The Ramsey number $r(K_5 - P_3, K_5)$

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $r(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let K_m denote a complete graph of order m and $K_n - P_3$ a complete graph of order n without two incident edges. In this paper, we prove that $r(K_5 - P_3, K_5) = 25$ without help of computer algorithms.

1 Introduction

All graphs considered in this paper are simple graphs without loops. For two given graphs G_1 and G_2 and a given integer n, let $R(G_1, G_2; n)$ denote the set of all graphs G of order n, such that G does not contain G_1 and $\overline{G}$ does not contain G_2, where $\overline{G}$ is the complement of G. The Ramsey number $r(G_1, G_2)$ is the smallest integer n such that $R(G_1, G_2; n)$ is empty.

The values of $r(G_1, G_2)$ for all graphs G_1 and G_2 of order at most five up to the three cases that G_1 is one of the graphs $K_5 - P_3$, $K_5 - e$ and K_5 and $G_2 = K_5$ are found in [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18].

Kalbfleisch [13] proved that $r(K_5 - P_3, K_5) \geq 25$ and McKay and Radziszowski [15] found 350904 graphs belonging to $R(K_4, K_5; 24) \subseteq R(K_5 - P_3, K_5; 24)$, but there might be more of them. Recently, Black, Leven and Radziszowski [2] proved that $r(K_5 - P_3, K_5) \leq 26$ and Clavert, Schuster and Radziszowski [4] computed the main result of the present paper.

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2 Main result

In this paper we find the value of $r(K_5 - P_3, K_5)$ without help of computer algorithms. The main result is the following:

Theorem 2.1 $r(K_5 - P_3, K_5) = 25.$

In order to prove Theorem 2.1, we proceed by reduction to the absurd. Suppose that there exists a graph $G \in R(K_5 - P_3, K_5; 25)$. Since $r(K_4, K_5) = 25$ [15] we have G contains K_4. Let $\mathcal{K}$ be the set of cliques of G of order 4, let $K \in \mathcal{K}$ be such that $\sum_{v \in V(K)} d(v) = \max\{\sum_{v \in V(X)} d(v) : X \in \mathcal{K}\}$ and let v_1, v_2, v_3 and v_4 be the vertices of K. We may suppose without loss of generality that $d(v_1) \geq d(v_2) \geq d(v_3) \geq d(v_4)$.

Let $|A|$ denote the cardinality of the set A. If F is a graph then $V(F)$ denotes its vertex set. The neighborhood $N_F(v)$ of a vertex v is the set of vertices adjacent to v in the graph F. If G_1 is isomorphic to a subgraph of G_2 then we use $G_1 \subseteq G_2$ to denote it. If A is a subset of $V(F)$, then $F[A]$ is the subgraph induced by A. If $v \in V(F)$, $d_F(v)$ is the degree of v in F. The maximum and minimum degree of F are denoted by $\Delta(F)$ and $\delta(F)$, respectively.

Let d, V and N denote d_G, $V(G)$ and N_G, respectively.

If k is a positive integer, $F \in R(K_m - P_3, K_n; k)$ and $v \in V(F)$ then $F[N_F(v)] \in R(K_{m-1} - P_3, K_n; d_F(v))$ and $F[N_{\overline{F}}(v)] \in R(K_m - P_3, K_{n-1}; k-1-d_F(v))$. Thus $\Delta(F) \leq r(K_{m-1} - P_3, K_n) - 1$ and $\delta(F) \geq k - r(K_m - P_3, K_{n-1})$. Since $r(K_5 - P_3, K_4) = 18$ [7] we have $\delta(G) \geq 7$.

In the rest of the paper, i and j are two different integers with $1 \leq i, j \leq 4$.

Let $A_i = N(v_i) - V(K)$ and $D = V(G) - V(K) - \bigcup_{k=1}^{4} A_k$. $A_i \cap A_j = \emptyset$, because in otherwise G should contain $K_5 - P_3$. Hence $\{V(K), A_1, A_2, A_3, A_4, D\}$ is a partition of V. Obviously, $|A_i| = d(v_i) - 3 \geq 4$.

If $u \in A_i$ or $u \in D$ then $|N(u) \cap A_i|$, the number of vertices belonging to A_j adjacent to u, is denoted by $e_j(u)$.

Let H_i denote the graph $G[V(G) - (A_i \cup V(K))] = G[N_{\overline{G}}(v_i)]$. Clearly $H_i \in R(K_5 - P_3, K_4; 21 - |A_i|)$.

If $u \in V - V(K) - D$, $g(u)$ will represent the integer k for which $u \in A_k$. Also, if $u \in D$, we define $g(u) = 0$.

In order to prove Theorem 2.1, we need the following results:

Lemma 2.1 *Let $u \in A_i$ and $w \in D$. Then $d(u) - 9 \leq e_j(u) \leq d(u) + |A_j| - 13$ and $d(w) - 8 \leq e_i(w) \leq d(w) + |A_i| - 12$.*

Proof. Since $H_j \in R(K_5 - P_3, K_4; 21 - |A_j|)$, $r(K_4 - P_3, K_4) = 9$ [6] and $r(K_5 - P_3, K_3) = 9$ [7], we have $d_{H_j}(u), d_{H_j}(w) \leq \Delta(H_j) \leq r(K_4 - P_3, K_4) - 1 = 8$ and $d_{H_j}(u), d_{H_j}(w) \geq \delta(H_j) \geq 21 - |A_j| - r(K_5 - P_3, K_3) = 12 - |A_j|$. The results follow on noting that $d(u) = d_{H_j}(u) + e_j(u) + 1$ and $d(w) = d_{H_i}(w) + e_i(w)$.

Corollary 2.1 *Let k be, with $1 \leq k \leq 4$ and $k \neq i$, and let $u \in A_i$. Then $e_j(u) + 4 - |A_j| \leq e_k(u)$.*

Proof. The result is obtained from $e_j(u) \leq d(u) + |A_j| - 13$ and $d(u) - 9 \leq e_k(u)$.

Corollary 2.2 *The degree in G of every vertex of D is* 10*.*

Proof. Let $w \in D$. Since $d(w) = d_D(w) + \sum_{k=1}^{4} e_k(w)$, by Lemma 2.1, we have $d_D(w) + 4(d(w) - 8) \leq d(w) \leq d_D(w) + \sum_{k=1}^{4}(d(w) + |A_k| - 12)$. Thus $3d(w) \geq 48 - \sum_{k=1}^{4} |A_k| - d_D(w) \geq 48 - \sum_{k=1}^{4} |A_k| - (|D| - 1) = 28$ and $3d(w) \leq 32 - d_D(w) \leq 32$. As $28 \leq 3d(w) \leq 32$, the result follows.

Lemma 2.2 *The vertices of degree* 7 *or* 8 *in G belong to K.*

Proof. Let $u \in V(G) - V(K)$. On the one hand, if $u \in D$ then, by Corollary 2.2, $d(u) = 10$, thus $d(u) \geq 9$. On the other hand, if $u \in A_i$ then, by Lemma 2.1, $d(u) = 1 + d_{G[A_i]}(u) + |D \cap N(u)| + \sum_{k=1,k\neq i}^{4} e_k(u) \leq 1 + (|A_i| - 1) + |D| + \sum_{k=1,k\neq i}^{4}(d(u) + |A_k| - 13) = 3d(u) + \sum_{k=1}^{4} |A_k| + |D| - 39 = 3d(u) + 21 - 39$. Therefore $d(u) \geq 9$.

Corollary 2.3 *G has exactly one subgraph isomorphic to K_4.*

Proof. Suppose, to the contrary, that there exists $K' \in \mathcal{K} - \{K\}$. Since $K_5 - P_3 \nsubseteq G$, we have $|V(K) \cap V(K')| \leq 1$ and, by Lemma 2.2, there are at least three vertices in K' with degree in G at least 9. Thus $\sum_{v \in V(K')} d(v) \geq 3 \cdot 9 + 1 \cdot 7 = 34$. As $21 \geq \sum_{k=1}^{4} |A_k| = \sum_{k=1}^{4}(d(v_k) - 3)$, we have $\sum_{k=1}^{4} d(v_k) \leq 33$, contradicting the definition of K.

Corollary 2.4 *Let $u \in V - V(K)$, then $K_3 \nsubseteq G[N(u)]$.*

Proof. If there is a clique of order 3 in $G[N(u)]$, let u_1, u_2 and u_3 be its vertices. $G[\{u, u_1, u_2, u_3\}]$ is a subgraph of G different of K isomorphic to K_4, contradicting Lemma 2.3.

Corollary 2.5 *$G[A_i] \in R(K_3, K_4; |A_i|)$.*

Proof. If $K_3 \subseteq G[A_i]$, then let u_1, u_2 and u_3 be the vertices of a clique of order 3 of $G[A_i]$. $G[\{u_1, u_2, u_3, v_i\}]$ is a subgraph of G isomorphic to K_4 different from K, contradicting Corollary 2.3.

If $K_4 \subseteq \overline{G[A_i]}$, then let u_1, u_2, u_3 and u_4 be the four vertices of a clique of order 4 of $\overline{G[A_i]}$. $\overline{G}[\{u_1, u_2, u_3, u_4, v_j\}]$ is a subgraph of $\overline{G}$ isomorphic to K_5, a contradiction. Thus $G[A_i] \in R(K_3, K_4; |A_i|)$.

Corollary 2.6 *$H_i \in R(K_4, K_4; 21 - |A_i|)$.*

Proof. K is not a subgraph of H_i, thus, by Corollary 2.3, $K_4 \nsubseteq H_i$. Since $H_i \in R(K_5 - P_3, K_4; 21 - |A_i|)$ we have $K_4 \nsubseteq \overline{H_i}$, concluding the proof.

Lemma 2.3 *$D = \emptyset$.*

Proof. Suppose, to the contrary, that there exists $w \in D$. Let $X = V(G) - V(K) - N(w) - \{w\}$. Since $N(w) \cap V(K) = \emptyset$, and, by Corollary 2.2, $|N(w)| = 10$, we have $|X| = 25 - 4 - 10 - 1 = 10$. As K is not a subgraph of $G[X]$, by Corollary 2.3, $K_4 \nsubseteq G[X] = \overline{\overline{G}[X]}$. $r(K_3, K_4) = 9$ [10], thus $R(K_3, K_4; 10) = \emptyset$ and $\overline{G}[X] \notin R(K_3, K_4; 10)$, hence $K_3 \subseteq \overline{G}[X]$. Let u_1, u_2 and u_3 be the vertices of a clique of order 3 of $\overline{G}[X]$ and let $k \in \{1, 2, 3, 4\} - \{g(u_1), g(u_2), g(u_3)\}$. Then $\overline{G}[\{w, v_k, u_1, u_2, u_3\}] \subseteq \overline{G}$ is isomorphic to K_5, a contradiction.

Lemma 2.4 *If $\overline{G}[A_i]$ contains a clique of order 3, with vertices w_1, w_2 and w_3, then $|(N(w_1) - N(w_2) - N(w_3)) \cap V(H_i)| \leq 2$.*

Proof. Let $Y = (N(w_1) - N(w_2) - N(w_3)) \cap V(H_i)$. By Corollary 2.4, $K_3 \nsubseteq G[Y] \subseteq G[N(w_1)]$. If $K_2 \subseteq \overline{G[Y]}$ then let u_1u_2 be an edge of $\overline{G[Y]}$ and let $k \in \{1, 2, 3, 4\} - \{i, g(u_1), g(u_2)\}$. $\overline{G[\{w_1, w_2, u_1, u_2, v_k\}]}$ is isomorphic to K_5, a contradiction. Thus $K_2 \nsubseteq \overline{G[Y]}$ and $G[Y] \in R(K_3, K_2; |Y|)$. Therefore $|(N(w_1) - N(w_2) - N(w_3)) \cap V(H_i)| = |Y| \leq r(K_3, K_2) - 1 = 2$.

Lemma 2.5 *If $\overline{G}[A_i]$ contains two adjacent vertices w_1 and w_2 then $|V(H_i) \cap N(w_1) \cap N(w_2)| \leq 3$.*

Proof. Let $Y = V(H_i) \cap N(w_1) \cap N(w_2)$. If $K_2 \subseteq G[Y]$ then let u_1 and u_2 be the vertices of an edge of $G[Y]$. $G[\{w_1, w_2, u_1, u_2\}]$ is a subgraph of G isomorphic to K_4 and different from K, contradicting Corollary 2.3. Thus $K_2 \nsubseteq G[Y]$.

By Corollary 2.6, $K_4 \nsubseteq \overline{G[Y]} \subseteq \overline{H_i}$. Thus $G[Y] \in R(K_2, K_4; |Y|)$ and $|V(H_i) \cap N(w_1) \cap N(w_2)| = |Y| \leq r(K_2, K_4) - 1 = 3$.

Lemma 2.6 *Let $u \in A_i$, then the following statements are verified: $d_{G[A_i]}(u) \leq 3$, $\sum_{k=1, k \neq i}^{4} e_k(u) \leq 8$ and $d(u) \leq 11$.*

Proof. Suppose, to the contrary, that $d_{G[A_i]}(u) \geq 4$. Let u_1, u_2, u_3 and u_4 be four different vertices belonging to $N_{G[A_i]}(u)$.

If $K_2 \subseteq G[\{u_1, u_2, u_3, u_4\}]$, then let $u_p, u_q \in \{u_1, u_2, u_3, u_4\}$ be two adjacent vertices. $G[\{u_p, u_q, u, v_i\}]$ is a subgraph of G isomorphic to K_4 different from K, contradicting Corollary 2.3. Thus $K_2 \nsubseteq G[\{u_1, u_2, u_3, u_4\}]$ and $\overline{G}[\{u_1, u_2, u_3, u_4\}] \subseteq \overline{H_i}$ is isomorphic to K_4, contradicting Corollary 2.6. Hence $d_{G[A_i]}(u) \leq 3$.

By Corollary 2.4, $K_3 \nsubseteq G[V(H_i) \cap N(u)] \subseteq G[N(u)]$ and, by Corollary 2.6, $K_4 \nsubseteq \overline{G[V(H_i) \cap N(u)]} \subseteq \overline{H_i}$. Therefore $G[V(H_i) \cap N(u)] \in R(K_3, K_4; |V(H_i) \cap N(u)|)$ and $8 = r(K_3, K_4) - 1 \geq |V(H_i) \cap N(u)| = \sum_{k=1, k \neq i}^{4} |V(A_k) \cap N(u)| = \sum_{k=1, k \neq i}^{4} e_k(u)$, completing the second part of the proof.

Finally, by Lemma 2.1, $d(u) - 9 \leq e_j(u)$, thus $3d(u) - 27 \leq \sum_{k=1, k \neq i}^{4} e_k(u) \leq 8$ and $d(u) \leq 11$.

Lemma 2.7 *Let $u \in A_i$, then $|A_i| \leq 2d(u) + d_{G[A_i]}(u) - 17$.*

Proof. By Lemma 2.1, $e_j(u) \leq d(u) - 13 + |A_j|$. Therefore $d(u) - d_{G[A_i]}(u) - 1 = \sum_{k=1,k\neq i}^{4} e_k(u) \leq 3d(u) - 39 + \sum_{k=1,k\neq i}^{4} |A_k| = 3d(u) - 39 + (21 - |A_i|)$ and $|A_i| \leq 2d(u) + d_{G[A_i]}(u) - 17$.

Corollary 2.7 *Let $u \in A_i$, then $d_{G[A_i]}(u) \geq 1$ and if $d_{G[A_i]}(u) = 1$ then $|A_i| = 4$.*

Proof. By Lemma 2.6, $d(u) = 1+\sum_{k=1,k\neq i}^{4} e_k(u)+d_{G[A_i]}(u) \leq 9+d_{G[A_i]}(u)$. If $d_{G[A_i]}(u) = 0$ then $d(u) \leq 9$ and, by Lemma 2.7, $|A_i| \leq 1$, contradicting $|A_i| \geq 4$. If $d_{G[A_i]}(u) = 1$ then $d(u) \leq 10$ and, by Lemma 2.7, $|A_i| \leq 4$, thus $|A_i| = 4$.

Corollary 2.8 *If $|A_i| = 7$ and $u \in A_i$ then $d(u) = 11$.*

Proof. By Lemmas 2.6 and 2.7, $7 = |A_i| \leq 2d(u) + d_{G[A_i]}(u) - 17 \leq 2d(u) + 3 - 17$, thus $2d(u) \geq 21$ and $d(u) \geq 11$. Since $d(u) \leq 11$, we conclude the proof.

In the rest of the paper if we assign the name W to an ordered set of vertices, $\{u_1, \dots, u_p\} \subseteq A_i$, with $p \geq 3$, then $|(N(u_k) - \bigcup_{t=1,t\neq k}^{p} N(u_t)) \cap V(H_i)|$ will be denoted by a_k, $|(N(u_h) \cap N(u_k) - \bigcup_{t=1,t\neq h,k}^{p} N(u_t)) \cap V(H_i)|$ by $b_{h,k}$, and $|(N(u_h) \cap N(u_k) \cap N(u_l) - \bigcup_{t=1,t\neq h,k,l}^{p} N(u_t)) \cap V(H_i)|$ by $c_{h,k,l}$.

Lemma 2.8 *If $\overline{G}[A_i]$ contains a clique of order 3, with vertices u_1, u_2 and u_3, then $39 - 2|A_i| \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3)$.*

Proof. Let $W = \{u_1, u_2, u_3\}$, let $w \in V(H_i)$, and let $k \in \{1, 2, 3, 4\} - \{i, g(w)\}$. Since $\overline{G}[\{u_1, u_2, u_3, w, v_k\}]$ is not isomorphic to K_5, w is adjacent to at least a vertex of W and, therefore, every vertex of $V(H_i)$ is adjacent to at least a vertex of W. Hence:

$$a_1 + a_2 + a_3 + b_{1,2} + b_{1,3} + b_{2,3} + c_{1,2,3} = |V(H_i)| = 21 - |A_i| \tag{1}$$

On the one hand, since u_1 is adjacent to $d(u_1) - d_{G[A_i]}(u_1) - 1$ vertices of H_i we have:

$$a_1 + b_{1,2} + b_{1,3} + c_{1,2,3} = d(u_1) - d_{G[A_i]}(u_1) - 1 \tag{2}$$

Analogously, u_2 and u_3 are adjacent to $d(u_2) - d_{G[A_i]}(u_2) - 1$ and $d(u_3) - d_{G[A_i]}(u_3) - 1$ vertices of H_i respectively, thus:

$$a_2 + b_{1,2} + b_{2,3} + c_{1,2,3} = d(u_2) - d_{G[A_i]}(u_2) - 1 \tag{3}$$

$$a_3 + b_{1,3} + b_{2,3} + c_{1,2,3} = d(u_3) - d_{G[A_i]}(u_3) - 1 \tag{4}$$

On the other hand, by Lemma 2.4:

$$a_1 = |(N(u_1) - N(u_2) - N(u_3)) \cap V(H_i)| \leq 2 \tag{5}$$

$$a_2 = |(N(u_2) - N(u_1) - N(u_3)) \cap V(H_i)| \leq 2 \tag{6}$$

$$a_3 = |(N(u_3) - N(u_1) - N(u_2)) \cap V(H_i)| \leq 2 \tag{7}$$

Finally, from (2) + (3) + (4) + (5) + (6) + (7) − 2(1) we have:

$$c_{1,2,3} \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3) - 39 + 2|A_i|.$$

We obtain the result noting that $c_{1,2,3}$ is non-negative.

Corollary 2.9 *If $\overline{G}[A_i]$ contains a clique of order 3, then $|A_i| \geq 6$ and if $|A_i| = 6$ then for any vertex u of the clique, $d_{G[A_i]}(u) = 2$ and $d(u) = 11$.*

Proof. Let u_1, u_2, and u_3 be the three vertices of the clique. By Corollary 2.7, $d_{G[A_i]}(u_1)$, $d_{G[A_i]}(u_2)$, $d_{G[A_i]}(u_3) \geq 1$ and, by Lemma 2.6, $d(u_1)$, $d(u_2)$, $d(u_3) \leq 11$.

Therefore, by Lemma 2.8, $39 - 2|A_i| \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3) \leq 11 + 11 + 11 - 1 - 1 - 1 = 30$ and $|A_i| \geq 5$. Hence, by Corollary 2.7, $d_{G[A_i]}(u_1), d_{G[A_i]}(u_2), d_{G[A_i]}(u_3) \geq 2$, and $39 - 2|A_i| \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3) \leq 11 + 11 + 11 - 2 - 2 - 2 = 27$, thus $|A_i| \geq 6$.

If $|A_i| = 6$, on the one hand, $27 = 39 - 2|A_i| \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3) \leq d(u) + 11 + 11 - 2 - 2 - 2 = d(u) + 16$, hence $d(u) = 11$. On the other hand, $27 = 39 - 2|A_i| \leq d(u_1) + d(u_2) + d(u_3) - d_{G[A_i]}(u_1) - d_{G[A_i]}(u_2) - d_{G[A_i]}(u_3) \leq 11 + 11 + 11 - d_{G[A_i]}(u) - 2 - 2 = 29 - d_{G[A_i]}(u)$, therefore $d_{G[A_i]}(u) = 2$.

Corollary 2.10 *Let $u \in A_i$, then $d(u) \geq 10$.*

Proof. By Lemma 2.6, $d_{G[A_i]}(u) \leq 3$. If $d(u) \leq 9$ then by Lemma 2.2, $d(u) = 9$ and, by Lemma 2.7, $|A_i| \leq 2d(u) + d_{G[A_i]}(u) - 17 \leq 18 + 3 - 17 = 4$, thus $|A_i| = 4$. By Lemma 2.1, $8 - d_{G[A_i]}(u) = d(u) - 1 - d_{G[A_i]}(u) = \sum_{k=1,k\neq i}^{4} e_k(u) \leq \sum_{k=1,k\neq i}^{4} (|A_k| - 13 + d(u)) = \sum_{k=1,k\neq i}^{4} |A_k| - 12 = (21 - 4) - 12 = 5$. Hence $d_{G[A_i]}(u) \geq 3$ and u is adjacent to the three vertices of $A_i - \{u\}$.

By Corollary 2.9, $K_3 \not\subseteq \overline{G[A_i]}$ and at least two of the three vertices of $A_i - \{u\}$ are adjacent. Let w_1 and w_2 denote them. Then u, w_1 and w_2 are the vertices of a clique of $G[A_i]$, contradicting Corollary 2.5 and completing the proof.

From Corollaries 2.5 and 2.9 and Lemma 2.6, it is easy to check the next result:

Corollary 2.11

1. *If $|A_i| = 4$ then $G[A_i]$ is isomorphic to $2K_2$, P_4 or C_4.*
2. *If $|A_i| = 5$ then $G[A_i]$ is isomorphic to C_5.*
3. *If $|A_i| = 6$ then $G[A_i]$ is isomorphic to C_6 or $SK_{2,3}$ (the graph obtained subdividing one edge of $K_{2,3}$).*

Now, we prove that $G[A_i]$ is isomorphic neither to C_5 nor to $SK_{2,3}$.

Lemma 2.9 *$G[A_i]$ is not isomorphic to C_5.*

Proof. Suppose, to the contrary, that $G[A_i]$ is isomorphic to C_5. Let $W = \{u_1, \ldots, u_5\}$ denote its vertices, with its edges being u_1u_2, u_2u_3, u_3u_4, u_4u_5 and u_1u_5.

Let $w \in V(H_i)$. By Lemmas 2.1 and 2.6 and Corollary 2.10, $1 \leq d(w) - 9 \leq e_i(w) \leq d(w) - 13 + |A_i| = d(w) - 8 \leq 3$, thus every vertex of H_i is adjacent to 1, 2 or 3 vertices of $G[A_i]$ and we have:

$$\sum_{k=1}^{5} a_k + \sum_{1 \leq k < m \leq 5} b_{k,m} + \sum_{1 \leq k < m < n \leq 5} c_{k,m,n} = |V(H_i)| = 16 \tag{8}$$

On the one hand, by Lemma 2.5:

$$b_{1,2} + c_{1,2,3} + c_{1,2,4} + c_{1,2,5} = |V(H_i) \cap N(u_1) \cap N(u_2)| \leq 3 \tag{9}$$

$$b_{2,3} + c_{1,2,3} + c_{2,3,4} + c_{2,3,5} = |V(H_i) \cap N(u_2) \cap N(u_3)| \leq 3 \tag{10}$$

$$b_{3,4} + c_{1,3,4} + c_{2,3,4} + c_{3,4,5} = |V(H_i) \cap N(u_3) \cap N(u_4)| \leq 3 \tag{11}$$

$$b_{4,5} + c_{1,4,5} + c_{2,4,5} + c_{3,4,5} = |V(H_i) \cap N(u_4) \cap N(u_5)| \leq 3 \tag{12}$$

$$b_{1,5} + c_{1,2,5} + c_{1,3,5} + c_{1,4,5} = |V(H_i) \cap N(u_1) \cap N(u_5)| \leq 3 \tag{13}$$

On the other hand, let $Y = V(H_i) - N(u_1) - N(u_3)$. By Corollary 2.6, $K_4 \nsubseteq G[Y] \subseteq H_i$. If $K_2 \subseteq \overline{G[Y]}$ then let w_1 and w_2 be the vertices of an edge of $\overline{G[Y]}$ and let $k \in \{1, 2, 3, 4\} - \{i, g(w_1), g(w_2)\}$. $G[\{u_1, u_3, w_1, w_2, v_k\}]$ is isomorphic to K_5, a contradiction. Thus $K_2 \nsubseteq \overline{G[Y]}$, $G[Y] \in R(K_4, K_2; |Y|)$ and:

$$\begin{aligned} a_2 + a_4 + a_5 + b_{2,4} + b_{2,5} + b_{4,5} + c_{2,4,5} = |V(H_i) - N(u_1) - N(u_3)| = \\ = |Y| \leq r(K_4, K_2) - 1 = 3 \end{aligned} \tag{14}$$

Similarly $G[V(H_i) - N(u_1) - N(u_4)] \in R(K_4, K_2; |V(H_i) - N(u_1) - N(u_4)|)$ and:

$$a_2 + a_3 + a_5 + b_{2,3} + b_{2,5} + b_{3,5} + c_{2,3,5} \leq 3 \tag{15}$$

$G[V(H_i) - N(u_2) - N(u_4)] \in R(K_4, K_2; |V(H_i) - N(u_2) - N(u_4)|)$ and:

$$a_1 + a_3 + a_5 + b_{1,3} + b_{1,5} + b_{3,5} + c_{1,3,5} \leq 3 \tag{16}$$

$G[V(H_i) - N(u_2) - N(u_5)] \in R(K_4, K_2; |V(H_i) - N(u_2) - N(u_5)|)$ and:

$$a_1 + a_3 + a_4 + b_{1,3} + b_{1,4} + b_{3,4} + c_{1,3,4} \leq 3 \tag{17}$$

$G[V(H_i) - N(u_3) - N(u_5)] \in R(K_4, K_2; |V(H_i) - N(u_3) - N(u_5)|)$ and:

$$a_1 + a_2 + a_4 + b_{1,2} + b_{1,4} + b_{2,4} + c_{1,2,4} \leq 3 \tag{18}$$

From $(9) + \cdots + (18)$ we have:

$$3\sum_{k=1}^{5} a_k + 2\sum_{1\leq k<m\leq 5} b_{k,m} + 2\sum_{1\leq k<m<n\leq 5} c_{k,m,n} \leq 30 \tag{19}$$

Finally, from $(19) - 2(8)$ we obtain $\sum_{k=1}^{5} a_k \leq -2$. This contradiction completes the proof.

Lemma 2.10 *$G[A_i]$ is not isomorphic to $SK_{2,3}$.*

Proof. Suppose, to the contrary, that $G[A_i]$ is isomorphic to $SK_{2,3}$. Let $W = \{u_1, \ldots, u_4\}$ be the set of vertices of A_i of degree 2 in $G[A_i]$, with its only edge being u_1u_2. Every vertex of W belongs to a clique of order 3 contained in $\overline{G}[A_i]$, thus, by Corollary 2.9, each vertex of W has degree 11 in G. Let $h = |V(H_i) \cap \bigcap_{n=1}^{4} N(u_n)|$.

Since $d(u_1) = 11$ and $d_{G[A_i]}(u_1) = 2$ we have that, $|N(u) \cap V(H_i)|$, the number of edges incident to u_1 and a vertex of H_i is:

$$a_1 + b_{1,2} + b_{1,3} + b_{1,4} + c_{1,2,3} + c_{1,2,4} + c_{1,3,4} + h = d(u_1) - d_{G[A_i]}(u_1) - 1 = 8 \tag{20}$$

On the one hand, by Lemma 2.4:

$$a_1 + b_{1,2} = |(N(u_1) - N(u_3) - N(u_4)) \cap V(H_i)| \leq 2 \tag{21}$$

On the other hand, let $Y = V(H_i) \cap N(u_1) \cap N(u_3)$. By Corollary 2.4, $K_3 \nsubseteq G[Y] \subseteq G[N(u_1)]$. If $K_2 \subseteq \overline{G[Y]}$ then let w_1 and w_2 be the vertices of an edge of $\overline{G[Y]}$ and let $k \in \{1,2,3,4\} - \{i, g(w_1), g(w_2)\}$. $\overline{G[\{u_1, u_3, w_1, w_2, v_k\}]}$ is isomorphic to K_5, a contradiction. Therefore $K_2 \nsubseteq \overline{G[Y]}$, $G[Y] \in R(K_3, K_2; |Y|)$ and:

$$b_{1,3} + c_{1,2,3} + c_{1,3,4} + h = |V(H_i) \cap N(u_1) \cap N(u_3)| = |Y| \leq r(K_3, K_2) - 1 = 2 \tag{22}$$

Similarly $G[V(H_i) \cap N(u_1) \cap N(u_4)] \in R(K_3, K_2; |V(H_i) \cap N(u_1) \cap N(u_4)|)$ and:

$$b_{1,4} + c_{1,2,4} + c_{1,3,4} + h \leq 2 \tag{23}$$

From $(21) + (22) + (23) - (20)$ we obtain $c_{1,3,4} + h \leq -2$, a contradiction.

Finally, we prove Theorem 2.1:

Proof. By Corollaries 2.8 and 2.11 and Lemmas 2.9 and 2.10, $|A_1| = 7$, $|A_2| = 6$, $|A_3| = |A_4| = 4$, $G[A_2]$ is isomorphic to C_6 and all vertices of $A_1 \cup A_2$ have degree 11 in G. Let s denote the number of edges of $G[A_3]$.

By Corollary 2.6, $H_4 \in R(K_4, K_4; 17)$. Kalbfleisch [14] proved that there is exactly one graph in the set $R(K_4, K_4; 17)$ and every vertex of this graph has degree 8, thus, for any $w \in V(H_4)$, $d_{H_4}(w) = 8$.

If $u \in A_1 \cup A_2$, by Lemma 2.1, $2 = d(u) - 9 \leq e_3(u) \leq d(u) - 13 + |A_3| = 2$, thus $e_3(u) = 2$ and the number of edges of H_4 with a vertex belonging to $A_1 \cup A_2$ and another vertex belonging to A_3 is $\sum_{w \in A_3}(e_1(w) + e_2(w)) = \sum_{u \in A_1 \cup A_2} e_3(u) = 2|A_1 \cup A_2| = 26$.

If $w \in A_3$, then $8 = d_{H_4}(w) = d_{G[A_3]}(w) + e_1(w) + e_2(w)$. Thus $32 = \sum_{w \in A_3} 8 = \sum_{w \in A_3} d_{G[A_3]}(w) + \sum_{w \in A_3}(e_1(w) + e_2(w)) = 2s + 26$ and $s = 3$. Hence, by Corollary 2.11, $G[A_3]$ is isomorphic to P_4.

In the following, we assume that $i = 3$. Let $W = \{u_1, \ldots, u_4\}$ be the vertices of A_3, with the edges of $G[A_3]$ being u_1u_2, u_2u_3 and u_3u_4.

Let $w \in V(H_3)$. By Lemmas 2.1 and 2.6 and Corollary 2.10, $1 \leq d(w) - 9 \leq e_3(w) \leq d(w) - 13 + |A_3| = d(w) - 9 \leq 2$, thus every vertex of H_3 is adjacent to 1 or 2 vertices of A_3 and $|N(u_2) \cap V(H_3)|$ is:

$$a_2 + b_{1,2} + b_{2,3} + b_{2,4} = d(u_2) - d_{G[A_3]}(u_2) - 1 \geq 10 - 2 - 1 = 7 \tag{24}$$

Let $Y_1 = V(H_3) \cap N(u_2) - N(u_1) - N(u_3)$. By Corollary 2.4, $K_3 \not\subseteq G[Y_1] \subseteq G[N(u_2)]$. If $K_2 \subseteq \overline{G[Y_1]}$ then let w_1 and w_2 be the vertices of an edge of $\overline{G[Y_1]}$ and let $k \in \{1, 2, 4\} - \{g(w_1), g(w_2)\}$. $\overline{G}[\{u_1, u_3, w_1, w_2, v_k\}]$ is isomorphic to K_5, a contradiction. Hence $K_2 \not\subseteq \overline{G[Y_1]}$, $G[Y_1] \in R(K_3, K_2; |Y_1|)$ and:

$$a_2 + b_{2,4} = |V(H_3) \cap N(u_2) - N(u_1) - N(u_3)| = |Y_1| \leq r(K_3, K_2) - 1 = 2 \tag{25}$$

On the one hand, by Lemma 2.5:

$$b_{1,2} = |V(H_3) \cap N(u_1) \cap N(u_2)| \leq 3 \tag{26}$$

On the other hand, let $Y_2 = V(H_3) \cap N(u_2) \cap N(u_3)$. If $K_2 \subseteq G[Y_2]$ then let w_1 and w_2 be the vertices of an edge of $G[Y_2]$. $G[\{w_1, w_2, u_2, u_3\}]$ is a subgraph of G isomorphic to K_4 and different from K, contradicting Corollary 2.3, thus $K_2 \not\subseteq G[Y_2]$. If $K_2 \subseteq \overline{G[Y_2]}$ then let w_1 and w_2 be the vertices of an edge of $\overline{G[Y_2]}$ and let $k \in \{1, 2, 4\} - \{g(w_1), g(w_2)\}$. $\overline{G}[\{u_1, u_4, w_1, w_2, v_{k_3}\}]$ is isomorphic to K_5, a contradiction. Hence $K_2 \not\subseteq \overline{G[Y_2]}$, $G[Y_2] \in R(K_2, K_2; |Y_2|)$ and:

$$b_{2,3} = |V(H_3) \cap N(u_2) \cap N(u_3)| = |Y_2| \leq r(K_2, K_2) - 1 = 1 \tag{27}$$

From (25)+(26)+(27) we obtain $a_2 + b_{1,2} + b_{2,3} + b_{2,4} \leq 6$, contradicting (24) and completing the proof of Theorem 2.1.

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