Which Chessboards have a Closed Knight's Tour within the Rectangular Prism?

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Abstract

A closed knight's tour of a chessboard uses legal moves of the knight to visit every square exactly once and return to its starting position. In 1991 Schwenk completely classified the $m \times n$ rectangular chessboards that admit a closed knight's tour. In honor of the upcoming twentieth anniversary of the publication of Schwenk's paper, this article extends his result by classifying the $i \times j \times k$ rectangular prisms that admit a closed knight's tour.

1 Introduction

The closed knight's tour of a chessboard is a classic problem in mathematics. Can the knight use legal moves to visit every square on the board and return to its starting position? The two dimensional movement of the knight makes its tour an intriguing problem which is trivial for other chess pieces. Euler presents solutions for the 8×8 board in a 1759 paper [4]. Martin Gardner discusses the knight's tour on rectangular boards and other mathematical problems involving the knight in his October 1967 column in Scientific American [5]. Papers exist analyzing the closed knight's tour on variant chessboards such as the cylinder [12], the torus [13], the sphere [1], the exterior of the cube [9] and the interior of the cube [3]. Donald Knuth generalizes the study of the $\{1, 2\}$-knight on a rectangular board to the $\{r, s\}$-leaper on a rectangular board [8]. *Across the Board: The Mathematics of Chessboard Problems* by John Watkins is an indispensable collection of knight's tour results as well as many other mathematically themed chessboard problems [11].

Generalizing away from the chessboard, the closed knight's tour is a subset of the well known problem of the existence of Hamiltonian cycles in graphs. Despite the prior appearance of a paper [7] by Thomas Kirkman posing the general question, this cycle's name originates from mathematician William Rowen Hamilton and his Icosian game of the late 1850's. Photographs of the actual game can be viewed at `http://puzzlemuseum.com/month/picm02/200207icosian.htm`, as hosted by The Puzzle Museum. Hamilton's Icosian game challenged players to visit every city on the board exactly once and return home.

Many results about closed knight's tours for rectangular boards had appeared in the literature throughout the years but no complete characterization of the solution was known until 1991. It was then that Schwenk completely answered the question: Which rectangular chessboards have a closed knight's tour [10]?

Theorem 1 (Schwenk) *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:*

$$\begin{aligned}&(a)\ m \text{ and } n \text{ are both odd;}\\ &(b)\ m \in \{1,2,4\}\,;\\ &(c)\ m = 3 \text{ and } n \in \{4,6,8\}\,.\end{aligned}$$

To honor the twentieth anniversary of Schwenk's Theorem, we extend the result to $i \times j \times k$ rectangular prisms for integers $i, j, k \geq 2$.

Theorem 2 *An $i \times j \times k$ chessboard for integers $i, j, k \geq 2$ has a closed knight's tour unless, without loss of generality, one or more of the following three conditions hold:*

$$\begin{aligned}&(a)\ i,\ j \text{ and } k \text{ are all odd;}\\ &(b)\ i = j = 2;\\ &(c)\ i = 2 \text{ and } j = k = 3.\end{aligned}$$

To begin, consider two views of a closed knight's tour on the $2 \times 5 \times 6$ board. When presenting a board for the first time, we will always display the slices as in Figure 1. Note that this three dimensional tour is not just a combination of two copies of a closed knight's tour of the 5×6 board.

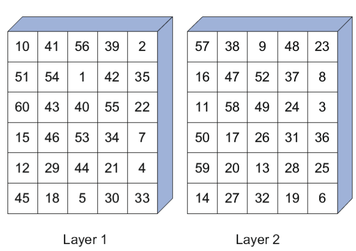

Figure 1: Slices of the $2 \times 5 \times 6$ board

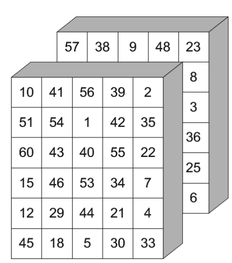

Figure 2: The 3-D view of the $2 \times 5 \times 6$ board

For the two-dimensional case the existence or non-existence of the $m \times n$ board clearly settles the question for the $n \times m$ board after a 90-degree rotation either clockwise or counterclockwise. The same holds true in three dimensions, although more options for rotations exist.

2 Boards without Tours

We first proceed by showing that the boards that satisfy at least one of the conditions of Theorem 2 do not contain a closed knight's tour. Parity conditions on i, j and k immediately dictate a necessary condition. A closed knight's tour does not exist on the $i \times j \times k$ rectangular prism for $i, j, k \equiv 1 \mod 2$. The moves of the knight alternate color on the chessboard as shown in Figure 3 by the $a - b$, $c - d$ and $e - f$ moves. Thus, the knight's graph is bipartite. A closed knight's tour is an alternating cycle of black and white cells. Clearly, the number of white cells must equal the number of black cells. However, if i, j and k are all odd then the number of cells on the board is odd and the number of black cells cannot equal the number of white cells. Thus, no closed knight's tour exists on the $i \times j \times k$ chessboard when i, j and k are all odd.

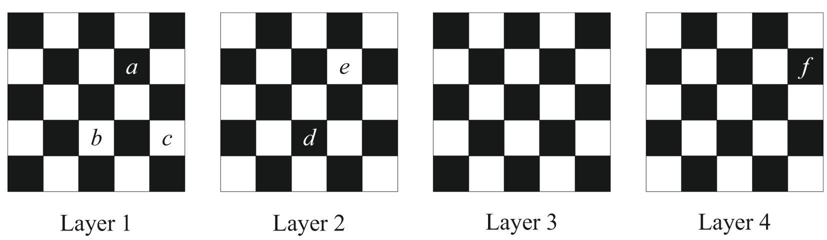

Figure 3: Knight moves on the $4 \times 5 \times 5$ board

It is a necessary condition that at least one of i, j and k be even. It is almost a sufficient condition as well. Almost, but not quite. A closed knight's tour does not exist on the $2 \times j \times 2$ board. The labeling of the cells in the $2 \times j \times 2$ board of Figure 4 shows that the knight's moves on the board are constrained. A knight can only move to and from cells of the same label. The knight's graph on the $2 \times j \times 2$ board is a disconnected graph. For the $3 \times 3 \times 2$ board, isolated vertices exist in the knight's graph as shown by the shaded cells in Figure 5. Naturally, a Hamiltonian cycle cannot exist in a disconnected graph or one with isolated vertices.

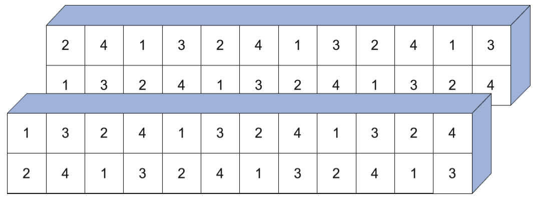

Figure 4: The $2 \times j \times 2$ rectangular prism

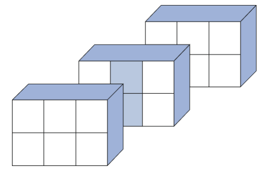

Figure 5: The $3 \times 3 \times 2$ Rectangular Prism

3 General Method to Create Tours

We now prove the existence of a closed knight's tour for all other boards by constructing a tour for each board. Proof of the existence of a closed knight's tour for all other boards will be constructive and use the strong form of induction. As a gentle introduction to the reader we'll begin with examples of the three types of constructions we employ for the $i \times j \times k$ boards. We will use multiple copies of a closed knight's tour on a $2 \times 4 \times 4$ board to illustrate the process. Figure 6 shows the two layers of the $2 \times 4 \times 4$ board while Figure 7 illustrates the $2 \times 4 \times 4$ board in three dimensions.

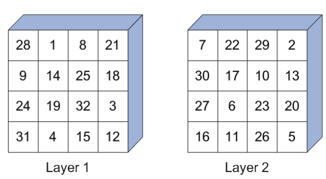

Figure 6: A $2 \times 4 \times 4$ closed knight's tour

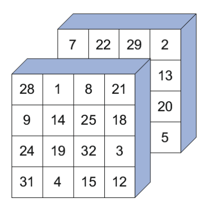

Figure 7: The 3-D view of the $2 \times 4 \times 4$ board

The constructions in our proof begin with two closed knight's tours on two boards sharing at least two common parameters. We place the boards adjacent to each other to create a larger board. By selectively deleting key edges and creating new edges, we create a single closed knight's tour that traverses the new larger board. We use three methods to extend boards that share a common parameter: vertical stacking, horizontal stacking and front stacking. In vertical stacking, we place copies of the $2 \times 4 \times 4$ board on top of each other as shown in Figure 8 to create a $2 \times 4 \times 8$ board. We now want to combine the two disjoint closed knight's tours into one tour that tours every cell of the new $2 \times 4 \times 8$

board exactly once. We achieve this by deleting the $3-4$ edge on the top $2\times4\times4$ board and the $8-9$ edge on the bottom $2\times4\times4$ board and then creating the $3-8$ and $4-9$ edges to connect the previously disjoint tours into one single closed knight's tour for the $2\times4\times8$ board.

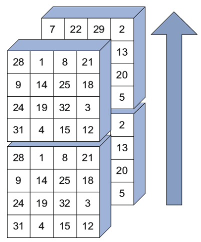

Figure 8: Vertical stacking of two copies of the $2\times4\times4$ board

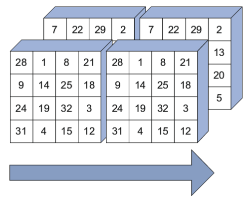

Figure 9: Horizontal stacking of two copies of the $2\times4\times4$ board

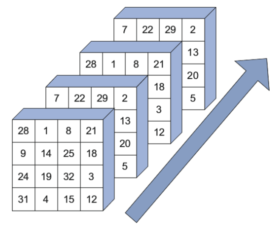

Figure 10: Front stacking of two copies of the $2\times4\times4$ board

Next we proceed with horizontal stacking of two copies of the $2\times4\times4$ board to create a $2\times8\times4$ board as illustrated in Figure 9. Delete the $25-26$ edge of the left $2\times4\times4$ board and the $27-28$ edge of the right $2\times4\times4$ board. Now create the $25-28$ and $26-27$ edges.

Finally we front stack two copies of the $2\times4\times4$ board to create a $4\times4\times4$ board. Delete the $10-11$ edge of the front $2\times4\times4$ board and the $14-15$ edge of the back $2\times4\times4$ board. Now create the $10-15$ and $11-14$ edges.

Using strong induction and the $2\times4\times4$ board, it is possible to construct a closed knight's tour on the $i\times j\times k$ for $i\equiv 0 \mod 2$ and $j,k\equiv 0 \mod 4$. If only a closed knight's tour existed for the $2\times2\times2$ board, our task would be much simpler! This clean and relatively simple example using only the $2\times4\times4$ board encompasses the range of techniques that constitute our entire proof. Many different boards will be required for the complete proof for all the possible values of i, j and k.

4 Boards with Tours

Using the technique demonstrated in the previous section, we need to show how to construct a tour for all other boards not forbidden by Theorem 2. This forthcoming process will be very detailed but conceptually no harder than what we have already done. Since not all three values for i, j and k can be odd we will without loss of generality assume that $i\equiv 0 \mod 2$. We will continue to utilize the $2\times4\times4$ board and introduce new boards as needed. We have already created a tour for any $i\times j\times k$ board where $i\equiv 0 \mod 2$ and $j,k\equiv 0 \mod 4$. Now we construct a tour for the other three values of k mod 4 while fixing $i \equiv 0 \mod 2$ and $j \equiv 0 \mod 4$. Figure 11 presents the $2 \times 4 \times 5$ base board for constructing the tours where $k \equiv 1 \mod 4$. From here on out, we provide the details of the three stacking methods via tables to conserve space.

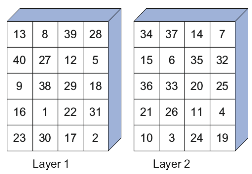

Figure 11: A $2 \times 4 \times 5$ closed knight's tour

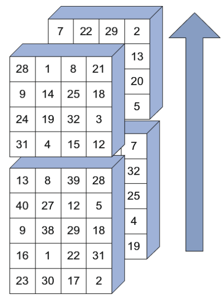

Figure 12: Vertical stacking of the $2 \times 4 \times 5$ board below the $2 \times 4 \times 4$ board of Figure 6

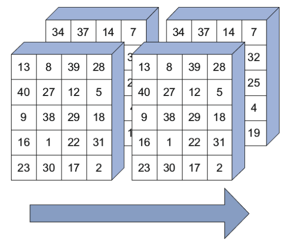

Figure 13: Horizontal stacking of two copies of the $2 \times 4 \times 5$ board

	Delete edges	Create edges
Vertical	$3-4$ top board, $39-40$ bottom board	$3-39$, $4-40$
Horizontal	$31-32$ left board, $37-38$ right board	$31-38$, $32-37$
Front	$10-11$ front board, $16-17$ back board	$10-17$, $11-16$

For $k \equiv 2 \mod 4$, Figure 14 provides the $2 \times 4 \times 6$ base board.

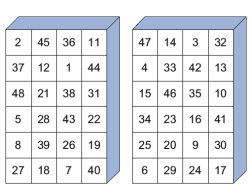

Figure 14: A $2 \times 4 \times 6$ closed knight's tour

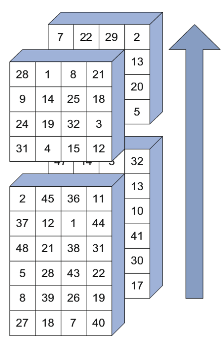

Figure 15: Vertical stacking of the $2 \times 4 \times 6$ board below the $2 \times 4 \times 4$ board of Figure 6

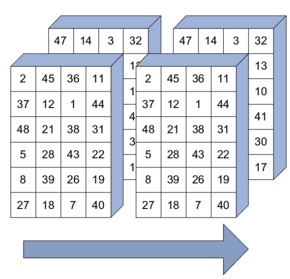

Figure 16: Horizontal stacking of two copies of the $2 \times 4 \times 6$ board

	Delete edges	Create edges
Vertical	$3-4$ top board, $36-37$ bottom board	$3-36$, $4-37$
Horizontal	$30-31$ left board, $28-29$ right board	$28-31$, $29-30$
Front	$29-30$ front board, $39-40$ back board	$29-40$, $30-39$

And finally, a $2\times 4\times 3$ base board for $k \equiv 3 \bmod 4$.

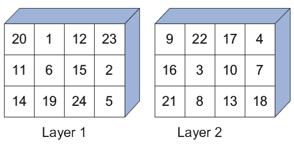

Figure 17: A $2\times 4\times 3$ closed knight's tour

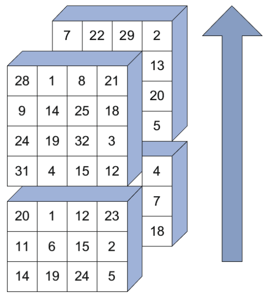

Figure 18: Vertical stacking of the $2\times 4\times 3$ board below the $2\times 4\times 4$ board of Figure 6

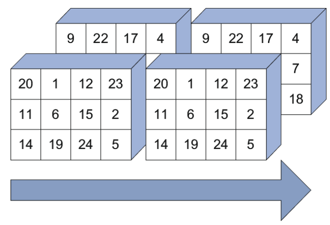

Figure 19: Horizontal stacking of two copies of the $2\times 4\times 3$ board

	Delete edges	Create edges
Vertical	$3-4$ top board, $11-12$ bottom board	$3-12$, $4-11$
Horizontal	$4-5$ left board, $20-21$ right board	$4-21$, $5-20$
Front	$7-8$ front board, $5-6$ back board	$5-8$, $6-7$

At this point we have constructed a closed knight's tour for any $i\times j\times k$ board for $i \equiv 0 \mod 2$, $j \equiv 0 \mod 4$ and $k > 2$. Next we cover the case of $i \equiv 0 \mod 2$, $j \equiv 1 \mod 4$ and $k > 2$. We will extend these boards in three dimensions using our usual techniques. Previously we've used the $2\times 4\times 4$ board of Figure 6 in the inductive step. Now we use the $2\times 4\times 5$ board from Figure 11 in the inductive step. We begin with a $2\times 5\times 5$ base board.

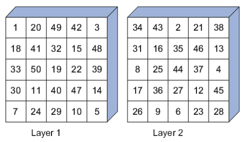

Figure 20: A $2 \times 5 \times 5$ closed knight's tour

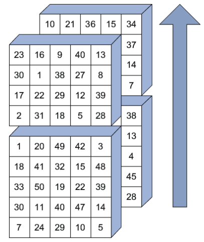
Figure 21: Vertical stacking of the $2 \times 5 \times 5$ board below the $2 \times 5 \times 4$ board of Figure 11

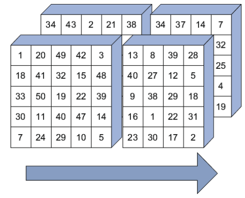
Figure 22: Horizontal stacking of the $2 \times 5 \times 5$ board and the $2 \times 5 \times 4$ board of Figure 11

	Delete edges	Create edges
Vertical	$4-5$ top board, $20-21$ bottom board	$4-21$, $5-20$
Horizontal	$47-48$ left board, $8-9$ right board	$8-48$, $9-47$
Front	$43-44$ front board, $49-50$ back board	$43-50$, $44-49$

Next, we create the $2 \times 5 \times 6$ base board and its extensions.

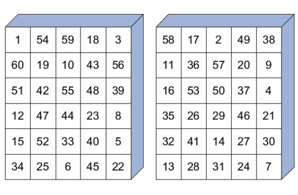

Figure 23: A $2 \times 5 \times 6$ closed knight's tour

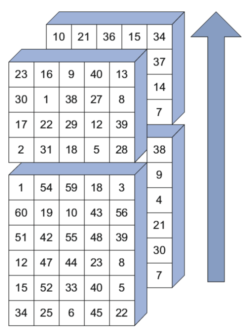
Figure 24: Vertical stacking of the $2 \times 5 \times 6$ board below the rotated $2 \times 5 \times 4$ board of Figure 11

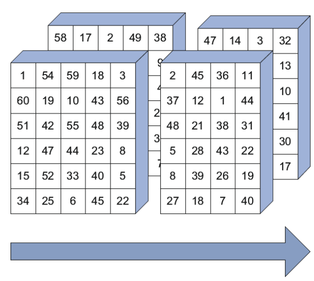
Figure 25: Horizontal stacking of the $2 \times 5 \times 6$ board and the $2 \times 4 \times 6$ board of Figure 14

	Delete edges	Create edges
Vertical	$11-12$ top board, $2-3$ bottom board	$2-11$, $3-12$
Horizontal	$37-38$ left board, $33-34$ right board	$33-38$, $34-37$
Front	$57-58$ front board, $59-60$ back board	$57-60$, $58-59$

Once more for the $2 \times 5 \times 3$ board.

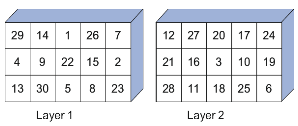

Figure 26: A $2 \times 5 \times 3$ closed knight's tour

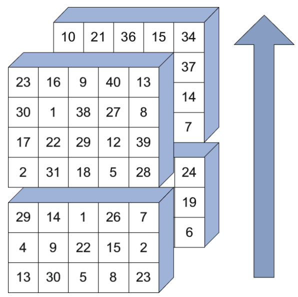

Figure 27: Vertical stacking of the $2 \times 5 \times 3$ board below the rotated $2 \times 4 \times 5$ board of Figure 11

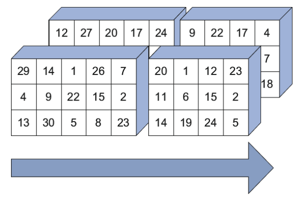

Figure 28: Horizontal stacking of the $2 \times 5 \times 3$ board and the $2 \times 4 \times 3$ board of Figure 17

	Delete edges	Create edges
Vertical	$17-18$ top board, $14-15$ bottom board	$14-17$, $15-18$
Horizontal	$23-24$ left board, $20-21$ right board	$20-23$, $21-24$
Front	$27-28$ front board, $29-30$ back board	$27-30$, $28-29$

There is no need for a $2 \times 5 \times 4$ board for the case of $i \equiv 0 \mod 2$, $j \equiv 1 \mod 4$ and $k \equiv 0 \mod 4$ as that case is covered by a rotation of the board created for the $i \equiv 0 \mod 2$, $j \equiv 0 \mod 4$ and $k \equiv 1 \mod 4$ case.

Continuing with our strategy, we proceed to create a closed knight's tour on all $i \times j \times k$ boards for $i \equiv 0 \mod 2$, $j \equiv 2 \mod 4$ and $k > 2$. We begin with a $2 \times 6 \times 6$ base board (since no $2 \times 2 \times k$ tour exists) and use the $2 \times 4 \times 6$ board of Figure 14 to extend.

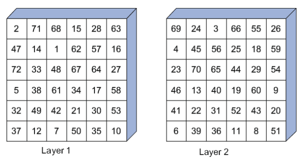

Figure 29: A $2 \times 6 \times 6$ closed knight's tour

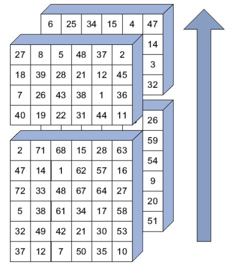

Figure 30: Vertical stacking of the $2 \times 6 \times 6$ board below the rotated $2 \times 4 \times 6$ board of Figure 14

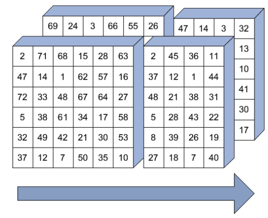

Figure 31: Horizontal stacking of the $2 \times 6 \times 6$ board and the $2 \times 4 \times 6$ board of Figure 14

	Delete edges	Create edges
Vertical	$43 - 44$ top board, $15 - 16$ bottom board	$15 - 43$, $16 - 44$
Horizontal	$29 - 30$ left board, $4 - 5$ right board	$4 - 29$, $5 - 30$
Front	$22 - 23$ front board, $32 - 33$ back board	$22 - 33$, $23 - 32$

Once again for the $2 \times 6 \times 3$ board.

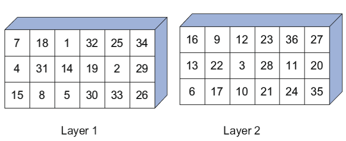

Figure 32: A $2 \times 6 \times 3$ closed knight's tour

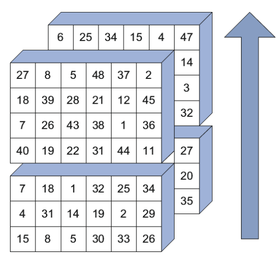

Figure 33: Vertical stacking of the $2 \times 6 \times 3$ board below the $2 \times 6 \times 4$ board of Figure 14

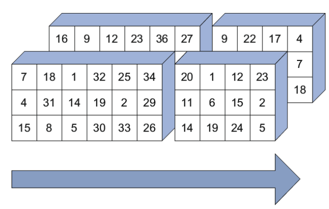

Figure 34: Horizontal stacking of the $2 \times 6 \times 3$ board and the $2 \times 4 \times 3$ board of Figure 17

	Delete edges	Create edges
Vertical	$9 - 10$ top board, $11 - 12$ bottom board	$9 - 12$, $10 - 11$
Horizontal	$26 - 27$ left board, $20 - 21$ right board	$20 - 26$, $21 - 27$
Front	$22 - 23$ front board, $18 - 19$ back board	$18 - 23$, $19 - 22$

Now on to the $i \times j \times k$ boards for $i \equiv 0 \mod 2$, $j \equiv 3 \mod 4$ and $k > 2$. The non-existence of a $2 \times 3 \times 3$ board forces us to use a $2 \times 7 \times 3$ closed knight's tour as a base case. To extend it in three dimensions, we vertically stack it with a $2 \times 7 \times 4$ board. This extension is a 90-degree rotation of the $2 \times 4 \times 7$ board which appeared in Figure 18. For clarity, we renumber the cells of the board from Figure 18 with labels 1 through 56 to create the board in Figure 36.

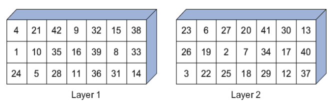

Figure 35: A $2 \times 7 \times 3$ closed knight's tour

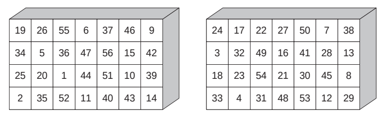

Figure 36: A $2 \times 7 \times 4$ closed knight's tour

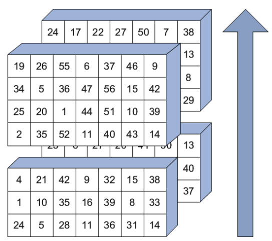

Figure 37: Vertical stacking of the $2 \times 7 \times 3$ board below the $2 \times 7 \times 4$ board of Figure 36

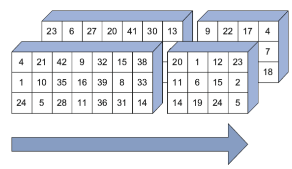

Figure 38: Horizontal stacking of the $2 \times 7 \times 3$ board and the $2 \times 4 \times 3$ board of Figure 17

	Delete edges	Create edges
Vertical	$20 - 21$ top board, $7 - 8$ bottom board	$7 - 20$, $8 - 21$
Horizontal	$13 - 14$ left board, $20 - 21$ right board	$13 - 21$, $14 - 20$
Front	$12 - 13$ front board, $14 - 15$ back board	$12 - 15$,$13 - 14$

We have almost demonstrated how to create all $2 \times j \times k$ boards that permit a closed knight's tour according to Theorem 2. At first glance it seems that we have covered all permitted combinations of i, j and k. However, the non-existence of the $2 \times 3 \times 3$ board prevented us from creating one particular case of boards; all $i \times 3 \times 3$ where $i \equiv 0 \bmod 2$. Constructing the $4 \times 3 \times 3$ and $6 \times 3 \times 3$ base boards and presenting the method to let i assume any even value allows us to complete the proof of Theorem 2. We can create all $i \times 3 \times 3$ boards where $i \equiv 0 \bmod 4$ with Figure 39.

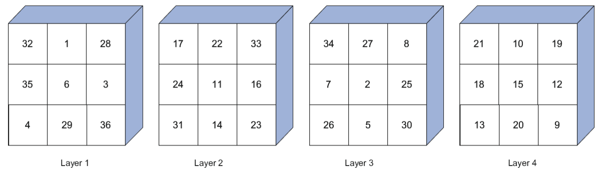

Figure 39: Slices of a $4 \times 3 \times 3$ closed knight's tour

	Delete edges	Create edges
Front	$9-10$ front board, $28-29$ back board	$9-28$, $10-29$

For $i \times 3 \times 3$ boards where $i \equiv 2 \bmod 4$ we begin with Figure 40 and continually front stack Figure 39.

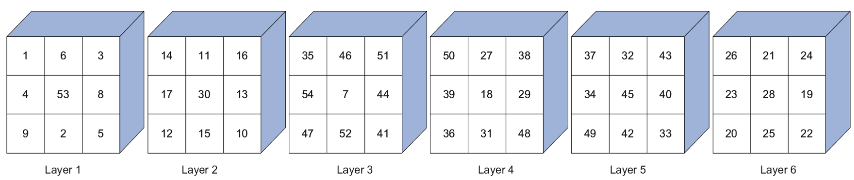

Figure 40: Slices of a $6 \times 3 \times 3$ closed knight's tour

	Delete edges	Create edges
Front	$21-22$ front board, $28-29$ back board	$21-29$, $22-28$

5 Future Work

One could pursue an increase in the number of dimensions with a $1-2$ knight by searching for closed knight's tours in four dimensions. Once we move to $n \geq 4$ dimensions we lose the ability to easily visualize the geometry of a closed knight's tour. One approach would use vectors of length n to represent the cells of the board. A legal move of the knight from one square to another square would change two of the vector coordinates from the initial square. One coordinate would change by ± 1 and the other by ± 2. A very ambitious project would be to find a general classification for the existence of a closed knight's tour in the n dimensional cube where Theorems 1 and 2 in this paper are just the specific cases for $n = 2, 3$.

Another option explores the nature of the move of the knight [9]. On the two dimensional board, the knight's move incorporates both directions in the $x-y$ plane. One could argue that the move of the knight on the three dimensional board should incorporate all three directions in the $x-y-z$ plane. How should the knight move in the three dimensional board? Perhaps, the obvious variant piece is a $1-2-3$ knight. Unfortunately, the move of the $1-2-3$ knight in the three-dimensional board is not bipartite. Such a move would leave the knight locked into one color like the bishop. See [2] for an example of a tour of the cube of side $n = 6$ with two $1-2-3$ knights, one for the black cells and one for the white cells.

Instead of a linear change in the number of cells the knight moves, let's consider an exponential change and use a $1-2-4$ knight in a three dimensional board. This move has an advantage over the $1-2-3$ knight since the graph of the $1-2-4$ knight is bipartite. As a teaser, in Figure 41, we leave you with a $1-2-4$ closed knight's tour of the cube of side 8, the smallest cube that admits such a tour.

349	278	339	176	157	18	391	234
120	501	60	451	40	415	206	413
347	62	435	44	437	196	421	42
74	433	96	395	72	393	20	337
497	38	489	200	507	46	485	64
6	439	70	399	98	341	88	125
251	140	151	22	447	86	149	268
204	1	36	483	58	301	110	471

Layer 1

408	195	498	103	454	63	368	45
71	386	225	364	89	360	95	294
156	23	448	233	410	283	372	175
35	310	219	458	191	472	69	326
262	179	358	277	346	273	366	17
59	332	187	470	205	450	183	312
244	47	322	185	420	43	428	197
97	292	73	126	7	336	27	328

Layer 2

39	416	255	490	221	508	65	464
158	223	390	325	348	311	340	173
321	252	369	446	315	504	303	144
4	229	494	291	466	257	492	111
317	350	319	338	245	432	323	396
478	119	402	327	476	293	414	207
57	300	123	436	227	438	211	398
170	75	148	271	250	113	190	21

Layer 3

80	167	94	357	320	343	16	365
403	258	477	188	511	208	495	102
90	473	92	463	142	419	226	459
353	272	371	174	263	178	379	26
192	503	212	453	104	481	220	445
387	274	405	224	363	136	359	138
8	443	314	411	282	373	240	127
309	34	429	198	457	50	285	52

Layer 4

61	318	121	502	213	452	41	324
264	177	382	279	394	253	370	19
287	488	247	444	201	412	235	392
500	169	496	99	422	199	426	189
141	376	161	434	85	400	289	342
2	203	512	231	486	267	506	109
37	482	217	440	171	484	143	302
152	5	352	139	150	269	124	87

Layer 5

12	155	24	377	288	259	82	131
423	232	409	266	499	168	455	68
218	407	194	385	216	367	160	241
331	276	333	430	361	182	295	284
166	461	214	449	186	469	248	441
305	374	261	180	381	510	345	28
78	135	286	297	308	329	162	335
281	54	243	48	427	184	313	30

Layer 6

215	116	159	14	133	84	145	238
202	3	230	509	256	487	108	505
389	254	417	222	491	172	465	66
10	401	316	351	304	375	246	129
307	122	475	228	493	210	467	100
76	265	290	383	280	397	112	431
147	270	249	118	153	236	355	114
106	479	56	299	32	425	164	181

Layer 7

474	91	462	67	468	101	418	209
93	260	81	130	11	380	25	344
134	79	146	237	356	117	334	15
55	404	105	480	165	460	31	384
154	13	354	115	378	239	132	83
33	424	163	442	107	456	51	298
406	193	388	29	306	53	242	49
275	330	9	362	77	128	137	296

Layer 8

Figure 41: A $1-2-4$ closed knight's tour of the cube of side 8

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The authors would like to thank Michael Schornak for his contributions to this project. Michael wrote a program that finds closed knight's tours in three dimensional boards. When constructing base boards for inductive purposes, Michael's program proved to be a priceless resource. We would also like to thank the Mentor-Protege program at Kennesaw State University for supporting this research. We appreciate the thoughtful and insightful comments by the anonymous referee which improved the clarity of the presentation of our work. And finally, we thank Dr. Philippe Laval for his help with the technical details of LaTeX and Scientific Workplace.

References

[1] G. Cairns, Pillow Chess, *Mathematics Magazine*, 75, no. 3. (June, 2002), 173–186.

[2] J. DeMaio, Closed Monochromatic Bishops' Tour, *Journal of Recreational Mathematics*, 34, no. 3, pp. 196–203

[3] J. DeMaio, Which Chessboards have a Closed Knight's Tour within the Cube?, *The Electronic Journal of Combinatorics,* 14, (2007) R32.

[4] L. Euler, Solutio d'une Question Curieuse qui ne Peroit Soumise a Aucune Analyse, *Mem. Acad. Sci.* Berlin 15 (1759), 310–337.

[5] M. Gardner, Problems that are built on the knight's move in chess, Scientific American, 217 (1967) no. 4, 128–132.

[6] R. J. Gould, Hamiltonian Graphs, *Handbook of Graph Theory*, CRC Press, Boca Raton, (2004), 261–278.

[7] T. P. Kirkman, On the Representation of Polyhedra, Philosophical Transactions of the Royal Society (London) 146 (1856), 413–418

[8] D. E. Knuth, Leaper Graphs, *Math. Gazette* 78 (1994), 274–297.

[9] Y. Qing and J.J. Watkins, Knight's Tours for Cubes and Boxes, *Congressus Numerantium* 181 (2006) 41–48.

[10] A. J. Schwenk, Which Rectangular Chessboards have a Knight's Tour? *Mathematics Magazine* 64:5 (December 1991) 325–332.

[11] J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, Princeton, 2004.

[12] J. J. Watkins, Knight's tours on cylinders and other surfaces, *Congr. Numer.* 143 (2000), 117–127.

[13] J. J. Watkins, Knight's tours on a torus, *Mathematics Magazine*, 70:3 (1997), 175–184.

[14] D. B. West, *Introduction to Graph Theory, Second Edition,* Prentice Hall, Upper Saddle River, 2001.