in\Omega, v\neq u}([u,v]-s-1)^2 = \sum_{v\in\Omega, v\neq u}([u,v])^2 - 2(s+1)\sum_{v\in\Omega, v\neq u}[u,v] + \sum_{v\in\Omega, v\neq u}(s+1)^2$$
$$= (s^2+s+1)(s+1)^2(s-1) - 2(s+1)(s^2+s+1)(s^2-1) + (s+1)^2(s^3-1) = 0.$$

Therefore $[u,v] = s+1$ for $v \in \Omega$ with $v \neq u$. Since u is arbitrary, this completes the proof. $\square$

We remark that orthogonal arrays with parameters $OA(3,s)$ have good connections with two bounds in coding theory. Actually, Lemma 1.4 shows that the code whose words are the rows of the OA (length s^2+s+1, number of codewords s^3) has constant distance s^2. This is a code which satisfies the Plotkin bound (Theorem 9.3 of [4]) with equality. Also, the OA itself satisfies the Bose-Bush bound(Theorem 9.6 of [4]) with equality. Thus the existence of orthogonal arrays $OA(3,s)$ yields many restrictions. So at first we expected that any $OA(3,s)$ is isomorphic to Rao-Hamming type. But we knew by Tonchev [3] that there are many nonisomorphic $OA(3,s)$ arrays. Next, we discovered a condition for an $OA(3,s)$ to be Rao-Hamming type, that is the condition α (see Definition 1.8).

Definition 1.5 Let s be a prime power and A an $OA(3,s)$ with entries from $GF(s)$. A is called to be *linear* if A satisfies

$$\lambda u + \mu v = (\lambda u(C) + \mu v(C))_{C\in\Gamma(A)} \in \Omega(A) \quad for\ \lambda,\mu \in GF(s) \quad and \quad u,v \in \Omega(A).$$

Definition 1.6 Let P and Q are orthogonal arrays with the same parameters. P and Q are isomorphic if Q can be obtained from P by permutation of the columns, the rows, and the symbols in each column.

Remark 1.7 Let $A = (a_{ij})_{1\leq i\leq s^3,\ 1\leq j\leq k(s)}$ be a linear $OA(3,s)$ with entries from $GF(s)$. Let φ be a permutation on $\{1,2,\cdots,k(s)\}$ and $\lambda_j \in GF(s)^*$ for $1 \leq j \leq k(s)$. Let $B = (b_{ij})_{1\leq i\leq s^3,\ 1\leq j\leq k(s)}$, where $b_{ij} = \lambda_j a_{i,\varphi(j)}$ for $1 \leq i \leq s^3$ and $1 \leq j \leq k(s)$. Then B is a linear $OA(3,s)$ which is isomorphic to A.

Definition 1.8 Let A be an $OA(3,s)$. A is called to be of *α-type* if

$$[u,v,w] \geq 1 \quad for \quad u,v,w \in \Omega(A).$$

We show later that this condition corresponds to a condition in affine space order s that "for any distinct three points there exists at least one plane containing them".

Proposition 1.9 *If A is a linear $OA(3,s)$ with entries from $GF(s)$, then A is of α-type.*

PROOF. Set $\Omega = \Omega(A)$ and $k = k(s)$. From the linearity of A, $o = (0,0,\cdots,0) \in \Omega$. For distinct $u_1,u_2,u_3 \in \Omega$, we have $[u_1,u_2,u_3] = [o,u_2-u_1,u_3-u_1]$. Therefore, it is enough to show that $[o,u,v] \geq 1$ for distinct $u,v \in \Omega - \{0\}$. Since $[u,o] = s+1$ by Lemma 1.4, u has exactly $s+1$ zeroes as entries. From Remark 1.7, we can assume that $u = (\underbrace{1,1,\cdots,1,}_{s^2}\underbrace{0,0,\cdots,0,}_{s+1}) \in \Omega(A)$. Then $\lambda u = (\underbrace{\lambda,\lambda,\cdots,\lambda,}_{s^2}\underbrace{0,0,\cdots,0}_{s+1})$ is an element of Ω for $\lambda \in GF(s)$. Let $v = (v(1),v(2),\cdots,v(k))$. Then there exists at least one zero in $v(s^2+1),v(s^2+2),\cdots,v(k)$. Suppose not. Since $s+1 = [\lambda u,v] = [(\underbrace{\lambda,\lambda,\cdots,\lambda,}_{s^2}\underbrace{0,0,\cdots,0}_{s+1}),(v(1),v(2),\cdots,v(k))]$, there are exactly $s+1$ λ's in $v(1),v(2),\cdots,v(s^2)$. We have $s^2 = \mid \{v(1),v(2),\cdots,v(s^2)\} \mid \geq (s+1)s$, since λ is arbitrary and $\mid GF(s) \mid = s$, This is a contradiction. This yields $[o,u,v] \geq 1$. $\square$

Proposition 1.10 *The orthogonal array $OA(3,s)$ of 3-dimensional Rao-Hamming type is of $\alpha-$type .*

PROOF. We consider the $OA(3,s)$ of 3-dimensional Rao-Hamming type stated in Construction 1 of Theorem 3.20 in [1] when $n=3$. Let π be a fixed plane of the projective geometry $PG(3,s)$. Let Ω be the set of points of $PG(3,s)$ excluding all points in π. Let Γ be the set of lines contained in π. Then the $OA(3,s)$ $A = (a_{ul})_{u\in\Omega,l\in\Gamma}$ is defined as follows. For each line $l \in \Gamma$, we label planes through l except π in some arbitrary way by $1,2,\cdots,s$. Then a_{ul} is the plane containing u and l. Let u_1,u_2, and u_3 be distinct elements in Ω. Let τ be the plane containing u_1,u_2 and u_3 and set $l = \tau\cap\pi \in \Gamma$. Then $a_{u_1,l} = a_{u_2,l} = a_{u_3,l}$ and therefore A is of $\alpha-$type. $\square$

Throughout the rest of this paper, we assume the following.

Hypothesis 1.11 A is an OA(3,s) of α-type. Set $\Omega = \Omega(A), \Gamma = \Gamma(A)$, and $k = k(s)$.

Lemma 1.12 $[u,v,w] = 1$ *or* $s+1$ *for distinct* $u,v,w \in \Omega$.

PROOF. Let u,v be distinct fixed elements of Ω. We may assume $u = (0,0,\cdots,0)$. From Lemma 1.4, v has $s+1$ zeroes in entries. Set $\Gamma_1 = \{C \mid v(C) = 0\}$. Then $\mid \Gamma_1 \mid = s+1$. We note $t_w = \mid \{C \mid w(C) = 0, C \in \Gamma_1\} \mid$ for any $w \in \Omega$. Then $\sum_{w\in\Omega} t_w = s^2(s+1)$. This is the total number of zeroes in Γ_1. Moreover since the array A has strength 2, $\sum_{w\in\Omega} t_w(t_w-1) = s(s+1)s = s^2(s+1)$. This is the total number of (0,0) tuples in any two columns in Γ_1. It follows that $\sum_{w\in\Omega}(t_w-1)(s+1-t_w) = 0$. By assumption, we have $t_w \geq 1$, therefore $(t_w-1)(s+1-t_w) \geq 0$. Hence $t_w = [u,v,w] \in \{1,s+1\}$. $\square$

Corollary 1.13 *For distinct $u, v \in \Omega$, there exist distinct $u_3, u_4, \cdots, u_s \in \Omega$ and $\Gamma_{uv} \subset \Gamma$ satisfying the following conditions:*
(1) $[u_1, u_2, u_3, u_4, \cdots, u_s] = s+1$, *where $u_1 = u$ and $u_2 = v$.*
(2) *If $C \in \Gamma_{uv}$ then $u_1(C) = u_2(C) = \cdots = u_s(C)$.*
(3) *If $C \in \Gamma - \Gamma_{uv}$ then $u_i(C) \neq u_j(C)$ for distinct $i, j \in \{1, 2, \cdots, s\}$.*
(4) $[u_1, u_2, u_3, u_4, \cdots, u_s, x] = 1$ *for $x \in \Omega - \{u_1, \cdots u_s\}$.*

Proof. We use the notations used in the proof of Lemma 1.12. Set $\Gamma_1 = \{C \in \Gamma \mid u(C) = v(C)\}$, $u_1 = u$, and $u_2 = v$. From the proof of Lemma 1.12 , we have $[u, x]_{\Gamma_1} = 1$ or $s+1$ for $x \in \Omega - \{u\}$. Set $r = \mid \{x \in \Omega \mid x \neq u, [u, x]_{\Gamma_1} = s+1\} \mid$. Then $\mid \{x \in \Omega \mid, [u, x]_{\Gamma_1} = 1\} \mid = s^3 - 1 - r$. Therefore, $r(s+1) + (s^3 - 1 - r) = \sum_{x \in \Omega, x \neq u} [u, x]_{\Gamma_1} = (s^2-1)(s+1)$. So $rs = (s^2-1)(s+1) - (s^3-1) = s(s-1)$. This yields $r = s-1$. Hence there exist $u_3, u_4, \cdots, u_s$ such that $[u, u_i]_{\Gamma_1} = s+1$ for $i \in \{3, 4, \cdots, s\}$. Therefore $u_1(C) = u_2(C) = \cdots = u_s(C)$ for $C \in \Gamma_1$. If there exists $C \notin \Gamma_1$ such that $u_i(C) = u_j(C)$ for some distinct $i, j \in \{1, 2, \cdots, s\}$, we have $[u_i, u_j] \geq s+2$, because $[u_i, u_j]_{\Gamma_1} = s+1$. This is contrary to Lemma 1.4. Hence $u_1(C), u_2(C), \cdots, u_s(C)$ are distinct if $C \notin \Gamma_1$. If we set $\Gamma_{uv} = \Gamma_1$, this completes the proof of (1), (2), and (3). From Lemma 1.12, for any $x \in \Omega - \{u_1, \cdots u_s\}$ there exists only one $C \in \Omega$ such that $u_1(C) = u_2(C) = x(C)$. By (2) and (3), C is in $\Gamma_1 (= \Gamma_{uv})$. Therefore $x(C) = u_1(C) = u_2(C) = \cdots = u_s(C)$. Since $x \notin \{u_1, u_2, \cdots, u_s\}$, we have $[u_1, u_2, u_3, u_4, \cdots, u_s, x] = 1$. □

2 A geometry

Under Hypothesis 1.11, we define the following.

Definition 2.1 (1) Elements of Ω are called *affine points.*
(2) Let $\Omega_1 = \{u_1, u_2, \cdots, u_s\} (\subseteq \Omega)$, $\Gamma_1 \subseteq \Gamma$, and $\mid \Gamma_1 \mid = s+1$. Then $\Omega_1 \cup \{\Gamma_1\}$ is called an *ordinary line* if $[u_1, u_2, \cdots, u_s] = s+1$ and $u_1(C) = u_2(C) = \cdots = u_s(C)$ for $C \in \Gamma_1$. Then Ω_1 and Γ_1 are called an *affine line* and an *infinite point* respectively.
(3) We denote the set of affine points by $\mathcal{P}_{\mathcal{F}}$ ($= \Omega$), the set of infinite points by $\mathcal{P}_\infty$, and the set of ordinary lines by $\mathcal{L}_{\mathcal{O}}$.
(4) The elements of $\mathcal{P} = \mathcal{P}_{\mathcal{F}} \cup \mathcal{P}_\infty$ are called *points* .

Lemma 2.2 *For any distinct $u, v \in \mathcal{P}_{\mathcal{F}}$, there exists only one $l \in \mathcal{L}_{\mathcal{O}}$ such that $u \in l$ and $v \in l$.*

Proof. The lemma is clear from Corollary 1.13 and Definition 2.1. □

Lemma 2.3 *Let C_1 and C_2 are fixed distinct elements of Γ.*
(1) *Set $\Omega(a, b) = \{u \in \Omega \mid u(C_1) = a,\ u(C_2) = b\}$ for $a, b \in S$. Then $\Omega(a, b)$ is an affine line.*
(2) *If $\Omega(a, b) \cup \{\Gamma_1\}$ and $\Omega(c, d) \cup \{\Gamma_2\}$ are ordinary lines, then $\Gamma_1 = \Gamma_2$.*

PROOF. (1) From the definition of $OA(3,s)$, we have $\mid \Omega(a,b) \mid = s$. Let $\Omega(a,b) = \{u_1, u_2, \cdots, u_s\}$. Let $u, v, w \in \Omega(a,b)$ be distinct elements. By Lemma 1.4, $[u,v] = [v,w] = [w,u] = s+1$. From Lemma 1.12 and $[u,v,w] \geq 2$, we have $[u,v,w] = s+1$. Therefore $[u_1, u_2, \cdots, u_s] = s+1$. This means that $\Omega(a,b)$ is an affine line.

(2) From (1), $\Omega(0,0)$ and $\Omega(0,b)$ $(b \neq 0)$ are affine lines. Let $\Omega(0,0) = \{u_1, u_2, \cdots, u_s\}$ and $\Omega(0,b) = \{v_1, v_2, \cdots, v_s\}$. Then $[u_1, u_2, \cdots, u_s] = s+1$ and $[v_1, v_2, \cdots, v_s] = s+1$. Let Γ_3 and Γ_4 be infinite points which correspond to $\Omega(0,0)$ and $\Omega(0,b)$ respectively. Let $\Gamma_3 = \{C_1, C_2, \cdots, C_{s+1}\}$ and set $a = u_1(C_3) = u_2(C_3) = \cdots = u_s(C_3)$. We prove $C_3 \in \Gamma_4$. Suppose that some value of $v_1(C_3), v_2(C_3), \cdots, v_s(C_3)$ is equal to a. We may assume that $v_1(C_3) = a$. Then $u_1(C_3) = u_2(C_3) = v_1(C_3) = a$. From these equations and $u_1(C_1) = u_2(C_1) = v_1(C_1) = 0$, we have $[u_1, u_2, v_1] \geq 2$. Therefore $[u_1, u_2, v_1] = s+1$ by Lemma 1.12. Hence $v_1 \in \Omega(0,0)$. This is a contradiction. Thus any value of $v_1(C_3), v_2(C_3), \cdots, v_s(C_3)$ is not equal to a. By the pigeonhole principle, there exist distinct v_i, v_j such that $v_i(C_3) = v_j(C_3)$. Therefore $v_1(C_3) = v_2(C_3) = \cdots = v_s(C_3)$, because $[v_1, v_2, \cdots, v_s] = s+1$, by Lemmas 1.12 and 1.4. Thus $C_3 \in \Gamma_4$. Similarly we can show that $C_4, C_5, \cdots, C_{s+1} \in \Gamma_4$. Moreover since $C_1, C_2 \in \Gamma_4$, we have $\Gamma_3 = \Gamma_4$. Similarly, it is shown that the infinite points corresponding to $\Omega(0,b)$ and $\Omega(a,b)$ are equal. Therefore the infinite points corresponding to $\Omega(0,0)$ and $\Omega(a,b)$ are equal. This completes the proof. □

Lemma 2.4 (1) *For any $C_1, C_2 \in \Gamma$ there exists an infinite point $\Gamma_1 (\in \mathcal{P}_\infty)$ uniquely such that $C_1, C_2 \in \Gamma_1$.*
(2) *For any $u \in \Omega$ and any infinite point Γ_1, there exists only one subset $\Omega_1 \subset \Omega$ such that $u \in \Omega_1$ and $\Omega_1 \cup \{\Gamma_1\}$ is an ordinary line.*
(3) $\mid \Gamma_1 \cap \Gamma_2 \mid = 1$ *for any distinct infinite points Γ_1 and Γ_2.*
(4) *Set $l_\infty(C) = \{\Gamma_1 \mid \Gamma_1$ is an infinite point such that $\Gamma_1 \ni C\}$ for $C \in \Gamma$. Then*
  (a) $\mid l_\infty(C) \mid = s+1$,
  (b) $\Gamma = \bigcup_{\Gamma_1 \in l_\infty(C)} (\Gamma_1 - \{C\}) \cup \{C\}$,
  (c) $(\Gamma_1 - \{C\}) \cap (\Gamma_2 - \{C\}) = \emptyset$ *for distinct $\Gamma_1, \Gamma_2 \in l_\infty(C)$.*

PROOF. (1) Let $C_1, C_2 \in \Gamma$. From (1) of Lemma 2.3, $\Omega_1 = \{u \in \Omega \mid u(C_1) = 0, u(C_2) = 0\}$ is an affine line. Let Γ_1 be the infinite point corresponding to Ω_1. Then $\Gamma_1 \ni C_1, C_2$. From (2) of Lemma 2.3, the infinite point containing C_1, C_2 is unique.

(2) Let $u \in \Omega$ and $\Gamma_1 \in \mathcal{P}_\infty$. Let $C_1, C_2 \in \Gamma_1$ and $\Omega_1 = \{v \in \Omega \mid v(C_1) = u(C_1), v(C_2) = u(C_2)\}$. From (1) of Lemma 2.3, Ω_1 is an affine line. Let Γ_2 be the infinite point corresponding to Ω_1. Then $\Gamma_1 \cap \Gamma_2 \supset \{C_1, C_2\}$. From (1) we have $\Gamma_1 = \Gamma_2$. Therefore $\Omega_1 = \{v \in \Omega \mid v(C) = u(C), C \in \Gamma_1\}$. Hence $\Omega_1 \cup \{\Gamma_1\}$ is a unique ordinary line containing u and Γ_1.

(3) Let Γ_1 and Γ_2 be distinct infinite points. For any $v \in \Omega$ and for $i \in \{1,2\}$, from (2), there exists only one ordinary line containing v and Γ_i. We denote it by $v\Gamma_i$ for $i \in \{1,2\}$. Let u and w be affine points such that $u \in v\Gamma_1 - \{v\}$ and $w \in v\Gamma_2 - \{v\}$. Since $\Gamma_1 \neq \Gamma_2$, by Lemma 1.12, $[u,v,w] = 1$. Therefore there exists $C \in \Gamma$ uniquely such that $u(C) = v(C) = w(C)$. Hence $\Gamma_1 \cap \Gamma_2 = \{C\}$ and so $\mid \Gamma_1 \cap \Gamma_2 \mid = 1$.

(4) Let C be a fixed element of Γ. For any $C_0 \in \Gamma - \{C\}$, from (1), there exists $\Gamma_0 \in \mathcal{P}_\infty$ uniquely such that $C, C_0 \in \Gamma_0$. Since $C \in \Gamma_0$, we have $\Gamma_0 \in l_\infty(C)$ and therefore $C_0 \in \Gamma_0 \in l_\infty(C)$. Thus we have $\Gamma = \bigcup_{\Gamma_1 \in l_\infty(C)} \Gamma_1$. Therefore $\Gamma = \bigcup_{\Gamma_1 \in l_\infty(C)} (\Gamma_1 - \{C\}) \cup \{C\}$. For distinct $\Gamma_1, \Gamma_2 \in l_\infty(C)$, by (3), $(\Gamma_1 - \{C\}) \cap (\Gamma_2 - \{C\}) = \emptyset$. Let $\mid l_\infty(C) \mid = r$. Then we have $r\{(s+1)-1\}+1 = s^2+s+1$. Hence $r = s+1$ and $\mid l_\infty(C) \mid = s+1$. □

Definition 2.5 (1) For any $C \in \Gamma$, $l_\infty(C) = \{\Gamma_1 \mid \Gamma_1$ an infinite point, $\Gamma_1 \ni C\}$ is called an *infinite line.* l is called a *line* if l is an ordinary or an infinite line.
(2) For any $a \in S$ and any $C \in \Gamma$, $\pi(a, C) = \{u \in \Omega \mid u(C) = a\}$, $\pi(a, C) \cup l_\infty(C)$, and $\pi_\infty = \bigcup_{C \in \Gamma} l_\infty(C)$ are called an *affine plane,* an *ordinary plane,* and an *infinite plane respectively.* π is called a *plane* if π is an ordinary or an infinite plane.
(3) The set of infinite lines and ordinary planes are denoted by $\mathcal{L}_\infty$ and $\mathcal{M}_0$ respectively. Moreover we set $\mathcal{L} = \mathcal{L}_o \cup \mathcal{L}_\infty$ and $\mathcal{M} = \mathcal{M}_o \cup \{\pi_\infty\}$.

Example 2.6 *The case of* $s = 2$.

$A =$

C_1	C_2	C_3	C_4	C_5	C_6	C_7	
0	0	0	0	0	0	0	u_1
1	0	0	1	1	0	1	u_2
0	1	0	1	0	1	1	u_3
0	0	1	0	1	1	1	u_4
1	1	0	0	1	1	0	u_5
1	0	1	1	0	1	0	u_6
0	1	1	1	1	0	0	u_7
1	1	1	0	0	0	1	u_8

is an $OA(3,2) = OA(2^3, 2^2+2+1, 2)$ $(s = 2)$ *of* α*-type.*

The affine points(the elements of $\mathcal{P}_F$*) are* $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$.
The infinite points(the elements of $\mathcal{P}_\infty$*) are* $\Gamma_1 = \{C_2, C_3, C_6\}$, $\Gamma_2 = \{C_1, C_3, C_5\}$, $\Gamma_3 = \{C_1, C_2, C_4\}$, $\Gamma_4 = \{C_3, C_4, C_7\}$, $\Gamma_5 = \{C_2, C_5, C_7\}$, $\Gamma_6 = \{C_1, C_6, C_7\}$, $\Gamma_7 = \{C_4, C_5, C_6\}$.
The ordinary lines (the elements of $\mathcal{L}_O$*) are*
$\{u_1, u_2\} \cup \{\Gamma_1\}$, $\{u_1, u_3\} \cup \{\Gamma_2\}$, $\{u_1, u_4\} \cup \{\Gamma_3\}$, $\{u_1, u_5\} \cup \{\Gamma_4\}$,
$\{u_1, u_6\} \cup \{\Gamma_5\}$, $\{u_1, u_7\} \cup \{\Gamma_6\}$, $\{u_1, u_8\} \cup \{\Gamma_7\}$, $\{u_2, u_3\} \cup \{\Gamma_4\}$,
$\{u_2, u_4\} \cup \{\Gamma_5\}$, $\{u_2, u_5\} \cup \{\Gamma_2\}$, $\{u_2, u_6\} \cup \{\Gamma_3\}$, $\{u_2, u_7\} \cup \{\Gamma_7\}$,
$\{u_2, u_8\} \cup \{\Gamma_6\}$, $\{u_3, u_4\} \cup \{\Gamma_6\}$, $\{u_3, u_5\} \cup \{\Gamma_1\}$, $\{u_3, u_6\} \cup \{\Gamma_7\}$,
$\{u_3, u_7\} \cup \{\Gamma_3\}$, $\{u_3, u_8\} \cup \{\Gamma_5\}$, $\{u_4, u_5\} \cup \{\Gamma_7\}$, $\{u_4, u_6\} \cup \{\Gamma_1\}$,
$\{u_4, u_7\} \cup \{\Gamma_2\}$, $\{u_4, u_8\} \cup \{\Gamma_4\}$, $\{u_5, u_6\} \cup \{\Gamma_6\}$, $\{u_5, u_7\} \cup \{\Gamma_5\}$,
$\{u_5, u_8\} \cup \{\Gamma_3\}$, $\{u_6, u_7\} \cup \{\Gamma_4\}$, $\{u_6, u_8\} \cup \{\Gamma_2\}$, $\{u_7, u_8\} \cup \{\Gamma_1\}$.
The infinite lines (the elements of $\mathcal{L}_\infty$*) are*
$l_\infty(C_1) = \{\Gamma_2, \Gamma_3, \Gamma_6\}$, $l_\infty(C_2) = \{\Gamma_1, \Gamma_3, \Gamma_5\}$, $l_\infty(C_3) = \{\Gamma_1, \Gamma_2, \Gamma_4\}$, $l_\infty(C_4) = \{\Gamma_3, \Gamma_4, \Gamma_7\}$, $l_\infty(C_5) = \{\Gamma_2, \Gamma_5, \Gamma_7\}$, $l_\infty(C_6) = \{\Gamma_1, \Gamma_6, \Gamma_7\}$, $l_\infty(C_7) = \{\Gamma_4, \Gamma_5, \Gamma_6\}$.
The ordinary planes (the elements of $\mathcal{M}_O$*) are*

$\pi(0, C_1) \cup l_\infty(C_1) = \{u_1, u_3, u_4, u_7\} \cup \{\Gamma_2, \Gamma_3, \Gamma_6\}$,
$\pi(0, C_2) \cup l_\infty(C_2) = \{u_1, u_2, u_4, u_6\} \cup \{\Gamma_1, \Gamma_3, \Gamma_5\}$,
$\pi(0, C_3) \cup l_\infty(C_3) = \{u_1, u_2, u_3, u_5\} \cup \{\Gamma_1, \Gamma_2, \Gamma_4\}$,
$\pi(0, C_4) \cup l_\infty(C_4) = \{u_1, u_4, u_5, u_8\} \cup \{\Gamma_3, \Gamma_4, \Gamma_7\}$,
$\pi(0, C_5) \cup l_\infty(C_5) = \{u_1, u_3, u_6, u_8\} \cup \{\Gamma_2, \Gamma_5, \Gamma_7\}$,
$\pi(0, C_6) \cup l_\infty(C_6) = \{u_1, u_2, u_7, u_8\} \cup \{\Gamma_1, \Gamma_6, \Gamma_7\}$,
$\pi(0, C_7) \cup l_\infty(C_7) = \{u_1, u_5, u_6, u_7\} \cup \{\Gamma_4, \Gamma_5, \Gamma_6\}$,
$\pi(1, C_1) \cup l_\infty(C_1) = \{u_2, u_5, u_6, u_8\} \cup \{\Gamma_2, \Gamma_3, \Gamma_6\}$,
$\pi(1, C_2) \cup l_\infty(C_2) = \{u_3, u_5, u_7, u_8\} \cup \{\Gamma_1, \Gamma_3, \Gamma_5\}$,
$\pi(1, C_3) \cup l_\infty(C_3) = \{u_4, u_6, u_7, u_8\} \cup \{\Gamma_1, \Gamma_2, \Gamma_4\}$,
$\pi(1, C_4) \cup l_\infty(C_4) = \{u_2, u_3, u_6, u_7\} \cup \{\Gamma_3, \Gamma_4, \Gamma_7\}$,
$\pi(1, C_5) \cup l_\infty(C_5) = \{u_2, u_4, u_5, u_7\} \cup \{\Gamma_2, \Gamma_5, \Gamma_7\}$,
$\pi(1, C_6) \cup l_\infty(C_6) = \{u_3, u_4, u_5, u_6\} \cup \{\Gamma_1, \Gamma_6, \Gamma_7\}$,
$\pi(1, C_7) \cup l_\infty(C_7) = \{u_2, u_3, u_4, u_8\} \cup \{\Gamma_4, \Gamma_5, \Gamma_6\}$.
The infinite plane is $\pi_\infty = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\}$.

Lemma 2.7 (Lemma A)
For $l \in \mathcal{L}$ *, we have* $| \, l \, | \geq 3$.

PROOF. From (2) of Definition 2.1 and (4) of Lemma 2.4, $| \, l \, |= s + 1$ for $l \in \mathcal{L}$. Since $s \geq 2$, we have the assertion. □

Lemma 2.8 (Lemma B)
For distinct points $\alpha, \beta \in \mathcal{P}$, *there exists a unique line* $l \in \mathcal{L}$ *such that* $\alpha \in l$ *and* $\beta \in l$. *We denote the line* l *by* $\alpha\beta$.

PROOF. Let α and β be distinct points. Then three cases (a) $\alpha, \beta \in \mathcal{P}_\mathcal{F}$, (b) $\alpha \in \mathcal{P}_\mathcal{F}$, $\beta \in \mathcal{P}_\infty$, and (c) $\alpha, \beta \in \mathcal{P}_\infty$ occur. For (a) or (b), the lemma holds by Lemma 2.2 and (2) of Lemma 2.4. We consider the case (c). Let $\alpha = \Gamma_1$ and $\beta = \Gamma_2$ be distinct infinite points. From (3) of Lemma 2.4, $| \, \Gamma_1 \cap \Gamma_2 \, |= 1$. Let $\Gamma_1 \cap \Gamma_2 = \{C\}$. Then $l_\infty(C) \ni \Gamma_1, \Gamma_2$. From the uniqueness of C, $l_\infty(C)$ is the unique line containing Γ_1 and Γ_2. □

Lemma 2.9 (1) *Let* $\alpha, \beta \in \mathcal{P}$ *be distinct points and* π *a plane containing* α *and* β. *Then every point on the line* $\alpha\beta$ *is a point on the plane* π.
(2) *Let* $\alpha, \beta, \gamma \in \mathcal{P}$ *be noncollinear points. Then there exists a unique plane* π *containing* α, β, *and* γ.

PROOF. (1) Let α is an affine point u. From Definition 2.5, any plane containing u is $\pi(u(C), C) \cup l_\infty(C)$ for some $C \in \Gamma$ and any line containing u is $\{v \in \Omega \mid v(C) = u(C), C \in \Gamma_1\} \cup \{\Gamma_1\}$ for some infinite point Γ_1. First, moreover let β be also an affine point v. Let Γ_1 be the infinite point corresponding to the line uv, then $\Gamma_1 = \{C \in \Gamma \mid u(C) = v(C)\}$, and $uv = \{w \in \Omega \mid w(C) = u(C), C \in \Gamma_1\} \cup \{\Gamma_1\}$. Let $\pi = \pi(u(C), C) \cup l_\infty(C)$ be a plane containing u and v. Then $C \in \Gamma_1$. Hence $\alpha\beta = uv \subset \pi$. Second, when β be an infinite point Γ_1, from Lemma 2.8, by a similar argument as stated above, we have the assertion in this case.

Next case, let α and β be both infinite points Γ_1 and Γ_2 respectively. From (3) of Lemma 2.4, there exists $C \in \Gamma$ such that $\Gamma_1 \cap \Gamma_2 = \{C\}$. Hence the line containing Γ_1 and Γ_2 is $l_\infty(C)$. A plane containing Γ_1 and Γ_2 is π_∞ or $\pi(a, C) \cup l_\infty(C)$ for some $a \in S$. Therefore every point on $l_\infty(C)$ is a point on a plane containing Γ_1 and Γ_2.

(2) Let Γ_1, Γ_2 be distinct infinite points. Let $\Gamma_1 \cap \Gamma_2 = \{C\}$. Then for any affine point u, $\pi = \pi(u(C), C) \cup l_\infty(C)$ is a unique plane containing u, Γ_1, and Γ_2. Next, let u and v be affine points, Γ_1 the infinite point corresponding to the line uv, and Γ_2 an infinite point. Then a plane containing u, Γ_1, and Γ_2 is the above plane π. Actually, from (1), the plane containing u, v, and Γ_2 is π. Hence we have the assertion in this case. Let u, v, and w be non collinear affine points. Then we can show that there exists exactly one plane containing u, v, and w by a similar argument. Finally, we can show that a plane containing any three infinite points is π_∞. Thus we have the assertion. $\square$

Lemma 2.10 *Let $\pi \in \mathcal{M}$ be a plane and $l, m \in \mathcal{L}$ distinct lines. If $l, m \subseteq \pi$ then $\mid l \cap m \mid = 1$.*

Proof. Let l and m be distinct lines. Since there exists only one line through distinct two points, we have $\mid l \cap m \mid \leq 1$. Therefore it is enough to show $l \cap m \neq \emptyset$. Then three cases (a) l and m are both ordinary lines, (b) l is an ordinary line and m is an infinite line, and (c) l and m are infinite lines, occur.

(a): Let Γ_1 and Γ_2 be the infinite points corresponding to lines l and m respectively. If $\Gamma_1 = \Gamma_2$, then $l \cap m = \{\Gamma_1\}$. Hence we may assume that $\Gamma_1 \neq \Gamma_2$. Let $\Gamma_1 \cap \Gamma_2 = \{C\}$. Then the plane containing l and m is $\pi(a, C) \cup l_\infty(C)$ for some $a \in S$. Let $l = \Omega_1 \cup \{\Gamma_1\}$, $m = \Omega_2 \cup \{\Gamma_2\}$, $C_1 \in \Gamma_1 - \{C\}$, $\Omega_1 = \{u_1, \cdots, u_s\}$, and $\Omega_2 = \{v_1, \cdots, v_s\}$. Then since $u_1(C_1) = \cdots = u_s(C_1)$, $v_1(C_1), v_2(C_1), \cdots, v_s(C_1)$ are not equal to each other. This means $\{v_1(C_1), v_2(C_1), \cdots, v_s(C_1)\} = S$. Since $S \ni u_1(C_1)$, there exists t such that $v_t(C_1) = u_1(C_1)(= \cdots = u_s(C_1))$. From this equation and $v_t(C) = u_1(C) = u_2(C)$, we have $[v_t, u_1, u_2] \geq 2$, and therefore $[v_t, u_1, u_2] = s + 1$ by Lemma 1.12. Thus $v_t \in \{u_1, \cdots, u_s\}$ and therefore $l \cap m = \{v_t\}$.

(b): Let $m = l_\infty(C)$. The plane containing l and $l_\infty(C)$ is an ordinary plane $\pi(a, C) \cup l_\infty(C)$ for some $a \in S$. Let $l = \{u_1, u_2, \cdots, u_s\} \cup \{\Gamma_1\}$. Then $u_1(C) = \cdots = u_s(C) = a$. Therefore $C \in \Gamma_1$ and so $\Gamma_1 \in l_\infty(C)$. Hence $l \cap l_\infty(C) = \{\Gamma_1\}$.

(c): Let $l = l_\infty(C_1)$ and $m = l_\infty(C_2)$. From (1) of Lemma 2.4, there exists an infinite point Γ_1 such that $C_1, C_2 \in \Gamma_1$. It follows that $l_\infty(C_1) \cap l_\infty(C_2) = \{\Gamma_1\}$. $\square$

Lemma 2.11 (Lemma C) *Let $P, Q, R \in \mathcal{P}$ be non collinear three points. Let $l \in \mathcal{L}$ be a line such that $P, Q, R \notin l$, $l \cap PQ \neq \emptyset$ and $l \cap PR \neq \emptyset$. Then, $l \cap QR \neq \emptyset$.*

Proof. From (2) of Lemma 2.9, there exists a unique plane π containing P, Q, and R. From Lemma 2.8, $\mid l \cap PQ \mid \leq 1$. Hence since $l \cap PQ \neq \emptyset$, we have $\mid l \cap PQ \mid = 1$. Let $l \cap PQ = \{X\}$. Similarly there exists a point Y such that $l \cap PR = \{Y\}$. From (1) of Lemma 2.9, all points on the line $XY(= l)$ are on the plane π. Similarly all points of the line QR are on the plane π . Hence by Lemma 2.10, $l \cap QR \neq \emptyset$. $\square$

Theorem 2.12 *Let A be an $OA(3, s)$ of α-type . Then*
(1) *s is a prime power and*
(2) *$(\mathcal{P}, \mathcal{L}, \mathcal{M})$ is isomorphic to $PG(3, s)$.*

PROOF. From Lemmas A, B, C and the theorem of Veblen and Young, we have the assertion. $\square$

3 The uniqueness

We denote the symmetric group of degree m by $Sym(m)$, and the identity element of $Sym(m)$ by 1_m. Let s and k be positive integers and $GOA(s,k) = \{f \mid f = (a_1, a_2, \cdots, a_k, \alpha) \quad a_i \in Sym(s)\ (i = 1, 2, \cdots, k),\ \alpha \in Sym(k)\}$. We define a product on $GOA(s,k)$ as follows. For $f = (a_1, a_2, \cdots, a_k, \alpha),\ g = (b_1, b_2, \cdots, b_k, \beta) \in GOA(s,k)$, $fg = (a_1, a_2, \cdots, a_k, \alpha)(b_1, b_2, \cdots, b_k, \beta) = (a_1 b_{\alpha(1)}, a_2 b_{\alpha(2)}, \cdots, a_k b_{\alpha(k)}, \beta\alpha)$.

Lemma 3.1 *$GOA(s,k)$ is a group.*

PROOF. Let $f = (a_1, a_2, \cdots, a_k, \alpha),\ g = (b_1, b_2, \cdots, b_k, \beta),\ h = (c_1, c_2, \cdots, c_k, \gamma) \in GOA(s,k)$. Then,

$$\begin{aligned}(fg)h &= (a_1 b_{\alpha(1)}, a_2 b_{\alpha(2)}, \cdots, a_k b_{\alpha(k)}, \beta\alpha)(c_1, c_2, \cdots, c_k, \gamma)\\ &= (a_1 b_{\alpha(1)} c_{\beta\alpha(1)}, \cdots, a_k b_{\alpha(k)} c_{\beta\alpha(k)}, \gamma\beta\alpha)\\ &= (a_1, a_2, \cdots, a_k, \alpha)(b_1 c_{\beta(1)}, \cdots, b_k c_{\beta(k)}, \gamma\beta) = f(gh).\end{aligned}$$

Set $e = (1_s, \cdots, 1_s, 1_k)$. Then we can easily show that $fe = ef = f$.
Let $f = (a_1, a_2, \cdots, a_k, \alpha) \in GOA(s,k)$ and set $g = ((a_{\alpha^{-1}(1)})^{-1}, \cdots, (a_{\alpha^{-1}(k)})^{-1}, \alpha^{-1})$. Then,

$$\begin{aligned}fg &= (a_1, a_2, \cdots, a_k, \alpha)((a_{\alpha^{-1}(1)})^{-1}, \cdots, (a_{\alpha^{-1}(k)})^{-1}, \alpha^{-1})\\ &= (a_1 (a_{\alpha^{-1}\alpha(1)})^{-1}, \cdots a_k (a_{\alpha^{-1}\alpha(k)})^{-1}, \alpha^{-1}\alpha)\\ &= (1_s, \cdots, 1_s, 1_k) = e.\\ gf &= ((a_{\alpha^{-1}(1)})^{-1}, \cdots, (a_{\alpha^{-1}(k)})^{-1}, \alpha^{-1})(a_1, a_2, \cdots, a_k, \alpha)\\ &= ((a_{\alpha^{-1}(1)})^{-1} a_{\alpha^{-1}(1)}, \cdots, ((a_{\alpha^{-1}(k)})^{-1} a_{\alpha^{-1}(k)}, \alpha\alpha^{-1})\\ &= (1_s, \cdots, 1_s, 1_k) = e.\end{aligned}$$

Therefore $GOA(s,k)$ is a group. $\square$

Let $S = \{1, 2, \cdots, s\}$ and $S^k = \underbrace{S \times S \times \cdots \times S}_{k}$. We define an operation of $GOA(s,k)$ on S^k as follows. For $u = (u(1), u(2), \cdots, u(k)) \in S^k$ and $f = (a_1, a_2, \cdots, a_k, \alpha) \in GOA(s,k)$, we define $fu = (a_1(u(\alpha(1))), \cdots, a_k(u(\alpha(k))))$. Let $g = (b_1, b_2, \cdots, b_k, \beta) \in GOA(s,k)$. Then,

$$\begin{aligned}g(fu) &= (b_1, b_2, \cdots, b_k, \beta)(a_1(u(\alpha(1))), \cdots, a_k(u(\alpha(k))))\\ &= (b_1(a_{\beta(1)} u(\alpha(\beta(1)))), \cdots, b_k(a_{\beta(k)} u(\alpha(\beta(k))))\ )\\ &= ((b_1 a_{\beta(1)}) u(\alpha\beta(1))), \cdots, (b_k a_{\beta(k)}) u(\alpha\beta(k))\ )\\ &= (b_1 a_{\beta(1)}, \cdots b_k a_{\beta(k)}, \alpha\beta)(u(1), u(2), \cdots, u(k))\\ &= (gf)u.\end{aligned}$$

We can state the definition of isomorphism of orthogonal arrays using the group $GOA(s,k)$.

Lemma 3.2 *Let A, B be two $OA(N,k,s,t)$s with entries from the set $S = \{1, 2, \cdots, s\}$ and $\Omega(A), \Omega(B)$ the sets of all rows of A, B respectively. Let $f(\Omega(A)) = \{fu \mid u \in \Omega(A),\ f \in GOA(s,k)\}$ for $f \in GOA(s,k)$. Then A and B are isomorphic if and only if there exists $f \in GOA(s,k)$ such that $f(\Omega(A)) = \Omega(B)$.*

Theorem 3.3 *The $OA(3,s)$s of α-type are isomorphic to each other.*

PROOF. Let $A^{(1)}, A^{(2)}$ be $OA(3,s)$s of α-type. Let $V^{(i)}$ be the $PG(3,s)$ defined by $A^{(i)}$, and $\pi_\infty^{(i)}$ the infinite plane of $V^{(i)}$ $(i = 1, 2)$. Then there exists an isomorphism $f; V^{(1)} \to V^{(2)}$ such that $f(\pi_\infty^{(1)}) = \pi_\infty^{(2)}$. Let $\Gamma^{(i)} = \{C_j^{(i)} \mid j = 1, 2, \cdots, s^2+s+1,\ C_j^{(i)}$ is a column of $A^{(i)}\}$ for $i = 1, 2$.

First, we prove that f induces a bijection from $\Gamma^{(1)}$ to $\Gamma^{(2)}$. Since $f(\pi_\infty^{(1)}) = \pi_\infty^{(2)}$, for any infinite line $l_\infty(C_i^{(1)})$ of $V^{(1)}$, there exists an infinite line $l_\infty(C_j^{(2)})$ of $V^{(2)}$ such that $f(l_\infty(C_i^{(1)})) = l_\infty(C_j^{(2)})$. Hence f yields a permutation $\sigma \in Sym(s^2+s+1)$ such that $f(l_\infty(C_{\sigma(j)}^{(1)})) = l_\infty(C_j^{(2)})$.

Second, we prove that for $j \in \{1, 2, \cdots, s^2+s+1\}$, f induces bijection from the entries of $C_{\sigma(j)}^{(1)}$ to the entries of $C_j^{(2)}$. For any infinite line $l_\infty(C_j^{(i)})$, a plane containing this line can be denote by $\pi(x, C_j^{(i)}) \cup l_\infty(C_j^{(i)})$ for some $x \in S$, where $\pi(x, C_j^{(i)}) = \{u \mid u(C_j^{(i)}) = x,\ u$ is an affine point$\}$. $(i = 1, 2)$ Fix $j \in \{1, 2, \cdots, s^2+s+1\}$. From $f(l_\infty(C_{\sigma(j)}^{(1)})) = l_\infty(C_j^{(2)})$, for any ordinary plane $\pi^{(1)} = \pi(x, C_{\sigma(j)}^{(1)}) \cup l_\infty(C_{\sigma(j)}^{(1)})$ on $V^{(1)}$ there exists an ordinary plane $\pi^{(2)} = \pi(y, C_j^{(2)}) \cup l_\infty(C_j^{(2)})$ on $V^{(2)}$ for some $y \in S$ such that $f(\pi^{(1)}) = \pi^{(2)}$. Hence f yields a permutation $\tau_j \in Sym(s)$ such that $f(\pi(x, C_{\sigma(j)}^{(1)}) \cup l_\infty(C_{\sigma(j)}^{(1)})) = \pi(\tau_j(x), C_j^{(2)}) \cup l_\infty(C_j^{(2)})$. Therefore $f(\pi(x, C_{\sigma(j)}^{(1)})) = \pi(\tau_j(x), C_j^{(2)}) \cdots\cdots [1]$.

We prove that f induces an element of $GOA(s, s^2+s+1)$. Let u and v be affine points of $V^{(1)}$ and $V^{(2)}$ respectively satisfy $f(u) = v$. Let $u = (u(1), u(2), \cdots, u(s^2+s+1))$, $v = (v(1), v(2), \cdots, v(s^2+s+1))$. From $u \in \pi(u(\sigma(j)), C_{\sigma(j)}^{(1)})$ and [1], we have $v = f(u) \in \pi(\tau_j(u(\sigma(j)), C_j^{(2)})$ for $j \in S$. Therefore $v(j) = v(C_j^{(2)}) = \tau_j(u(\sigma(j))$. Hence $v = (\tau_1(u(\sigma(1)), \tau_2(u(\sigma(2)), \cdots \tau_{s^2+s+1}(u(\sigma(s^2+s+1))$. Let $\varphi = (\tau_1, \tau_2, \cdots, \tau_{s^2+s+1}, \sigma) \in GOA(s, s^2+s+1)$. Then $\varphi u = v$. φ is independent of a choice of u. From Lemma 3.2, $A^{(1)}$ and $A^{(2)}$ are isomorphic as $OA(3,s)$s. This completes the proof. $\square$

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