A q-analogue of some binomial coefficient identities of Y. Sun

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Abstract

We give a q-analogue of some binomial coefficient identities of Y. Sun [Electron. J. Combin. 17 (2010), #N20] as follows:

$$\sum_{k=0}^{\lfloor n/2\rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_{q} q^{\binom{n-2k}{2}} = \begin{bmatrix} m+n \\ n \end{bmatrix}_{q},$$

$$\sum_{k=0}^{\lfloor n/4\rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_{q} q^{\binom{n-4k}{2}} = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_{q},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ stands for the q-binomial coefficient. We provide two proofs, one of which is combinatorial via partitions.

1 Introduction

Using the Lagrange inversion formula, Mansour and Sun [2] obtained the following two binomial coefficient identities:

$$\sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{2k+1}\binom{3k}{k}\binom{n+k}{3k} = \frac{1}{n+1}\binom{2n}{n}, \tag{1.1}$$

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{1}{2k+1}\binom{3k+1}{k+1}\binom{n+k}{3k+1} = \frac{1}{n+1}\binom{2n}{n} \quad (n \geqslant 1). \tag{1.2}$$

In the same way, Sun [3] derived the following binomial coefficient identities

$$\sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{3k+a}\binom{3k+a}{k}\binom{n+a+k-1}{n-2k} = \frac{1}{2n+a}\binom{2n+a}{n}, \tag{1.3}$$

$$\sum_{k=0}^{\lfloor n/4\rfloor} \frac{1}{4k+1}\binom{5k}{k}\binom{n+k}{5k} = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k}{n+1}\binom{n+k}{k}\binom{2n-2k}{n}, \tag{1.4}$$

$$\sum_{k=0}^{\lfloor n/4\rfloor} \frac{n+a+1}{4k+a+1}\binom{5k+a}{k}\binom{n+a+k}{5k+a} = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k\binom{n+a+k}{k}\binom{2n+a-2k}{n+a}. \tag{1.5}$$

It is not hard to see that both (1.1) and (1.2) are special cases of (1.3), and (1.4) is the $a=0$ case of (1.5). A bijective proof of (1.1) and (1.3) using binary trees and colored ternary trees has been given by Sun [3] himself. Using the same model, Yan [4] presented an involutive proof of (1.4) and (1.5), answering a question of Sun.

Multiplying both sides of (1.3) by $n+a$ and letting $m=n+a-1$, we may write it as

$$\sum_{k=0}^{\lfloor n/2\rfloor} \binom{m+k}{k}\binom{m+1}{n-2k} = \binom{m+n}{n}, \tag{1.6}$$

while letting $m=n+a$, we may write (1.5) as

$$\sum_{k=0}^{\lfloor n/4\rfloor} \binom{m+k}{k}\binom{m+1}{n-4k} = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k\binom{m+k}{k}\binom{m+n-2k}{m}. \tag{1.7}$$

The purpose of this paper is to give a q-analogue of (1.6) and (1.7) as follows:

$$\sum_{k=0}^{\lfloor n/2\rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_{q} q^{\binom{n-2k}{2}} = \begin{bmatrix} m+n \\ n \end{bmatrix}_{q}, \tag{1.8}$$

$$\sum_{k=0}^{\lfloor n/4\rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_{q} q^{\binom{n-4k}{2}} = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_{q}, \tag{1.9}$$

where the *q-binomial coefficient* $\begin{bmatrix} x \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{cases} \displaystyle\prod_{i=1}^{k} \frac{1-q^{x-i+1}}{1-q^i}, & \text{if } k \geqslant 0, \\ 0, & \text{if } k < 0. \end{cases}$$

We shall give two proofs of (1.8) and (1.9). One is combinatorial and the other algebraic.

2 Bijective proof of (1.8)

Recall that a *partition* λ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ in decreasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. A nonzero λ_i is called a *part* of λ. The number of parts of λ, denoted by $\ell(\lambda)$, is called the *length* of λ. Write $|\lambda| = \sum_{i=1}^{m} \lambda_i$, called the *weight* of λ. The sets of all partitions and partitions into distinct parts are denoted by $\mathscr{P}$ and $\mathscr{D}$ respectively. For two partitions λ and μ, let $\lambda \cup \mu$ be the partition obtained by putting all parts of λ and μ together in decreasing order.

It is well known that (see, for example, [1, Theorem 3.1])

$$\sum_{\substack{\lambda_1 \leqslant m+1 \\ \ell(\lambda)=n}} q^{|\lambda|} = q^n \begin{bmatrix} m+n \\ n \end{bmatrix}_q,$$

$$\sum_{\substack{\lambda \in \mathscr{D} \\ \lambda_1 \leqslant m+1 \\ \ell(\lambda)=n}} q^{|\lambda|} = \begin{bmatrix} m+1 \\ n \end{bmatrix}_q q^{\binom{n+1}{2}}.$$

Therefore,

$$\sum_{\substack{\mu \in \mathscr{D} \\ \lambda_1, \mu_1 \leqslant m+1 \\ 2\ell(\lambda)+\ell(\mu)=n}} q^{2|\lambda|+|\mu|} = q^n \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+1 \\ n-2k \end{bmatrix}_q q^{\binom{n-2k}{2}},$$

where $k = \ell(\lambda)$. Let

$$\begin{aligned} \mathscr{A} &= \{\lambda \in \mathscr{P} \colon \lambda_1 \leqslant m+1 \text{ and } \ell(\lambda) = n\}, \\ \mathscr{B} &= \{(\lambda, \mu) \in \mathscr{P} \times \mathscr{D} \colon \lambda_1, \mu_1 \leqslant m+1 \text{ and } 2\ell(\lambda) + \ell(\mu) = n\}. \end{aligned}$$

We shall construct a weight-preserving bijection ϕ from $\mathscr{A}$ to $\mathscr{B}$. For any $\lambda \in \mathscr{A}$, we associate it with a pair $(\overline{\lambda}, \mu)$ as follows: If λ_i appears r times in λ, then we let λ_i appear $\lfloor r/2 \rfloor$ times in $\overline{\lambda}$ and $r - 2\lfloor r/2 \rfloor$ times in μ. For example, if $\lambda = (7,5,5,4,4,4,4,2,2,2,1)$, then $\overline{\lambda} = (5,4,4,2)$ and $\mu = (7,2,1)$. Clearly, $(\overline{\lambda}, \mu) \in \mathscr{B}$ and $|\lambda| = 2|\overline{\lambda}| + |\mu|$. It is easy to see that $\phi : \lambda \mapsto (\overline{\lambda}, \mu)$ is a bijection. This proves that

$$\sum_{\lambda \in \mathscr{A}} q^{|\lambda|} = \sum_{(\lambda,\mu) \in \mathscr{B}} q^{2|\lambda|+|\mu|}.$$

Namely, the identity (1.8) holds.

3 Involutive proof of (1.9)

It is easy to see that

$$\begin{aligned}
q^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m+n-2k \\ n-2k \end{bmatrix}_q &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \sum_{\substack{\lambda_1 \leqslant m+1 \\ \ell(\lambda)=k}} q^{2|\lambda|} \sum_{\substack{\mu_1 \leqslant m+1 \\ \ell(\mu)=n-2k}} q^{|\mu|} \\
&= \sum_{\substack{\lambda_1, \mu_1 \leqslant m+1 \\ 2\ell(\lambda)+\ell(\mu)=n}} (-1)^{\ell(\lambda)} q^{2|\lambda|+|\mu|}, \qquad (3.1)
\end{aligned}$$

and

$$q^n \sum_{k=0}^{\lfloor n/4 \rfloor} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q^4} \begin{bmatrix} m+1 \\ n-4k \end{bmatrix}_q q^{\binom{n-4k}{2}} = \sum_{\substack{\mu \in \mathscr{D} \\ \lambda_1, \mu_1 \leqslant m+1 \\ 4\ell(\lambda)+\ell(\mu)=n}} q^{4|\lambda|+|\mu|}. \qquad (3.2)$$

Let

$$\begin{aligned}
\mathscr{U} &= \{(\lambda, \mu) \in \mathscr{P} \times \mathscr{P} \colon \lambda_1, \mu_1 \leqslant m+1 \text{ and } 2\ell(\lambda) + \ell(\mu) = n\}, \\
\mathscr{V} &= \{(\lambda, \mu) \in \mathscr{U} : \text{each } \lambda_i \text{ appears an even number of times and } \mu \in \mathscr{D}\}.
\end{aligned}$$

We shall construct an involution θ on the set $\mathscr{U} \setminus \mathscr{V}$ with the properties that θ preserves $2|\lambda| + |\mu|$ and reverses the sign $(-1)^{\ell(\lambda)}$.

For any $(\lambda, \mu) \in \mathscr{U} \setminus \mathscr{V}$, notice that either some λ_i appears an odd number of times in λ, or some μ_j is repeated in μ, or both are true. Choose the largest such λ_i and μ_j if they exist, denoted by λ_{i_0} and μ_{j_0} respectively. Define

$$\theta((\lambda, \mu)) = \begin{cases} ((\lambda \setminus \lambda_{i_0}), \mu \cup (\lambda_{i_0}, \lambda_{i_0})), & \text{if } \lambda_{i_0} \geqslant \mu_{j_0} \text{ or } \mu \in \mathscr{D}, \\ ((\lambda \cup \mu_{j_0}), \mu \setminus (\mu_{j_0}, \mu_{j_0})), & \text{if } \lambda_{i_0} < \mu_{j_0} \text{ or } \lambda_{i_0} \text{ does not exist.} \end{cases}$$

For example, if $\lambda = (5, 5, 4, 4, 4, 3, 3, 3, 1, 1)$ and $\mu = (5, 3, 2, 2, 1)$, then

$$\theta(\lambda, \mu) = ((5, 5, 4, 4, 3, 3, 3, 1, 1), (5, 4, 4, 3, 2, 2, 1)).$$

It is easy to see that θ is an involution on $\mathscr{U} \setminus \mathscr{V}$ with the desired properties. This proves that

$$\begin{aligned}
\sum_{(\lambda,\mu) \in \mathscr{U}} (-1)^{\ell(\lambda)} q^{2|\lambda|+|\mu|} &= \sum_{(\lambda,\mu) \in \mathscr{V}} (-1)^{\ell(\lambda)} q^{2|\lambda|+|\mu|} \\
&= \sum_{\substack{\mu \in \mathscr{D} \\ \tau_1, \mu_1 \leqslant m+1 \\ 4\ell(\tau)+\ell(\mu)=n}} q^{4|\tau|+|\mu|}, \qquad (3.3)
\end{aligned}$$

where $\lambda = \tau \cup \tau$. Combining (3.1)–(3.3), we complete the proof of (1.9).

4 Generating function proof of (1.8) and (1.9)

Recall that the *q-shifted factorial* is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n = 1,2,\dots.$$

Then we have

$$\frac{1}{(z^2;q^2)_{m+1}}(-z;q)_{m+1} = \frac{1}{(z;q)_{m+1}}, \tag{4.1}$$

$$\frac{1}{(z^4;q^4)_{m+1}}(-z;q)_{m+1} = \frac{1}{(z;q)_{m+1}}\frac{1}{(-z^2;q^2)_{m+1}}. \tag{4.2}$$

By the q-binomial theorem (see, for example, [1, Theorem 3.3]), we may expand (4.1) and (4.2) respectively as follows:

$$\left(\sum_{k=0}^{\infty}\begin{bmatrix}m+k\\k\end{bmatrix}_{q^2} z^{2k}\right)\left(\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}_{q} q^{\binom{k}{2}} z^{k}\right) = \sum_{k=0}^{\infty}\begin{bmatrix}m+k\\k\end{bmatrix}_{q} z^{k}, \tag{4.3}$$

$$\begin{aligned}&\left(\sum_{k=0}^{\infty}\begin{bmatrix}m+k\\k\end{bmatrix}_{q^4} z^{4k}\right)\left(\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}_{q} q^{\binom{k}{2}} z^{k}\right)\\ &= \left(\sum_{k=0}^{\infty}\begin{bmatrix}m+k\\k\end{bmatrix}_{q} z^{k}\right)\left(\sum_{k=0}^{\infty}\begin{bmatrix}m+k\\k\end{bmatrix}_{q^2} (-1)^k z^{2k}\right).\end{aligned} \tag{4.4}$$

Comparing the coefficients of z^n in both sides of (4.3) and (4.4), we obtain (1.8) and (1.9) respectively.

Finally, we give the following special cases of (1.8):

$$\sum_{k=0}^{\lfloor n/2\rfloor}\begin{bmatrix}n+k\\k\end{bmatrix}_{q^2}\begin{bmatrix}n+1\\2k+1\end{bmatrix}_{q} q^{\binom{n-2k}{2}} = \begin{bmatrix}2n\\n\end{bmatrix}_{q}, \tag{4.5}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\begin{bmatrix}n+k\\k+1\end{bmatrix}_{q^2}\begin{bmatrix}n\\2k+1\end{bmatrix}_{q} q^{\binom{n-2k-1}{2}} = \begin{bmatrix}2n\\n-1\end{bmatrix}_{q}. \tag{4.6}$$

When $q = 1$, the identities (4.5) and (4.6) reduce to (1.1) and (1.2) respectively.

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