Integral Quartic Cayley Graphs on Abelian Groups

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Abstract

A graph is called integral, if its adjacency eigenvalues are integers. In this paper we determine integral quartic Cayley graphs on finite abelian groups. As a side result we show that there are exactly 27 connected integral Cayley graphs up to 11 vertices.

1 Introduction and Results

A graph is called integral if all the eigenvalues of its adjacency matrix are integers. The notion of integral graphs was first introduced by Harary and Schwenk in 1974 [13].

It is known that the number of non-isomorphic k-regular integral graphs is finite (See e.g. [10]). Bussemaker and Cvetković [8] and independently Schwenk [20], proved that there are exactly 13 connected cubic integral graphs. It is shown in [4] and [5] that there are exactly 263 connected integral graphs on up to 11 vertices.

Radosavljević and Simić in [19] determined all thirteen nonregular nonbipartite connected integral graphs with maximum degree four. Stevanović [22] determined all connected 4-regular integral graphs avoiding ± 3 in the spectrum. A survey of results on integral graphs may be found in [6].

Omidi [17] identified integral graphs with at most two cycles with no eigenvalues 0. Sander [18] proved that Sudoku graphs are integral. In [2] it is shown that the total number of adjacency matrices of integral graphs with n vertices is less than or equal to $2^{\frac{n(n-1)}{2}-\frac{n}{400}}$ for a sufficiently large n. Let G be a non-trivial group with the identity element 1 and let S be a non-empty subset of $G \setminus \{1\}$ such that $S = S^{-1} := \{s^{-1} | s \in S\}$. The

Cayley graph of G with respect to S which is denoted by $\Gamma(S : G)$ is the graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$.

Klotz and Sander [15] proved that $\Gamma(U_n : \mathbb{Z}_n)$ is integral, where $\mathbb{Z}_n$ is the cyclic group of order n and U_n is the subset of all elements of Z_n of order n. W. So [21] characterizes integral graphs among circulant graphs. In [1] we determined integral cubic Cayley graphs.

In this paper we study integral quartic Cayley graphs on finite abelian groups. Our main results are the following.

Theorem 1.1 *Let G be an abelian group such that $\Gamma(S : G)$ is integral, 4-regular and connected for some $S \subseteq G$. Then*

$$|G| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\}.$$

As a side result, we show that

Theorem 1.2 *There are exactly 27 connected integral Cayley graphs up to 11 vertices.*

2 Preliminaries

First we give some facts that are needed in the next section. Let n be a positive integer. Then $B(1, n)$ denotes the set $\{j \mid 1 \le j < n, (j, n) = 1\}$. Let $\omega = e^{\frac{2\pi i}{n}}$ and

$$C(r, n) = \sum_{j \in B(1,n)} \omega^{jr}, \qquad 0 \le r \le n-1. \tag{2.1}$$

The function $C(r, n)$ is a Ramanujan sum. For integers r and n, $(n > 0)$, Ramanujan sums have only integral values (See [16] and [23]).

First we give some facts that are needed in the next section.

Lemma 2.1 *Let $\omega = e^{\frac{\pi i}{n}}$, where $i^2 = -1$. Then*

i) $\displaystyle\sum_{j=1}^{2n-1} \omega^j = -1.$

ii) If l is even, then $\displaystyle\sum_{j=1}^{n-1} \omega^{lj} = -1.$

iii) If l is odd, then $\displaystyle\sum_{j=1}^{n-1} \omega^{lj} + \omega^{-lj} = 0.$

Proof. The proof is straightforward. $\square$

Lemma 2.2 *Let G be a finite group and $a \in G$. If χ is a linear character of G and $o(a) = 2$, then $\chi(a) = \pm 1$.*

Proof. We know that each linear character is a homomorphism. So $(\chi(a))^2 = \chi(a^2) = \chi(1) = 1$. Hence $\chi(a) = \pm 1$.

Lemma 2.3 *[3] Let G be a finite group of order n whose irreducible characters (over $\mathbb{C}$) are $\chi_1, \ldots, \chi_h$ with respective degree $n_1, \ldots, n_h$. Then the spectrum of the Cayley graph $\Gamma(S : G)$ can be arranged as $\Lambda = \{\lambda_{ijk} \mid i = 1, \ldots, h; j, k = 1, \ldots, n_i\}$ such that $\lambda_{ij1} = \ldots = \lambda_{ijn_i}$ (this common value will be denoted by λ_{ij}), and*

$$\lambda_{i1}^t + \ldots + \lambda_{in_i}^t = \sum_{s_1, \ldots, s_t \in S} \rho_i(\Pi_{l=1}^t s_l) \tag{2.2}$$

for any natural number t.

Lemma 2.4 *[14] Let C_n be the cyclic group generated by a of order n. Then the irreducible characters of C_n are $\rho_j(a^k) = \omega^{jk}$, where $j, k = 0, 1, \ldots, n-1$.*

Lemma 2.5 *[14] Let $G = C_{n_1} \times \cdots \times C_{n_r}$ and $C_{n_i} = \langle a_i \rangle$, so that for any $i, j \in \{1, \ldots, r\}$, $(n_i, n_j) \neq 1$. If $\omega_t = e^{\frac{2\pi i}{n_t}}$, then $n_1 \cdots n_r$ irreducible characters of G are*

$$\rho_{l_1 \ldots l_r}(a_1^{k_1}, \ldots, a_r^{k_r}) = \omega_1^{l_1 k_1} \omega_2^{l_2 k_2} \cdots \omega_r^{l_r k_r} \tag{2.3}$$

where $l_i = 0, 1, \ldots, n_i - 1$ and $i = 1, 2, \ldots, r$.

Lemma 2.6 (Lemma 2.6 of [1]) *Let G be a group and $G = \langle S \rangle$, where $S = S^{-1}$ and $1 \notin S$. If $a \in S$ and $o(a) = m > 2$, then $\Gamma(S : G)$ has the cycle with m vertices as a subgraph.*

Lemma 2.7 (Lemma 2.7 of [1]) *Let $G = \langle S \rangle$ be a group, $|G| = n$, $|S| = 2$, $S = S^{-1} \not\ni 1$. Then $\Gamma(S : G)$ is an integral graph if and only if $n \in \{3, 4, 6\}$.*

Lemma 2.8 (Lemma 2.9 of [1]) *Let G be the cyclic group $\langle a \rangle$, $|G| = n > 3$ and let S be a generating set of G such that $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is an integral graph if and only if $n \in \{4, 6\}$.*

Lemma 2.9 *Let G_1 and G_2 be two non-trivial abelian groups and $G = G_1 \times G_2$ such that $\Gamma(S : G)$ is integral, $G = \langle S \rangle$, $S = S^{-1} \not\ni 1$ and $|S| = 4$. If $S_1 = \{s_1 \mid (s_1, g_2) \in S$ for some $g_2 \in G_2\} \setminus \{1\}$, then $\Gamma(S_1 : G_1)$ is a connected integral graph.*

Proof. Since S generates G and $S = S^{-1} \not\ni 1$ with four elements, S_1 generates G_1, $S_1 = S_1^{-1} \not\ni 1$ and $|S_1| \in \{1, 2, 3, 4\}$. It is easy to see that if $|S_1| = 1$, then $|G_1| = 2$ and so $\Gamma(S_1 : G_1)$ is the complete graph K_2 with two vertices which is an integral graph.
Let χ_0 and ρ_0 be the trivial irreducible characters of G_1 and G_2, respectively. Let λ_{i0} and λ_i be the eigenvalues of $\Gamma(S : G)$ and $\Gamma(S_1 : G_1)$ corresponding to irreducible characters of $\chi_i \times \rho_0$ and χ_i, respectively. By Lemma 2.3,

$$\lambda_{i0} = \sum_{(g_1, g_2) \in S} (\chi_i \times \rho_0)(g_1, g_2).$$

We have the following cases:
Case 1: If $|S_1| = 4$, then $\lambda_{i0} = \lambda_i$. It follows that $\Gamma(S_1 : G_1)$ is an integral graph.
Case 2: Let $|S_1| = 3$ and suppose that

$$S = \{(a,x),(a^{-1},x^{-1}),(b,y),(b,y^{-1})\} \;\text{ or }\; S = \{(a,x),(a^{-1},x^{-1}),(b,y),(1,y^{-1})\},$$

where $o(b) = 2$. Then $\lambda_{i0} = \lambda_i + \chi_i(b)$ or $\lambda_{i0} = \lambda_i + 1$, respectively. Since $2 \mid \chi_i(b) - \chi_i(1)$, $\chi_i(b)$ is integer and So $\Gamma(S_1 : G_1)$ is an integral graph.
Case 3: Let $|S_1| = 2$ and $S = \{(a,x),(a^{-1},x^{-1}),(1,y),(1,y^{-1})\}$. Then $\lambda_{i0} = \lambda_i + 2$ and so $\Gamma(S_1 : G_1)$ is an integral graph.
Case 4: Let $|S_1| = 2$ and $S = \{(a,x),(a^{-1},x^{-1}),(a,y),(a^{-1},y^{-1})\}$. Then G_1 is a cyclic group and

$$\lambda_{i0} = 2\sum_{s_1 \in S_1} \chi_i(s_1) = 2\lambda_i.$$

Since $\lambda_{i0} \in \{-4, \pm 3, \pm 2, \pm 1, 0\}$ $(i \neq 0)$, λ_1 and $\lambda_2 \in \{-2, \pm 3/2, \pm 1, \pm 1/2, 0\}$. By Lemmas 2.3 and 2.4, $\lambda_1 = 2\cos(\frac{2\pi}{n})$ and $\lambda_2 = 2\cos 2(\frac{2\pi}{n})$, where $|G_1| = n$. By using $\cos 2x = 2\cos^2 x - 1$ we conclude that $\lambda_1^2 = \lambda_2 + 2$. Hence $(\lambda_1, \lambda_2) \in \{(0,-2),(-1,-1),(1,-1)\}$.
If $(\lambda_1, \lambda_2) = (1,-1)$, then $\cos(\frac{2\pi}{n}) = \frac{1}{2}$. So $n = 6$. By [1, Lemma 2.7], $\Gamma(S_1 : G_1)$ is an integral graph.
If $(\lambda_1, \lambda_2) = (0,-2)$, then $\cos(\frac{2\pi}{n}) = 0$. So $n = 4$. By [1, Lemma 2.7] $\Gamma(S_1 : G_1)$ is an integral graph.
If $(\lambda_1, \lambda_2) = (-1,-1)$, then $\cos(\frac{2\pi}{n}) = \frac{-1}{2}$. So $n = 3$. By [1, Lemma 2.7], $\Gamma(S_1 : G_1)$ is an integral graph. □

Lemma 2.10 (Lemma 2.11 of [1]) *Let G be a finite non-cyclic abelian group and let $G = \langle S \rangle$, where $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is an integral graph if and only if $|G| \in \{4, 8, 12\}$.*

Theorem 2.11 (Theorem 1.1 of [1]) *There are exactly seven connected cubic integral Cayley graphs. In particular, for a finite group G and a subset $S = S^{-1} \not\ni 1$ with three elements, $\Gamma(S : G)$ is integral graph if and only if G is isomorphic to one the following groups: C_2^2, C_4, C_6, S_3, C_2^3, $C_2 \times C_4$, D_8, $C_2 \times C_6$, D_{12}, A_4, S_4, $D_8 \times C_3$, $D_6 \times C_4$ or $A_4 \times C_2$.*

We denote as usual the complete graph on n vertices by K_n and the complete bipartite graph with parts of sizes m and n by $K_{m,n}$.

Lemma 2.12 *There are exactly 40 connected, regular, integral graphs up to 10 vertices.*

Proof. By [4], connected, regular, integral graphs are of type Γ_i $(i = 1, \ldots, 40)$, where $\Gamma_1 = K_1$, $\Gamma_2 = K_2$, $\Gamma_3 = K_3$, $\Gamma_4 = K_4$, $\Gamma_5 = K_{2,2}$, $\Gamma_6 = K_5$, $\Gamma_7 = K_7$ and connected, regular, integral graphs with 6, 8, 9 and 10 vertices are displayed in tables 1, 2, 3 and 4 respectively. Graphs in these tables are represented in the form

$$\Gamma_i \;\; a_{12}a_{13}a_{23}a_{14}a_{24}a_{34}\cdots a_{1n}a_{2n}\cdots a_{(n-1)n},$$

where Γ_i is the name of the corresponding integral graph and

$$a_{12}a_{13}a_{23}a_{14}a_{24}a_{34}\cdots a_{1n}a_{2n}\cdots a_{(n-1)n},$$

is the upper diagonal part of its adjacency matrix $[a_{ij}]_{n\times n}$ of the graph Γ_i. Also spectra of these graphs are displayed in tables 5, 6, 7 and 8 respectively. $\square$

Lemma 2.13 *Let $G = \langle a\rangle$ be a finite cyclic group of order $n > 4$ and let S be a generating set of G such that $|S| = 4$, $S = S^{-1}$ and $1 \notin S$. Then there exist two relatively prime, positive integers $r, s < n/2$ $(r \neq s)$ such that $S = \{a^r, a^{-r}, a^s, a^{-s}\}$.*

Proof. Since G is cyclic, G has at most one element of order 2. Therefore S cannot contain elements of order 2 as $|S| = 4$ and $S = S^{-1}$. Since $a^{-\ell} = a^{n-\ell}$ for any integer ℓ, it follows that there exist two positive integers $r, s < n/2$ $(r \neq s)$ such that $S = \{a^r, a^{-r}, a^s, a^{-s}\}$. Since S generates G, $\gcd(r, s) = 1$. This completes the proof. $\square$

Lemma 2.14 *Let $G = \langle a\rangle$ be a finite cyclic group of order $n > 4$ and let S be a generating set of G such that $|S| = 4$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is integral if and only if one the following holds:*

1. $n = 5$ *and* $S = \{a, a^{-1}, a^2, a^{-2}\}$;
2. $n = 6$ *and* $S = \{a, a^{-1}, a^2, a^{-2}\}$;
3. $n = 8$ *and* $S = \{a, a^{-1}, a^3, a^{-3}\}$;
4. $n = 10$ *and* $S = \{a, a^{-1}, a^3, a^{-3}\}$;
5. $n = 12$ *and* $S = \{a, a^{-1}, a^5, a^{-5}\}$;
6. $n = 12$ *and* $S = \{a^2, a^{-2}, a^3, a^{-3}\}$;
7. $n = 12$ *and* $S = \{a^4, a^{-4}, a^3, a^{-3}\}$.

Proof. We need So's theorem [21, Theorem 7.1] and some knowledge about Euler's totient function φ:

$$\varphi(n) = 2 \Longleftrightarrow n \in \{3, 4, 6\}, \quad \varphi(n) = 4 \Longleftrightarrow n \in \{5, 8, 10, 12\}.$$

Let $\Gamma = \Gamma(S : G)$. According to So's theorem [21, Theorem 7.1] we have to consider two main cases.
Case 1. Γ is a unitary Cayley graph.
Then the degree of regularity of Γ is $\varphi(n) = 4$, which implies $n \in \{5, 8, 10, 12\}$. This gives graphs (1), (3), (4), (5) in the list of Lemma 2.14.
Case 2. It follows from Lemma 2.13 that there are proper divisors r, s of n, $1 \leq r < s < n/2$, such that

$$S = \{a^r, a^s, a^{-r}, a^{-s}\},\ \varphi(\frac{n}{r}) = 2,\ \varphi(\frac{n}{s}) = 2 \text{ and } \gcd(r, s) = 1.$$

This means that there is an integer k such that $n = krs$ and

$$\frac{n}{r} = ks, \frac{n}{s} = kr \in \{3, 4, 6\};\ k, r, s \in \{1, 2, 3, 4, 6\}.$$

We distinguish two subcases:
Case 2.1. $r = 1$.
Then we have $\frac{n}{r} = n \in \{3, 4, 6\}$, which implies $n = 6$.
Now $\frac{n}{s} = \frac{6}{s} \in \{3, 4, 6\}$ implies $s = 2$. This gives graph (2).
Case 2.2. $r \geq 2$.
In this case only two subcases remain: $r = 2$, $s = 3$ and $r = 3$, $s = 4$. This leads to graphs (6) and (7) of the list.
To show that the determined graphs are not isomorphic, we need only care for the graphs on 12 vertices, i.e. graphs (5), (6), (7).
Graph (5) as a unitary Cayley graph of even order is bipartite, while (6) and (7) are not bipartite. Graph (7) contains a triangle. Graph (6) contains a circuit of length 5, but no triangle. □

Corollary 2.15 *Let $G = \langle a \rangle$ be a finite cyclic group of even order $n > 4$. Let $S_1 = \{a^r, a^{-r}, a^s, a^{-s}\}$ and $S_2 = \{a^r, a^{-r}, a^s, a^{-s}, a^{n/2}\}$, where $r, s < n/2$ $(r \neq s)$, $S_t = S_t^{-1} \not\ni 1$ and $G = \langle S_t \rangle$ for $t = 1, 2$. Then $\Gamma(S_1 : G)$ is an integral graph if and only if $\Gamma(S_2 : G)$ is an integral graph.*

Proof. Let λ_j and μ_j $j \in \{0, 1, 2, \ldots, n-1\}$ be the eigenvalues of $\Gamma(S_1 : G)$ and $\Gamma(S_2 : G)$, respectively. By Lemmas 2.3 and 2.4, $\lambda_j = \omega^{jr} + \omega^{-jr} + \omega^{js} + \omega^{-js}$ and $\mu_j = \omega^{jr} + \omega^{-jr} + \omega^{js} + \omega^{-js} + (-1)^j$ for $j \in \{0, 1, 2, \ldots, n-1\}$. This completes the proof. □

Corollary 2.16 *Let $G = \langle a \rangle$ be a finite cyclic group of order even $n > 4$ and let S be a generating set of G such that $|S| = 5$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is an integral graph if and only if one the following holds:*

1. *$n = 6$ and $S = \{a, a^{-1}, a^2, a^{-2}, a^3\}$;*
2. *$n = 8$ and $S = \{a, a^{-1}, a^3, a^{-3}, a^4\}$;*
3. *$n = 10$ and $S = \{a, a^{-1}, a^3, a^{-3}, a^5\}$;*
4. *$n = 12$ and $S = \{a, a^{-1}, a^5, a^{-5}, a^6\}$;*
5. *$n = 12$ and $S = \{a^2, a^{-2}, a^3, a^{-3}, a^6\}$;*
6. *$n = 12$ and $S = \{a^4, a^{-4}, a^3, a^{-3}, a^6\}$.*

Proof. Note that $a^{n/2} \in S$. Now Lemmas 2.13 and 2.14 complete the proof. □

The following result follows from [21, Theorem 7.1].

Theorem 2.17 (Theorem 7.1 of [21]) *Let G be a finite group of order prime $p > 1$. Then $\Gamma(S : G)$ is an integral graph if and only if $|S| = p - 1$, where $S = S^{-1} \not\ni 1$ and $G = \langle S \rangle$.*

Definition 2.18 *A group G is called Cayley simple if $\Gamma(S : G)$ is not integral, where $G = \langle S \rangle$, $1 \notin S = S^{-1}$ and $S \neq G \setminus \{1\}$.*

So by Theorem 2.17, we have

Corollary 2.19 *Any finite group of prime order is Cayley simple.*

Let us to put forward the following questions.

Question 2.20 *Which finite groups are Cayley simple?*

Question 2.21 *Is any finite simple group, Cayley simple?*

Lemma 2.22 *There are exactly five connected, integral Cayley graphs with ten vertices.*

Proof. We show that the graphs Γ_{26}, Γ_{27}, Γ_{35}, Γ_{36} and Γ_{37} are Cayley graphs and others are not (See table 4).
By table 8, $|S| \in \{3, 4, 5, 6, 7, 8, 9\}$. By Theorem 2.11, the graphs Γ_{38}, Γ_{39} and Γ_{40} are not Cayley graphs.
It is clear that if G is a finite group of order 10 and $|S| = 9$, then $\Gamma(S : G)$ is the complete graph K_{10} and so $\Gamma(S : G) = \Gamma_{26}$. Let G be a finite group of order 10. Then G is isomorphic to C_{10} or D_{10}. So we have the following two cases:
Case 1: Let $G = C_{10} = \langle a \rangle$.
If $|S| = 4$ and $S = \{a, a^3, a^7, a^9\}$, then by easy calculations, one can see $\Gamma(S : C_{10})$ is an integral graph with the spectrum $[-4, -1^4, 1^4, 4]$. Hence $\Gamma(S : C_{10}) = \Gamma_{37}$.
Let $|S| = 5$. It is clear that $a^5 \in S$. If $S = \{a, a^3, a^5, a^7, a^9\}$, then by a straightforward computation, one can see that $\Gamma(S : C_{10})$ is an integral graph with the spectrum $[-5, 0^8, 5]$. Thus $\Gamma(S : C_{10}) = \Gamma_{36}$.
Let $\Gamma(S : C_{10}) = \Gamma_{35}$. By table 8, $|S| = 5$ and so $a^5 \in S$. It is clear that $S \neq \{a, a^3, a^5, a^7, a^9\}$. By Lemmas 2.2 and 2.3, $\Gamma(S \setminus \{a^5\} : C_{10})$ is an integral graph and so $\Gamma(S \setminus \{a^5\} : C_{10}) = \Gamma_{37}$. If χ is the irreducible character of C_{10} corresponding to the eigenvalue 3 in graph $\Gamma(S : C_{10}) = \Gamma_{35}$, then by Lemmas 2.2 and 2.3, the eigenvalue of $\Gamma(S \setminus \{a^5\} : C_{10})$ corresponding to χ is 2 or 4, which is impossible. Therefore Γ_{35} is not a Cayley graph of C_{10}.
If $|S| = 6$, then $a^5 \notin S$ and so there is exactly one integer r $(1 \leq r \leq 4)$ such that $a^r \notin S$. Without loss of generality we can assume $r = 4$ so that $S = \{a, a^2, a^3, a^7, a^8, a^9\}$. Then by a straightforward computation $\Gamma(S : C_{10})$ is not an integral graph. Thus the graphs Γ_{30}, Γ_{31}, Γ_{32}, Γ_{33} and Γ_{34} are not Cayley graphs of C_{10}.
If $|S| = 7$, then $a^5 \in S$ and so there is exactly one integer $r \in \{1, 2, 3, 4\}$ such that $a^r \notin S$. Without loss of generality we can assume $r = 4$ so that $S = \{a, a^2, a^3, a^5, a^7, a^8, a^9\}$. Then by a straightforward computation $\Gamma(S : C_{10})$ is not an integral graph. Thus the graphs

Γ_{28} and Γ_{29} are not Cayley graphs of C_{10}.
If $|S| = 8$, then $a^5 \notin S$ and so $S = G \setminus \{1, a^5\}$. Now easy calculations show that $\Gamma(S : C_{10})$ is an integral graph with the spectrum $[-2^4, 0^5, 8]$. Therefore $\Gamma(S : C_{10}) = \Gamma_{27}$.
Case 2: Let $G = D_{10} = \langle a, b \mid a^5 = b^2 = 1, (ab)^2 = 1\rangle$.
If $|S| = 4$ and $S = \{b, ab, a^2b, a^3b\}$, then by a straightforward computation, $\Gamma(S : D_{10})$ is an integral graph with the the spectrum $[-4, -1^4, 1^4, 4]$. Therefore $\Gamma(S : C_{10}) = \Gamma_{37}$.
If $S_1 = \{b, ab, a^2b, a^3b, a^4b\}$ and $S_2 = \{a, a^2, a^3, a^4, b\}$, then $\Gamma(S_1 : D_{10})$ and $\Gamma(S_2 : D_{10})$ are integral graphs with the spectra $[-5, 0^8, 5]$ and $[-2^4, 0^4, 3, 5]$, respectively. Thus $\Gamma(S_1 : D_{10}) = \Gamma_{36}$ and $\Gamma(S_2 : D_{10}) = \Gamma_{35}$.
Let $|S| = 6$. Since D_{10} has exactly two linear characters, it follows from Lemma 2.3 that $\Gamma(S : D_{10})$ has exactly two simple eigenvalues. So the graphs Γ_{30}, Γ_{31} and Γ_{34} are not Cayley graphs of D_{10}. Since $|S| = 6$, $S \cap \langle a\rangle = \langle a\rangle$ or $|S \cap \langle a\rangle| = 2$. Suppose λ be the eigenvalue of $\Gamma(S : D_{10})$ corresponding to the linear character χ_3 of D_{10}. Then by Lemma 2.3, $\lambda = 2$ if $S \cap \langle a\rangle = \langle a\rangle$ and $\lambda = -2$ if $|S \cap \langle a\rangle| = 2$. On the contrary, let $\Gamma(S : D_{10}) = \Gamma_{32}$ or $\Gamma(S : D_{10}) = \Gamma_{33}$. Since $Spec(\Gamma_{32}) = [-2^5, 1^4, 6]$ and $Spec(\Gamma_{33}) = [-3, -2^3, 0^2, 1^3, 6]$, $|S \cap \langle a\rangle| = 2$. Now if λ_{11} and λ_{12} are the eigenvalues of $\Gamma(S : D_{10})$ corresponding to χ_1, then by Lemma 2.3 and using the character table of D_{10}, $\lambda_{11} + \lambda_{12} = 4\cos(\frac{2\pi}{5})$ or $4\cos(\frac{4\pi}{5})$. But the latter is not an integer, which is a contradiction. Hence Γ_{32} and Γ_{33} are not Cayley graphs of D_{10}.
Let $|S| = 7$ and $\Gamma(S : D_{10}) = \Gamma_{28}$. Since $\lambda = -3$ is a simple eigenvalue of $\Gamma(S : D_{10}) = \Gamma_{28}$, it follows from Lemma 2.3 and the character table of D_{10} that $\{b, ab, a^2b, a^3b, a^4b\} \subseteq S$. Thus $S = \{a, a^{-1}, b, ab, a^2b, a^3b, a^4b\}$ or $\{a^2, a^{-2}, b, ab, a^2b, a^3b, a^4b\}$. By easy calculations one finds that $\Gamma(S : D_{10}) = \Gamma_{28}$ is not integral graph, a contradiction. Therefore Γ_{28} is not a Cayley graph of D_{10}.
Let $|S| = 7$ and $\Gamma(S : D_{10}) = \Gamma_{29}$. Since $\lambda = 1$ is a simple eigenvalue of $\Gamma(S : D_{10}) = \Gamma_{29}$, it follows from Lemma 2.3 and the character table of D_{10} that $S \cap \langle a\rangle = \langle a\rangle$. If λ_{k1} and λ_{k2} are the eigenvalues of $\Gamma(S : D_{10})$ corresponding to χ_k for $k \in \{1, 2\}$, then Lemma 2.3 and the character table of D_{10} implies that $\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = -2$. Since the multiplicity 0 as an eigenvalue of $\Gamma(S : D_{10}) = \Gamma_{29}$ is 4, $\lambda_{11} = 0$, $\lambda_{12} = -2$, $\lambda_{21} = 0$ and $\lambda_{22} = -2$ or $\lambda_{11} = -2$, $\lambda_{12} = 0$, $\lambda_{21} = -2$ and $\lambda_{22} = 0$ (Each one, two times). This shows that -2 is an eigenvalue of Γ_{29}, which is impossible. Therefore Γ_{29} is not a Cayley graph of D_{10}.
Therefore there are exactly five connected, integral Cayley graphs with 10 vertices. □

Lemma 2.23 *There are exactly three connected, integral Cayley graphs with nine vertices.*

Proof. We show that the graphs Γ_{19}, Γ_{21} and Γ_{24} are Cayley graphs and others are not (See table 3). It follows from table 7 that $|S| \in \{4, 6, 8\}$. Clearly if G is a finite group of order nine and $|S| = 8$, then $\Gamma(S : G)$ is the complete graph K_9 and so $\Gamma(S : G) = \Gamma_{19}$.

Let G be a finite group of order 9. Then G is isomorphic to C_9 or C_3^2. We distinguish the following two cases:

Case 1: Let $G = C_9 = \langle a\rangle$. If $|S| = 4$, then by Lemma 2.14, $\Gamma(S : C_9)$ is not an integral graph. Since the graphs Γ_{22}, Γ_{23}, Γ_{24} and Γ_{25} are 4-regular integral graphs with 9 vertices, they are not Cayley graphs of C_9.

Let $\Gamma(S:C_9)$ be an integral graph and λ be the eigenvalue $\Gamma(S:C_9)$ corresponding to the irreducible character $\chi(a^j)=\omega^j$, where $|S|=6$ and $\omega=e^{\frac{2\pi i}{9}}$. Since λ and $\sum_{j=1}^{8}\omega^j$ are integers and $\omega^r+\omega^{-r}$ $(r\neq 3)$ is not an integer, it follows from Lemma 2.3 that $S=\{a,a^2,a^4,a^5,a^7,a^8\}$. By an easy calculation one can see that $\Gamma(S:C_9)$ is an integral graph with the spectrum $[-3^2,0^6,6]$. Thus $\Gamma(S:C_9)=\Gamma_{21}$. This shows that Γ_{20} is not a Cayley graph of C_9.

Case 2: Let $G=C_3^2=\langle b\rangle\times\langle b\rangle$. It is clear that $\omega+\omega^2=\omega^2+\omega^4=-1$, where $\omega=e^{\frac{2\pi i}{3}}$.

If $|S|=4$, then by Lemmas 2.3 and 2.5, all eigenvalues of $\Gamma(S:C_3^2)$ are in $\{-2,1,4\}$. Therefore Γ_{22}, Γ_{23} and Γ_{25} are not Cayley graphs of C_3^2. If

$$S=\{(b,1),(b^2,1),(1,b),(1,b^2)\},$$

then by a straightforward computation one can see that $\Gamma(S:C_3^2)$ is an integral graph with the spectrum $[-2^4,1^4,4]$. Hence $\Gamma(S:C_3^2)=\Gamma_{24}$.

If $|S|=6$, then by Lemmas 2.3 and 2.5, all the eigenvalues of $\Gamma(S:C_3^2)$ are in $\{-3,0,6\}$. Thus Γ_{20} is not a Cayley graph of C_3^2. If

$$S=\{(b,1),(b^2,1),(1,b),(1,b^2),(b,b),(b^2,b^2)\},$$

then $\Gamma(S:C_3^2)$ is an integral graph with the spectrum $[-3^2,0^6,6]$ and so $\Gamma(S:C_3^2)=\Gamma_{21}$. Therefore, the graphs Γ_{19}, Γ_{21} and Γ_{24} are the only Cayley graphs with nine vertices. This completes the proof. □

Lemma 2.24 *There are exactly six connected, integral Cayley graphs with eight vertices.*

Proof. There are exactly six connected, regular integral graphs with eight vertices (See table 2). We show that these graphs are Cayley graphs. Clearly if $|S|=7$, then $\Gamma(S:G)$ is the complete graph K_8 and so $\Gamma(S:G)=\Gamma_{13}$.

Suppose $G=C_8=\langle a\rangle$ and

$$S_1=\{a,a^3,a^5,a^7\},\ S_2=\{a,a^3,a^4,a^5,a^7\},\ S_3=\{a,a^2,a^3,a^5,a^6,a^7\}.$$

Then $\Gamma(S_1:C_8)$, $\Gamma(S_2:C_8)$ and $\Gamma(S_3:C_8)$ are integral graphs with the spectra $[-4,0^6,4]$, $[-3,-1^4,1^2,5]$ and $[-2^3,0^4,6]$, respectively. Thus $\Gamma(S_1:C_8)=\Gamma_{17}$, $\Gamma(S_2:C_8)=\Gamma_{15}$ and $\Gamma(S_3:C_8)=\Gamma_{14}$.

Let $G=D_8=\langle a,b\mid a^4=b^2=1,(ab)^2=1\rangle$, $S_1=\{a,a^3,b\}$ and $S_2=\{a,a^2,a^3,b\}$. Then $\Gamma(S_1:D_8)$ and $\Gamma(S_2:D_8)$ are integral graphs with the spectra $[-3,-1^3,1^3,3]$ and $[-2^3,0^3,2,4]$, respectively. Hence $\Gamma(S_1:D_8)=\Gamma_{18}$ and $\Gamma(S_2:D_8)=\Gamma_{16}$.

Hence all of the connected, regular integral graphs with eight vertices are Cayley graphs. This completes the proof. □

3 Proofs of Main Results

In this section we prove our main results.

Proof of Theorem 1.1. Let $\Gamma(S:G)$ be integral. If G is a cyclic group, then by Lemma 2.14, $n \in \{5,6,8,10,12\}$. Let G be a finite abelian group, which is not cyclic. Suppose all of the elements of S are of order two. Then $|G| = 8$ or 16. Otherwise since $S = S^{-1}$ and $1 \notin S$, the proof falls naturally into two parts.

i) There are exactly two elements of order two in S. Thus G is isomorphic to $C_m \times C_2^2$. Let $S_1 = \{s_1 \in C_m \mid \exists x \in C_2^2, (s_1, x) \in S\} \setminus \{1\}$. Since $\Gamma(S:G)$ is an integral graph, by Lemma 2.9, $\Gamma(S_1 : C_m)$ is an integral graph. By Lemmas 2.7, 2.8 and 2.14, $m \in \{3,4,5,6,8,10,12\}$. Hence $n \in \{12,16,20,24,32,40,48\}$.

ii) There is no elements of order two in S. Thus G is isomorphic to $C_{m_1} \times C_{m_2}$, where $(m_1, m_2) \neq 1$. Let $S_1 = \{s_1 \in C_{m_1} \mid \exists x \in C_{m_2}, (s_1, x) \in S\} \setminus \{1\}$ and $S_2 = \{s_2 \in C_{m_2} \mid \exists x \in C_{m_1}, (x, s_2) \in S\} \setminus \{1\}$. By Lemma 2.9, $\Gamma(S_1 : C_{m_1})$ and $\Gamma(S_2 : C_{m_2})$ are integral graphs. It follows from Lemmas 2.7, 2.8 and 2.14 that $m_1, m_2 \in \{3,4,5,6,8,10,12\}$. Since $(m_1, m_2) \neq 1$, we have:
$n \in \{9,16,18,24,25,32,36,40,48,50,60,64,72,80,96,100,120,144\}$. □

Proof of Theorem 1.2. It is easy to see that the graphs Γ_i $(1 \leq i \leq 8)$ are Cayley graphs.
Let $G = C_6 = \langle a \rangle$, $S_1 = \{a, a^5\}$, $S_2 = \{a, a^3, a^5\}$, $S_3 = \{a^2, a^3, a^4\}$ and $S_4 = \{a, a^2, a^4, a^5\}$. Then $\Gamma(S_1 : C_6)$, $\Gamma(S_2 : C_6)$, $\Gamma(S_3 : C_6)$ and $\Gamma(S_4 : C_6)$ are integral with the spectra $[-2, -1^2, 1^2, 2]$, $[-3, 0^4, 3]$, $[-2^2, 0^2, 1, 3]$, and $[-2^2, 0^3, 4]$, respectively. Thus $\Gamma(S_1 : C_6) = \Gamma_9$, $\Gamma(S_2 : C_6) = \Gamma_{10}$, $\Gamma(S_2 : C_6) = \Gamma_{11}$ and $\Gamma(S_3 : C_6) = \Gamma_{12}$. Hence all of the connected, regular integral graphs up to seven vertices are Cayley graphs. In other words there are exactly 12 connected, integral Cayley graphs up to seven vertices.
It follows from Lemmas 2.22, 2.23, 2.24 and Theorems 2.17 and 1.1, there are exactly 27 connected, integral Cayley graphs up to 11 vertices. □

Table 1: Connected regular graphs with 6 vertices

Γ_8	1	11	111	1111	11111
Γ_9	1	01	001	0001	10001
Γ_{10}	1	01	101	0101	10101
Γ_{11}	1	11	001	0101	10011
Γ_{12}	1	11	011	1011	11011

Table 2: Connected regular graphs with 8 vertices

Γ_{13}	1	11	111	1111	11111	111111	1111111
Γ_{14}	1	10	011	1111	11110	111111	1111110
Γ_{15}	1	10	110	1010	01111	111110	0111110
Γ_{16}	1	10	010	1010	01110	101110	0101110
Γ_{17}	1	10	100	1000	01111	011110	0111100
Γ_{18}	1	10	010	0010	00011	100110	0110010

Table 3: Connected regular graphs with 9 vertices

Γ_{19}	1	11	111	1111	11111	111111	1111111	11111111
Γ_{20}	1	10	101	0110	01011	111111	1111110	11111100
Γ_{21}	1	11	110	1100	10111	101110	0111111	01111110
Γ_{22}	1	00	001	1100	00110	100111	0110110	11110000
Γ_{23}	1	00	001	0011	11001	110010	0011011	11110000
Γ_{24}	1	11	100	1001	01010	010011	0011010	00101011
Γ_{25}	1	11	100	0010	01000	010111	0011110	10011100

Table 4: Connected regular graphs with 10 vertices

Γ_{26}	1	11	111	1111	11111	111111	1111111	11111111	111111111
Γ_{27}	1	11	111	1111	11110	111011	1101111	10111111	011111111
Γ_{28}	1	00	001	1111	11111	111110	1111101	11110110	111101011
Γ_{29}	1	00	001	1111	11110	111111	1111110	11110011	111111001
Γ_{30}	1	11	111	1100	11001	001101	0011101	00111111	110011110
Γ_{31}	1	11	000	0001	00011	111111	1111110	11111100	111111000
Γ_{32}	1	11	110	1101	10110	101011	0111010	01101011	000111111
Γ_{33}	1	00	110	1011	01110	111011	1110001	10111101	011111010
Γ_{34}	1	10	010	0010	00011	111111	1111110	11111100	111111000
Γ_{35}	1	10	101	1011	10111	011000	0101001	01001011	010001111
Γ_{36}	1	10	100	1000	10000	011111	0111110	01111100	011111000
Γ_{37}	1	10	100	1000	01110	011010	0101100	00111000	000001111
Γ_{38}	0	10	010	1010	01010	001100	0000110	00000011	110000001
Γ_{39}	0	00	000	1100	00111	001100	0011000	11000010	110000010
Γ_{40}	1	10	100	0100	01000	001010	0010010	00011001	000101100

Spectra of connected, regular, integral graphs with 6, 8, 9 and 10 vertices:

Table 5: Spectra of connected regular graphs with 6 vertices

Γ_8			5	-1^5
Γ_9	2	1^2	-1^2	-2
Γ_{10}		3	0^4	-3
Γ_{11}	3	1	0^2	-2^2
Γ_{12}		4	0^3	-2^2

Table 6: Spectra of connected regular graphs with 8 vertices

Γ_{13}			7	-1^7
Γ_{14}		6	0^4	-2^3
Γ_{15}	5	1^2	-1^4	-3
Γ_{16}	4	2	0^3	-2^3
Γ_{17}		4	0^6	-4
Γ_{18}	3	1^3	-1^3	-3

Table 7: Spectra of connected regular graphs with 9 vertices

Γ_{19}						8	-1^8	
Γ_{20}			6	1	0^4	-2^2	-3	
Γ_{21}					6	0^6	-3^2	
Γ_{22}			4	2	1^2	-1^2	-2^3	
Γ_{23}	4		2	1	0^2	-1^2	-2	-3
Γ_{24}					4	1^4	-2^4	
Γ_{25}		4		1^3	0^2	-2^2	-3	

Table 8: Spectra of connected regular graphs with 10 vertices

Γ_{26}						9	-1^9
Γ_{27}					8	0^5	-2^4
Γ_{28}		7	1^2	0^2	-1^2	-2^2	-3
Γ_{29}			7	1	0^4	-1^2	-3^2
Γ_{30}	6	2	1	0^2	-1^2	-2^2	-3
Γ_{31}			6	2	0^3	-1^4	-4
Γ_{32}					6	1^4	-2^5
Γ_{33}			6	1^3	0^2	-2^3	-3
Γ_{34}		6	1^2	0^3	-1^2	-2	-4
Γ_{35}				5	3	0^4	-2^4
Γ_{36}					5	0^8	-5
Γ_{37}				4	1^4	-1^4	-4
Γ_{38}			3	2	1^3	-1^2	-2^3
Γ_{39}	3	2	1^2	0^2	-1^2	-2	-3
Γ_{40}					3	1^5	-2^4

Character Table of D_{2n}, $n = 2m + 1$ odd

	1	a^r	b
χ_j	2	$\omega^{jr} + \omega^{-jr}$	0
χ_{m+1}	1	1	-1
χ_{m+2}	1	1	1

$\omega = e^{\frac{2\pi i}{n}}$, $1 \leq j \leq m$ and $1 \leq r \leq m$

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