Unimodality problems of multinomial coefficients and symmetric functions

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Abstract

In this note we consider unimodality problems of sequences of multinomial coefficients and symmetric functions. The results presented here generalize our early results for binomial coefficients. We also give an answer to a question of Sagan about strong q-log-concavity of certain sequences of symmetric functions, which can unify many known results for q-binomial coefficients and q-Stirling numbers of two kinds.

Keywords: Unimodality; Log-concavity; Log-convexity; q-log-concavity; Strong q-log-concavity; Multinomial coefficients; Symmetric functions

1 Introduction

Let $a_0, a_1, a_2, \dots$ be a sequence of nonnegative numbers. It is called *unimodal* if $a_0 \le a_1 \le \cdots \le a_{m-1} \le a_m \ge a_{m+1} \ge \cdots$ for some m. It is called *log-concave* (resp. *log-convex*) if $a_{i-1}a_{i+1} \le a_i^2$ (resp. $a_{i-1}a_{i+1} \ge a_i^2$) for all $i \ge 1$. Clearly, a sequence $\{a_i\}$ of positive numbers is log-concave (resp. log-convex) if and only if $a_{i-1}a_{j+1} \le a_ia_j$ (resp. $a_{i-1}a_{j+1} \ge a_ia_j$) for $1 \le i \le j$. So the log-concavity of a sequence of positive numbers implies the unimodality.

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Unimodality problems, including unimodality, log-concavity and log-convexity of sequences, arise naturally in combinatorics and other branches of mathematics (see, e.g., [1, 2, 6, 7, 9, 12, 14, 15]). In particular, many sequences of binomial coefficients enjoy various unimodality properties. For example, the sequence of binomial coefficients along any finite transversal of Pascal's triangle is log-concave and the sequence along any infinite downwards-directed transversal is asymptotically log-convex. More precisely, we have the following result.

Theorem 1 ([13])**.** *Let n_0, k_0, d, δ be four nonnegative integers and $n_0 \geq k_0$. Define*

$$B_i = \binom{n_0 + id}{k_0 + i\delta}, \qquad i = 0, 1, 2, \ldots.$$

Then

(i) *if $\delta \geq d \geq 0$, the sequence $\{B_i\}$ is log-concave; and*

(ii) *if $0 < \delta < d$, the sequence $\{B_i\}$ is asymptotically log-convex, i.e., there exists a nonnegative integer t such that $B_t, B_{t+1}, B_{t+2}, \ldots$ is log-convex.*

The object of the present paper is to generalize the above result for binomial coefficients to multinomial coefficients and symmetric functions. In §2 we give a generalization of Theorem 1 to multinomial coefficients. In §3 we give an answer to a question of Sagan about strong q-log-concavity of certain sequences of symmetric functions, which can unify many known results for q-binomial coefficients and q-Stirling numbers of two kinds.

2 Unimodality of multinomial coefficients

Binomial coefficients can be generalized to multinomial coefficients, which are defined by

$$\binom{m_1 + m_2 + \cdots + m_n}{m_1, m_2, \ldots, m_n} = \begin{cases} \frac{(m_1+m_2+\cdots+m_n)!}{m_1! m_2! \cdots m_n!}, & \text{if } m_k \in \mathbb{N} \text{ for all } k; \\ 0, & \text{otherwise.} \end{cases}$$

The case $n = 2$ gives binomial coefficients:

$$\binom{m_1 + m_2}{m_1, m_2} = \binom{m_1 + m_2}{m_1}.$$

The following result gives a generalization of Theorem 1 to multinomial coefficients.

Theorem 2. *Let $m_k \in \mathbb{N}$ and $d_k \in \mathbb{Z}$ for $k = 1, 2, \ldots, n$. Define*

$$M_i = \binom{\sum_{k=1}^{n} m_k + i \sum_{k=1}^{n} d_k}{m_1 + id_1, m_2 + id_2, \ldots, m_n + id_n}, \quad i = 0, 1, 2, \ldots.$$

Then

(i) *if* $d_1 \geq \sum_{k=1}^n d_k \geq 0$*, then the sequence* $\{M_i\}$ *is log-concave; and*

(ii) *if* $d_k > 0$ *for all* k*, then the sequence* $\{M_i\}$ *is asymptotically log-convex. Furthermore,*

$$\frac{M_{i-1}M_{i+1}}{M_i^2} \sim \left(\frac{i^2}{i^2-1}\right)^{\frac{n-1}{2}} \quad as \quad i \to \infty. \tag{1}$$

Proof. (i) Clearly, to prove the log-concavity of $\{M_i\}$, it suffices to prove the inequality

$$H := \frac{\binom{\sum_{k=1}^n x_k - \sum_{k=1}^n d_k}{x_1-d_1,x_2-d_2,\ldots,x_n-d_n}\binom{\sum_{k=1}^n x_k + \sum_{k=1}^n d_k}{x_1+d_1,x_2+d_2,\ldots,x_n+d_n}}{\binom{\sum_{k=1}^n x_k}{x_1,x_2,\ldots,x_n}^2} \leq 1$$

for $x_k \geq |d_k|$. Denote $X = \sum_{k=1}^n x_k$ and $D = \sum_{k=1}^n d_k$. Then

$$H = \frac{(X-D)!(X+D)!}{(X-d_1)!(X+d_1)!}\prod_{k=2}^n \frac{x_k!^2}{(x_k-d_k)!(x_k+d_k)!}\prod_{j=1}^{d_1}\frac{(x_1-j+1)(X+d_1-j+1)}{(X-j+1)(x_1+d_1-j+1)}.$$

Since the factorial $\{i!\}$ is log-convex and any subsequence of a log-convex sequence is still log-convex, we have

$$x_k!^2 \leq (x_k-d_k)!(x_k+d_k)!$$

and

$$(X-D)!(X+D)! \leq (X-d_1)!(X+d_1)!$$

for $d_1 \geq D \geq 0$. Also,

$$(x_1-j+1)(X+d_1-j+1)-(X-j+1)(x_1+d_1-j+1) = (x_1-X)d_1 \leq 0.$$

Thus $H \leq 1$, as desired.

(ii) To prove the asymptotic estimate (1), we need the Stirling's approximation for the factorial function

$$i! \sim \sqrt{2\pi i}\left(\frac{i}{e}\right)^i.$$

From the above formula we have as $i \to +\infty$,

$$(m+id)! \sim \sqrt{2\pi(m+id)}\left(\frac{m+id}{e}\right)^{m+id} \sim \sqrt{2\pi id}\left(\frac{id}{e}\right)^{m+id}$$

for $d > 0$.

Now assume $d_k > 0$ for all k. Denote $M = \sum_{k=1}^n m_k$ and $D = \sum_{k=1}^n d_k$. Then

$$M_i = \frac{(M+iD)!}{\prod_{k=1}^n (m_k+id_k)!} \sim \frac{D^{M+iD+\frac{1}{2}}}{d_1^{m_1+id_1+\frac{1}{2}}\cdots d_n^{m_n+id_n+\frac{1}{2}}}\left(\frac{1}{2\pi i}\right)^{\frac{n-1}{2}}.$$

It follows that as $i \to +\infty$,

$$\frac{M_{i-1}M_{i+1}}{M_i^2} \sim \left(\frac{i^2}{i^2-1}\right)^{\frac{n-1}{2}}.$$

Thus $M_{i-1}M_{i+1} \geq M_i^2$ for sufficiently large i. In other words, the sequence $\{M_i\}$ is asymptotically log-convex. This completes the proof of the theorem. $\square$

Remark 3. An infinite sequence $a_0, a_1, \ldots$ is called a *Pólya frequency* (PF, for short) sequence if every minor of the matrix $(a_{i-j})_{i,j\geq 0}$ is nonnegative, where $a_k = 0$ if $k < 0$. A finite sequence $a_0, a_1, \ldots, a_n$ is PF if the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$ is PF. Clearly, a PF sequence is log-concave. Recently, Yu [16] showed the following strengthening of Theorem 1, which was conjectured by the present authors in [13]:

(i) If $\delta > d > 0$, then the sequence $\{B_i\}$ is PF.

(ii) If $0 < \delta < d$, then there exists a nonnegative integer t such that $\{B_i\}$ is log-concave for $0 \leq i \leq t$ and log-convex for $i \geq t$.

It would be interesting to know whether similar results hold for multinomial coefficients.

3 Unimodality of symmetric functions

A natural problem is to consider the q-analogue of Theorem 1. We first demonstrate some necessary concepts. Many combinatorial sequences $\{a_k\}$ admit q-analogues, that is, polynomial sequences $\{a_k(q)\}$ in a variable q such that $a_k(1) = a_k$. Gaussian polynomials, also called q-binomial coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{[n]!}{[k]![n-k]!}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

where $[m]! = [1][2]\cdots[m]$ and $[i] = 1 + q + \cdots + q^{i-1}$. Clearly, Gaussian polynomials are the q-analogs of common binomial coefficients. Following Sagan [10], we introduce the definition of q-log-concavity. Given two real polynomials $f(q)$ and $g(q)$, we write $f(q) \leq_q g(q)$ if $g(q) - f(q)$ has nonnegative coefficients as a polynomial in q. Let $\{f_i(q)\}_{i\geq 0}$ be a sequence of real polynomials with nonnegative coefficients. It is called *q-log-concave* if $f_{i-1}(q)f_{i+1}(q) \leq_q f_i^2(q)$ for all $i \geq 1$, and *strongly q-log-concave* if $f_{i-1}(q)f_{j+1}(q) \leq_q f_i(q)f_j(q)$ for all $1 \leq i \leq j$. Clearly, the q-log-concavity for $q = 1$ reduces to the ordinary log-concavity, and the strong q-log-concavity implies the q-log-concavity. But a q-log-concave polynomial sequence need not be strongly q-log-concave (see, e.g., Sagan [10]).

There have been various results on the (strong) q-log-concavity of q-binomial coefficients. For example, it is known that $\left\{\begin{bmatrix} n \\ k \end{bmatrix}\right\}_{n\geq k}$, $\left\{\begin{bmatrix} n \\ k \end{bmatrix}\right\}_{0\leq k\leq n}$ and $\left\{\begin{bmatrix} n+i \\ k+i \end{bmatrix}\right\}_{i\geq 0}$ are strongly q-log-concave respectively (Sagan [11, Theorem 1.1]). It is also well known that q-binomial coefficients can be expressed as specializations of symmetric function. So it is natural to consider log-concavity of symmetric functions.

Let $\mathbb{X} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. The elementary and complete homogeneous symmetric functions of degree k in $x_1, x_2, \ldots, x_n$ are defined by

$$\begin{aligned} e_k(n) := e_k(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \\ h_k(n) := h_k(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \end{aligned}$$

where $e_0(n) = h_0(n) = 1$ and $e_k(n) = 0$ for $k > n$. Set $e_k(n) = h_k(n) = 0$ unless $k, n \ge 0$, and $e_k(0) = h_k(0) = \delta_{0,k}$ where $\delta_{0,k}$ is the Kronecker delta. Then for $n \ge 1$ and $k \in \mathbb{Z}$,

$$\begin{aligned} e_k(n) &= e_k(n-1) + x_n e_{k-1}(n-1), \\ h_k(n) &= h_k(n-1) + x_n h_{k-1}(n). \end{aligned}$$

In [11], Sagan showed that the sequences

$$\{e_k(n)\}_{n \ge 0}, \quad \{h_k(n)\}_{n \ge 0}, \quad \{e_{k-i}(n+i)\}_{i \ge 0}, \quad \{h_{k-i}(n+i)\}_{i \ge 0}$$

are all strongly q-log-concave if $\{x_i\}_{i \ge 1}$ is a strongly q-log-concave sequence of polynomials in q. He also gave the following ([11, Theorem 4.4]).

Proposition 1. *Let $\{x_i\}_{i \ge 1}$ be a sequence of polynomials in q with nonnegative coefficients. Then for $k \le l$ and $m \le n$,*

(i) $e_{k-1}(n) e_{l+1}(m) \le_q e_k(n) e_l(m)$;

(ii) $h_{k-1}(n) h_{l+1}(m) \le_q h_k(n) h_l(m)$.

Moreover, if the sequence $\{x_i\}_{i \ge 1}$ is strongly q-log-concave, then

(iii) $e_k(n+1) e_l(m-1) \le_q e_k(n) e_l(m)$;

(iv) $h_k(n+1) h_l(m-1) \le_q h_k(n) h_l(m)$.

Furthermore, Sagan asked that for which linear relations between n and k, $e_k(n)$ and $h_k(n)$ are strongly q-log-concave respectively ([11, §6]). Here we give an answer to this question by means of Proposition 1.

Theorem 4. *Let $\{x_i\}_{i \ge 1}$ be a sequence of polynomials in q with nonnegative coefficients. If the sequence $\{x_i\}_{i \ge 1}$ is strongly q-log-concave, then for the fixed integers a, b, n_0 and k_0 satisfying $ab \ge 0$, $n_0 \ge k_0$, the sequences*

$$\{e_{k_0 - ib}(n_0 + ia)\}_{i \in \mathbb{Z}}, \qquad \{h_{k_0 - ib}(n_0 + ia)\}_{i \in \mathbb{Z}}$$

are strongly q-log-concave respectively.

Proof. We may assume, without loss of generality, that both a and b are positive. Then, to prove the strong q-log-concavity of $\{e_{k_0-ib}(n_0+ia)\}$, it suffices to show that for $k \le l$ and $m \le n$,

$$e_{k-b}(n+a)e_{l+b}(m-a) \le_q e_k(n)e_l(m). \tag{2}$$

Applying Proposition 1 (i) and (iii) repeatedly, we have

$$e_{k-b}(n+a)e_{l+b}(m-a) \le_q e_{k-b}(n+a-1)e_{l+b}(m-a+1) \le_q \cdots \le_q e_{k-b}(n)e_{l+b}(m)$$

and

$$e_{k-b}(n)e_{l+b}(m) \le_q e_{k-b+1}(n)e_{l+b-1}(m) \le_q \cdots \le_q e_k(n)e_l(m).$$

Thus (2) follows.

Similarly, the strong q-log-concavity of $\{h_{k_0-ib}(n_0+ia)\}$ follows from Proposition 1 (ii) and (iv). $\square$

Theorem 4 can unify many known results. In what follows we present some applications of Theorem 4 for q-binomial coefficients and q-Stirling numbers of two kinds.

It is well known that the q-binomial coefficients satisfy the recursions

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Thus the q-binomial coefficients can be expressed as specializations of symmetric function:

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-\binom{k}{2}} e_k(1, q, \ldots, q^{n-1}) = h_k(1, q, \ldots, q^{n-k}).$$

Similar results hold for the q-Stirling numbers of $c[n,k]$ and $S[n,k]$ two kinds:

$$\begin{aligned} c[n,k] &= e_{n-k}([1],[2],\ldots,[n-1]), \\ S[n,k] &= h_{n-k}([1],[2],\ldots,[k]), \end{aligned}$$

where the q-Stirling numbers are defined by the recursions:

$$\begin{aligned} c[n,k] &= c[n-1,k-1] + [n-1]c[n-1,k] \qquad \text{for } n \ge 1, \text{ with } c[0,k] = \delta_{0,k}, \\ S[n,k] &= S[n-1,k-1] + [k]S[n-1,k] \qquad \text{for } n \ge 1, \text{ with } S[0,k] = \delta_{0,k}. \end{aligned}$$

See Sagan [10, 11] for details. Note that two sequences $\{q^{i-1}\}_{i\ge1}$ and $\{[i]\}_{i\ge1}$ are strongly q-log-concave respectively. Hence the following corollary is an immediate consequence of Theorem 4.

Corollary 1. *Let n_0, k_0, a, b be four nonnegative integers, where $n_0 \ge k_0$. The following sequences are all strongly q-log-concave:*

(i) $\left\{ \begin{bmatrix} n_0-ia \\ k_0+ib \end{bmatrix} \right\}_{i\ge0}$, *with $a, b \ge 0$;*

(ii) $\{c[n_0+ia, k_0+ib]\}_{i\ge0}$, *with $b \ge a \ge 0$;*

(iii) $\{S[n_0 - ia, k_0 + ib]\}_{i\geq 0}$, *with* $a, b \geq 0$*;*

(iv) $\{S[n_0 + ia, k_0 + ib]\}_{i\geq 0}$, *with* $b \geq a \geq 0$.

Remark 5. Many special cases of Corollary 1 have occurred in the literature ([3, 4, 5, 10, 11]).

Remark 6. The q-binomial coefficients, as well as q-Stirling numbers of two kinds, can be arranged in a triangle like Pascal's triangle for the binomial coefficients. Each sequence in Proposition 1 locates exactly on a transversal of the triangle. In particular, by the symmetry of the q-binomial coefficients, each sequence located on a transversal of q-Pascal triangle has a form as $\left\{\begin{bmatrix} n_0-ia \\ k_0+ib \end{bmatrix}\right\}_{i\geq 0}$. The sequence $\left\{\begin{bmatrix} n_0+ia \\ k_0+ib \end{bmatrix}\right\}_{i\geq 0}$ with $a, b \geq 0$ is not strongly q-log-concave in general. For example, the sequence $\{\begin{bmatrix} 2i \\ i \end{bmatrix}\}$ is not strongly q-log-concave.

Remark 7. The sequence $\{c[n_0 - ia, k_0 + ib]\}$ with $a, b \geq 0$ is not strongly q-log-concave in general. For example, the sequence $c[11, 1], c[7, 2], c[3, 3]$ is not even q-log-concave.

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