

Growth of graph powers

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Submitted: Nov 11, 2010; Accepted: Mar 23, 2011; Published: Apr 14, 2011

Mathematics Subject Classification: 05C12

Abstract

For a graph G , its r th power is constructed by placing an edge between two vertices if they are within distance r of each other. In this note we study the amount of edges added to a graph by taking its r th power. In particular we obtain that, for $r \geq 3$, either the r th power is complete or “many” new edges are added. In this direction, Hegarty showed that there is a constant $\epsilon > 0$ such $e(G^3) \geq (1 + \epsilon)e(G)$. We extend this result in two directions. We give an alternative proof of Hegarty’s result with an improved constant of $\epsilon = \frac{1}{6}$. We also show that for general r , $e(G^r) \geq (\lceil \frac{r}{3} \rceil - 1) e(G)$.

1 Introduction

This note addresses some questions raised by P. Hegarty in [4]. In that paper he studied results about graphs inspired by the Cauchy-Davenport Theorem.

All graphs in this paper are simple and loopless. For two vertices $u, v \in V(G)$, denote the length of the shortest path between them by $d(u, v)$. For $v \in V(G)$, define its i th neighborhood as $N_i(v) = \{u \in V(G) : d(u, v) = i\}$. The r th power of a graph G , denoted G^r , is constructed from G by adding an edge between two vertices x and y when they are within distance r in G . Define the diameter of G , $\text{diam}(G)$, as the minimal r such that G^r is complete (alternatively, the maximal distance between two vertices). Denote the number of edges of G by $e(G)$. For $v \in V(G)$ and a set of vertices S , define $e^r(v, S) = |\{u \in S : d(v, u) \leq r\}|$.

The Cayley graph of a subset $A \subseteq \mathbb{Z}_p$ is constructed on the vertex set \mathbb{Z}_p . For two distinct vertices $x, y \in \mathbb{Z}_p$, we define xy to be an edge whenever $x - y \in A$ or $y - x \in A$.

*Research supported by the LSE postgraduate research studentship scheme.

The following is a consequence of the Cauchy-Davenport Theorem (usually stated in the language of additive number theory [1, 2]).

Theorem 1. *Let p be a prime, A a subset of \mathbb{Z}_p , and G the Cayley graph of A . Then for any integer $r < \text{diam}(G)$:*

$$e(G^r) \geq r e(G).$$

If we take A to be the arithmetic progression $\{a, 2a, \dots, ka\}$, then equality holds in this theorem for all $r < \text{diam}(G)$. We might look for analogues of Theorem 1 for more general graphs G . In particular since these Cayley graphs are always regular and (when p is prime) connected, we might focus on regular, connected G . In [4] Hegarty proved the following theorem:

Theorem 2. *Suppose G is a regular, connected graph with $\text{diam}(G) \geq 3$. Then we have*

$$e(G^3) \geq (1 + \epsilon) e(G),$$

with $\epsilon \approx 0.087$

In other words, the cube of G retains the original edges of G and gains a positive proportion of new ones. In Section 3 we prove this theorem with an improved constant of $\epsilon = \frac{1}{6}$. Since we announced this note, DeVos and Thomassé [3] further improved the constant in Theorem 2 to $\epsilon = \frac{3}{4}$. They also show that the constant cannot be improved further by exhibiting a sequence of regular graphs G_n , such that $\frac{e(G_n^r)}{e(G_n)} \rightarrow \frac{7}{4}$ as $n \rightarrow \infty$.

Theorem 2 leads to the question of how the growth behaves for other powers of the G . Note that Theorem 2 cannot be used recursively to obtain such a result – since the cube of a regular graph is not necessarily regular. In [4] it was shown that no equivalent of Theorem 2 exists with G^3 replaced by G^2 , and it was asked what happens for higher powers. In this note we address that question.

2 Main Result

We prove the following theorem:

Theorem 3. *Suppose G is a regular, connected graph, and $r \leq \text{diam}(G)$. Then we have:*

$$e(G^r) \geq \left(\left\lceil \frac{r}{3} \right\rceil - 1 \right) e(G).$$

Proof. Let the degree of each vertex be d . Fix some v with $N_{\text{diam}(G)}(v)$ nonempty.

Consider any vertex $u \in V(G)$. Then for any j satisfying $d(u, v) - r < j \leq d(u, v)$, there is a $w_j \in N_j(v)$ such that $d(u, w_j) < r$. For such a w_j , all vertices $x \in N_1(w_j)$ have $d(u, x) \leq r$. All such x are contained in $N_{j-1}(v) \cup N_j(v) \cup N_{j+1}(v)$, hence

$$e^r(u, N_{j-1}(v) \cup N_j(v) \cup N_{j+1}(v)) \geq d. \tag{1}$$

Note that each $j \in \{d(u, v) - 3, d(u, v) - 6, \dots, d(u, v) - 3(\lceil \frac{1}{3} \min\{d(u, v), r\} \rceil - 1)\}$ satisfies $d(u, v) - r < j \leq d(u, v)$. Summing the bound (1) over all these j , noting that any edge is counted at most once, we obtain

$$e^r(u, N_0(v) \cup \dots \cup N_{d(u,v)-2}(v)) \geq \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil d - d.$$

Now we sum this over all $u \in G$. Note that since the edges counted above go from some $N_i(v)$ to $N_j(v)$ with $j < i$, each edge is counted at most once. Also we haven't yet counted any of the original edges of G , so we might as well add them. Hence

$$\begin{aligned} e(G^r) &\geq \sum_{u \in G} e^r(u, N_0(v) \cup \dots \cup N_{d(u,v)-2}(v)) + e(G) \\ &\geq \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil d - |V(G)|d + e(G) \\ &= \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil d - e(G). \end{aligned} \tag{2}$$

Obviously there was nothing particularly special about v . We can get a similar expression using $v' \in N_{\text{diam}(G)}(v)$, namely

$$e(G^r) \geq \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil d - e(G). \tag{3}$$

Averaging (2) and (3) we get

$$e(G^r) \geq \frac{1}{2} \sum_{u \in G} \left(\left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil + \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil \right) d - e(G). \tag{4}$$

Note that for any $u \in V(G)$ we have

$$\left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil + \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil \geq \left\lceil \frac{r}{3} \right\rceil. \tag{5}$$

This is because $d(u, v) + d(u, v') \geq d(v, v') = \text{diam}(G) \geq r$. Putting the bound (5) into the sum (4) we obtain

$$e(G^r) \geq \frac{|V(G)|d}{2} \left\lceil \frac{r}{3} \right\rceil - e(G) = \left\lceil \frac{r}{3} \right\rceil e(G) - e(G).$$

Thus the theorem is proven. □

3 Cubes

Note that for $r \leq 6$ the bounds in Theorem 3 are trivial. In particular it says nothing about the increase in the number of edges of the cube of a regular, connected graph. Such an increase was already demonstrated by Hegarty in Theorem 2. Here we give an alternative proof of that theorem, yielding a slightly better constant.

Theorem 4. *Suppose G is a regular, connected graph with $\text{diam}(G) \geq 3$. Then we have*

$$e(G^3) \geq \left(1 + \frac{1}{6}\right) e(G).$$

Proof. Let the degree of each vertex be d . Note that as G is regular, and not complete, every $v \in V(G)$ will have a non-neighbour in G . Together with connectedness this implies that each $v \in V(G)$ has at least one new neighbour in G^2 . This implies the theorem for $d \leq 6$. For the remainder of the proof, we assume that $d > 6$. The proof rests on the following colouring of the edges of G : For an edge uv in G , colour

$$\begin{aligned} uv \text{ red} & \text{ if } |N_1(u) \cap N_1(v)| > \frac{2}{3}d, \\ uv \text{ blue} & \text{ if } |N_1(u) \cap N_1(v)| \leq \frac{2}{3}d. \end{aligned}$$

Notice that if uv is a blue edge, then there are at least $\frac{4}{3}d - 1$ neighbours of u in G^2 . This is because u will be connected to everything in $N_1(u) \cup N_1(v)$ except itself, and $|N_1(u) \cup N_1(v)| \geq \frac{4}{3}d$ for uv blue. If, in addition, we have some x connected to u by an edge (of any colour), then x will be at distance at most 3 from everything in $N_1(u) \cup N_1(v) \setminus \{x\}$. Hence x will have at least $\frac{4}{3}d - 1$ neighbours in G^3 .

Partition the vertices of G as follows:

$$\begin{aligned} B &= \{v \in V(G) : v \text{ has a blue edge coming out of it}\}, \\ R &= \{v \in V(G) : v \notin B \text{ and there is a } u \in B \text{ such that } uv \text{ is an edge}\}, \\ S &= V(G) \setminus (B \cup R). \end{aligned}$$

By the above argument, if v is in $B \cup R$, then $e^3(v, V(G)) \geq \frac{4}{3}d - 1$. Recall that each $u \in S$ will have at least one new neighbour in G^2 , giving $e^3(u, V(G)) \geq d + 1$. Summing these two bounds over all vertices in G , noting that any edge is counted twice, gives

$$\begin{aligned} 2e(G^3) &\geq \left(\frac{4}{3}d - 1\right) |B \cup R| + (d + 1)|S| \\ &= \left(\frac{4}{3}d - 1\right) |B \cup R| + (d + 1) (|V(G)| - |B \cup R|) \\ &= \frac{7}{6}d|V(G)| + \frac{1}{3} \left(|B \cup R| - \frac{1}{2}|V(G)| \right) (d - 6) \\ &= \frac{7}{3}e(G) + \frac{1}{3} \left(|B \cup R| - \frac{1}{2}|V(G)| \right) (d - 6). \end{aligned}$$

Recall that we are considering the case when $d > 6$. Thus to prove that $e(G^3) \geq \frac{7}{6}e(G)$, it suffices to show that $|B \cup R| \geq \frac{1}{2}|V(G)|$. To this end we shall demonstrate that $|S| \leq |R|$. First however we need a proposition helping us to find blue edges in G .

Proposition 5. *For any $v \in V(G)$ there is some $b \in B$ such that $d(v, b) \leq 2$.*

Proof. Suppose $d(v, u) = 3$. Then there are vertices x and y such that $\{v, x, y, u\}$ forms a path between u and v . We will show that one of the edges vx , xy or yu is blue. This will prove the proposition assuming that there are *any* blue edges to begin with. However, it also shows the existence of blue edges because $\text{diam}(G) \geq 3$.

So, suppose that the edges vx and uy are red. Then we have $|N_1(v) \cap N_1(x)| > \frac{2}{3}d$, and $|N_1(u) \cap N_1(y)| > \frac{2}{3}d$. Using this and $N_1(u) \cap N_1(v) = \emptyset$ gives

$$\begin{aligned} |N_1(x) \cup N_1(y)| &\geq |(N_1(x) \cup N_1(y)) \cap N_1(v)| + |(N_1(x) \cup N_1(y)) \cap N_1(u)| \\ &\geq |N_1(x) \cap N_1(v)| + |N_1(y) \cap N_1(u)| \\ &> \frac{4}{3}d. \end{aligned}$$

Therefore $|N_1(x) \cap N_1(y)| = 2d - |N_1(x) \cup N_1(y)| \leq \frac{2}{3}d$. Hence xy is blue, proving the proposition. \square

Now we will show that $|S| \leq |R|$. Suppose $r \in R$. By the definition of R , there is a $b \in B$ such that rb is an edge. This edge is necessarily red as $r \notin B$. Using $N_1(b) \subseteq B \cup R$, we have $|N_1(r) \cap (B \cup R)| \geq |N_1(r) \cap N_1(b)| > \frac{2}{3}d$. Hence

$$|N_1(r) \cap S| \leq \frac{1}{3}d. \tag{6}$$

Suppose $s \in S$. Proposition 5 implies that there is some $r \in R$ such that sr is an edge. Since sr is red, we have $|N_1(s) \cap N_1(r)| > \frac{2}{3}d$. Using this, the fact that $N_1(s) \subseteq R \cup S$, and (6), gives

$$\begin{aligned} |N_1(s) \cap R| &\geq |N_1(s) \cap N_1(r) \cap R| \\ &= |N_1(s) \cap N_1(r)| - |N_1(s) \cap N_1(r) \cap S| \\ &\geq |N_1(s) \cap N_1(r)| - |N_1(r) \cap S| \\ &> \frac{1}{3}d. \end{aligned} \tag{7}$$

Double-counting the edges between S and R using the bounds (6) and (7) gives a contradiction unless $|S| \leq |R|$. Therefore $|B \cup R| \geq \frac{1}{2}|V(G)|$ as required. \square

4 Discussion

Theorem 3 answers the question of giving a lower bound on the number of edges that are gained by taking higher powers of a graph. We obtain growth that is linear with r – just as in Theorem 1.

- The constant $\lceil \frac{1}{3}r \rceil$ in Theorem 3 cannot be improved to something of the form λr with $\lambda > \frac{1}{3}$. To see this, consider the following sequence of graphs $H_r(d)$ as d tends to infinity:

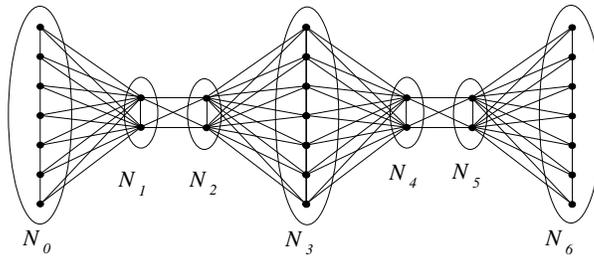


Figure 1: The graph $H_6(8)$.

Take disjoint sets of vertices N_0, \dots, N_r , with $|N_i| = d - 1$ if $i \equiv 0 \pmod{3}$ and $|N_i| = 2$ otherwise. Add all the edges within each set and also between neighboring ones. So if $u \in N_i$, $v \in N_j$, then uv is an edge whenever $|i - j| \leq 1$ (see Figure 1).

The number of edges in $H_r(d)$ is at least the number of edges in the larger classes which is $\left\lceil \frac{1}{3}(r + 1) \right\rceil \binom{d-1}{2}$.

The r th power $H_r(d)^r$ has less than $\binom{|V(G)|}{2}$ edges which is less than $\binom{\lceil \frac{1}{3}(r+1) \rceil (d+3)}{2}$. Therefore,

$$\limsup_{d \rightarrow \infty} \frac{e(H_r(d)^r)}{e(H_r(d))} \leq \lim_{d \rightarrow \infty} \frac{\binom{\lceil \frac{1}{3}(r+1) \rceil (d+3)}{2}}{\left\lceil \frac{1}{3}(r + 1) \right\rceil \binom{d-1}{2}} = \left\lceil \frac{1}{3}(r + 1) \right\rceil.$$

The graphs $H_r(d)$ are not regular, but if $r \not\equiv 2 \pmod{3}$, it is possible to remove a small (less than $|V(G)|$) number of edges from the graphs and make them d -regular without losing connectedness (any cycle passing through all the vertices in $N_1 \cup \dots \cup N_{r-1}$ would work). Call these new graphs $\hat{H}_r(d)$. By the same argument as before we have

$$\limsup_{d \rightarrow \infty} \frac{e(\hat{H}_r(d)^r)}{e(\hat{H}_r(d))} \leq \left\lceil \frac{1}{3}(r + 1) \right\rceil.$$

If $r \equiv 2 \pmod{3}$, a similar trick can be performed, but we'd need to start with $|N_i| = d - 1$ if $i \equiv 1 \pmod{3}$ and $|N_i| = 2$ otherwise.

So the factor of $\frac{1}{3}$ cannot be improved for regular graphs. All these examples are inspired by one given in [4] to show that for any ϵ there are regular graphs G with $e(G^2) < (1 + \epsilon)e(G)$.

- All the questions from this paper and [4] could be asked for directed graphs. In particular one can define directed Cayley graphs for a set $A \subseteq \mathbb{Z}_p$ by letting xy be a directed edge whenever $x - y \in A$. Then the Cauchy-Davenport Theorem implies an identical version of Theorem 1 for directed Cayley graphs. In this setting it is easy to show that there is growth even for the square of an out-regular oriented graph D (a directed graph where for a pair of vertices u and v , uv and vu are not

both edges). In particular, we have

$$e(D^2) \geq \frac{3}{2} e(D). \quad (8)$$

This occurs because every vertex v has $|N_2^{out}(v)| \geq \frac{1}{2}|N_1^{out}(v)|$ in an out-regular oriented graph. It's easy to see that this is best possible for such graphs. One can construct out-regular oriented graphs with an arbitrarily large proportion of vertices v satisfying $|N_2^{out}(v)| = \frac{1}{2}|N_1^{out}(v)|$.

However if we insist on *both* in and out-degrees to be constant, (8) no longer seems tight. Such graphs are always Eulerian. In [5] there is a conjecture attributed to Jackson and Seymour that if an oriented graph D is Eulerian, then $e(D^2) \geq 2 e(D)$ holds. If this conjecture were proved, it would be an actual generalization of the directed version of Theorem 1, as opposed to the mere analogues proved above.

Acknowledgement

The author would like to thank his supervisors Jan van den Heuvel and Jozef Skokan for helpful advice and discussions.

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