Rainbow paths with prescribed ends

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Abstract

It was conjectured in [S. Akbari, F. Khaghanpoor, and S. Moazzeni. Colorful paths in vertex coloring of graphs. Preprint] that, if G is a connected graph distinct from C_7, then there is a $\chi(G)$-coloring of G in which every vertex $v \in V(G)$ is an initial vertex of a path P with $\chi(G)$ vertices whose colors are different. In [S. Akbari, V. Liaghat, and A. Nikzad. Colorful paths in vertex coloring of graphs. Electron. J. Combin. 18(1):P17, 9pp, 2011] this was proved with $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices instead of $\chi(G)$ vertices. We strengthen this to $\chi(G) - 1$ vertices. We also prove that every connected graph with at least one edge has a proper k-coloring (for some k) such that every vertex of color i has a neighbor of color $i+1$ (mod k). C_5 shows that k may have to be greater than the chromatic number. However, if the graph is connected, infinite and locally finite, and has finite chromatic number, then the k-coloring exists for every $k \geq \chi(G)$. In fact, the k-coloring can be chosen such that every vertex is a starting vertex of an infinite path such that the color increases by 1 (mod k) along each edge. The method is based on the circular chromatic number $\chi_c(G)$. In particular, we verify the above conjecture for all connected graphs whose circular chromatic number equals the chromatic number.

Keywords: Chromatic number, circular coloring, rainbow path.

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1 Introduction

The Gallai-Roy theorem implies that, for any k-coloring of a k-chromatic graph G, there exists a path with k vertices, all of distinct colors. As pointed out by Hossein Hajiabolhassan, (private communication), it follows from Minty's characterization of the chromatic number [8] (see also [4]) that, if $k \geq 3$, then G even contains a cycle which contains a subpath with k vertices and with all the k colors occurring in the order $1, 2, \ldots, k$. In particular, if G is connected, then each vertex is the starting vertex of a path containing all colors, as also proved in [6, 7].

[1, 2, 3, 6, 7] study k-colorings with long rainbow paths starting with any prescribed vertex. In this note we apply the circular chromatic number to refine some of those results.

A *proper k-coloring* of a graph G is a function $c : V(G) \longrightarrow [k] = \{1, 2, \ldots, k\}$ such that, for any two adjacent vertices u and v, $c(u) \neq c(v)$. The least number k that G admits a proper k-coloring is called the *chromatic number* of G, and is denoted by $\chi(G)$. If n and d are positive integers with $n \geq 2d$ and $\gcd(n, d) = 1$, then an (n, d)-coloring of G is a function $c : V(G) \longrightarrow [n]$ such that, for any edge $uv \in E(G)$, $d \leq |c(u) - c(v)| \leq n - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of those ratios $\frac{n}{d}$ such that G admits an (n, d)-coloring. As a proper k-coloring is the same as a $(k, 1)$-coloring, it is clear that $\chi_c(G) \leq \chi(G)$. If c is an (n, d)-coloring of G, then we obtain a proper $\lceil \frac{n}{d} \rceil$-coloring by giving each vertex v the color $\lceil \frac{c(v)}{d} \rceil$. Hence $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.

We shall use the following two facts about the circular chromatic number:

Vince [9] (see also [10]) introduced the circular chromatic number and proved that

(1): The infimum can be replaced by minimum.

Guichard [5] proved that

(2): If $\chi_c(G) = \frac{n}{d} > 2$ then, for every (n, d)-coloring c, G contains a cycle $C : v_1 v_2 \ldots v_m v_1$ (where the indices are expressed modulo m) such that, for each i, the difference $c(v_{i+1}) - c(v_i)$ is equal to $d \pmod{n}$ (that is, the difference is either d or $d - n$). (As $\gcd(n, d) = 1$, any such cycle has length divisible by n.)

Lin [7] raised the question if a connected graph G has a proper $\chi(G)$-coloring such that every vertex of G is on a path with $\chi(G)$ vertices, all of different colors.

The following stronger conjecture was proposed in [1].

Conjecture 1. *Let G be a connected graph, and $G \neq C_7$. Then there exists a proper $\chi(G)$-coloring of G such that, for every vertex $v \in V(G)$, there is a path starting at v containing all $\chi(G)$ colors.*

2 Application of circular coloring to rainbow paths

We begin by extending an elegant coloring lemma by Akbari, Liaghat, and Nikzad [2] to circular colorings.

Lemma 1. *Let G be a connected graph, and let f be an (n, d)-coloring of G. Let H be any nonempty subgraph of G.*

Then there is an (n, d)-coloring c of G such that:

a) if $v \in V(H)$, then $f(v) = c(v)$ and

b) for every vertex $v \in V(G) \setminus V(H)$ there is a path $v_0v_1 \ldots v_m$ such that $v_0 = v$, v_m is in H, and $c(v_{i+1}) - c(v_i)$ is equal to $d \pmod n$ for $i = 0, 1, \ldots, m-1$.

Proof. Let c be an (n, d)-coloring with maximum number of vertices v satisfying the conclusion of the Lemma. Let S be the set of all those vertices v. We claim that $V(G) = S$. Suppose therefore (reductio ad absurdum) that this is not the case.

Consider two vertices $u \in S$ and $v \notin S$ such that uv is an edge in G. Define $x_{uv} = c(u) - c(v) \pmod n$ such that x_{uv} is an integer in $\{1, 2, \ldots, n-1\}$. Let t is the minimum such number x_{uv}. Then $t > d$. Now define a coloring c' such that c' and c are the same on S, and, for every vertex $x \in V(G) \setminus S$, $c'(x) = c(x) + t - d$. It is easy to see that c' is an (n, d)-coloring with a bigger set S, a contradiction. ∎

In [2] Conjecture 1 was verified for all graphs G satisfying $\chi(G) = \omega(G)$ by letting H in Lemma 1 be a complete subgraph with $\omega(G)$ vertices. We shall here extend this result to the larger class of graphs for which $\chi(G) = \chi_c(G)$.

Theorem 1. *Let G be a connected graph with $\chi_c(G) = \frac{n}{d}$. Then G has an (n, d)-coloring c of G such that, for every vertex $v \in V(G)$, there is a path $P = v_0v_1 \ldots v_{n-1}$ such that $v = v_0$ and for each $i \in \{0, 1, \ldots n-2\}$, $c(v_{i+1}) - c(v_i) = d \pmod n$.*

Proof. If G is bipartite, the statement is trivial. So assume that $\frac{n}{d} > 2$. Consider an (n, d)-coloring of G. Let H be the subgraph induced by the cycle in (2) found by Guichard [5]. Now apply Lemma 1. ∎

It was also shown in [2] that there is some $\chi(G)$-coloring of any given graph G such that each vertex $v \in V(G)$ is an initial vertex of a rainbow path (that is, a path in which no two vertices have the same color) with the number of vertices at least $\lfloor \frac{\chi(G)}{2} \rfloor$. We extend that as follows.

Theorem 2. *Let G be a connected graph. Then there is a $\chi(G)$-coloring of G such that for every vertex $v \in V(G)$, there exists a path with $\lfloor \chi_c(G) \rfloor$ vertices, all of different colors.*

Proof. Assume that $\chi_c(G) = \frac{n}{d}$, and let c be an (n, d)-coloring as in Theorem 1. Define $f : V(G) \longrightarrow \{1, 2, \ldots, \lceil \frac{n}{d} \rceil\}$ such that $f(v) = \lceil \frac{c(v)}{d} \rceil$. Note that $\chi(G) = \lceil \frac{n}{d} \rceil$. It is easy to check that f is a proper $\chi(G)$-coloring of G. Consider an arbitrary vertex $v \in V(G)$. Let $P = v_0v_1 \ldots v_{n-1} (v = v_0)$ be a path as in Theorem 1. Then all f-colors of $v_0, v_1, \ldots, v_{\chi(G)-1}$ are distinct except that possibly $f(v_0) = f(v_{\chi(G)-1})$, as desired. ∎

Corollary 1. *For any connected graph G, there is a $\chi(G)$-coloring of G such that every vertex of G is an initial vertex of a rainbow path with $\chi(G) - 1$ vertices.*

Corollary 2. *Let G be a connected graph. Then there exists a natural number k and a proper k-coloring of G such that every vertex of color i, say, has a neighbor of color $i + 1$ $(\mathrm{mod}\ k)$.*

Proof. By Theorem 1, there is an (n, d)-coloring c of G such that every vertex $v \in V(G)$ of color i has a neighbor w of color $i+d \pmod n$. Since $\gcd(n, d) = 1$, there exists a naturel number q such that $qd = 1 \pmod n$. We define a new n-coloring f of G by letting $f(v)$ be $qc(v)$ reduced modulo n. If $c(v) - c(w) = d \pmod n$, then $f(v) - f(w) = 1 \pmod n$, and hence every vertex $v \in V(G)$ of color i has a neighbor w of color $i + 1 \pmod n$. ■

As mentioned earlier, there are examples where we must have $k > \chi(G)$. We do not know if Corollary 2 holds for $k = \chi(G) + 1$. For infinite graphs stronger results hold, as we prove in the next section.

3 A colored version of Kőnig's Infinity Lemma

In the previous sections all graphs are finite. In this section graphs are allowed to be infinite. Kőnig's Infinity Lemma says that an infinite connected graph G which is also locally finite (that is, all vertices have finite degree) has a one-way infinite path. We now extend this to a colored version which also verifies Conjecture 1 for connected infinite, locally finite graphs with finite chromatic number.

Theorem 3. *Let G be a connected infinite, locally finite graph with finite chromatic number $\chi(G)$. Then for any $k \geq \chi(G)$, there is a k-coloring c of G such that any vertex of G is an initial vertex of an infinite path $v_1 v_2 \dots$ such that $c(v_{i+1}) = c(v_i) + 1$ $(\mathrm{mod}\ k)$ for each $i = 1, 2, \dots$.*

Proof. Assume that $k \geq \chi(G)$ is a positive integer. Since G is a locally finite and connected graph, $V(G)$ can be written as $\{v_1, v_2, \dots\}$ such that, for any $m \in \mathbb{N}$, the subgraph G_m induced by $V_m = \{v_1, v_2, \dots, v_m\}$ is connected. For each $m \in \mathbb{N}$, let c_m be an (n, d)-coloring of G_m satisfying the conclusion of Lemma 1, where $n = k, d = 1, H = \{v_m\}$. For infinitely many m, $c_m(v_1)$ has the same color which we call $c(v_1)$. For infinitely many of those colorings, $c_m(v_2)$ has the same color which we call $c(v_2)$. We continue like this, defining a coloring c of G. This is clearly a proper k-coloring. Now consider any vertex of G, say v_1. If c_m is one of the colorings used in defining c, then G_m has a path P_m from v_1 to v_m such that the colors (in c_m) increase by 1 $(\mathrm{mod}\ k)$ along each edge. (Note that c and c_m may not agree on all the vertices $v_1, v_2, \dots v_m$.) Infinitely many of these P_m share the first edge e_1 (because v_1 has finite degree). Infinitely many of those paths share the second edge e_2, etc. Now the infinite sequence $e_1, e_2, \dots$ is the edge sequence of an infinite path whose colors increase by one $(\mathrm{mod}\ k)$ along each edge. ■

In Theorem 3 it is important that the graph be locally finite even if we seek a coloring c such that every vertex has a neighbor of color $c(v) + 1 \pmod k$. To see this, take an infinite collection of pairwise disjoint 7-cycles C_7. Select a vertex in each of them, and identify all those vertices so that we obtain a graph whose blocks are all copies of C_7.

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