Continued fractions related to (t, q)-tangents and variants

Helmut Prodinger

Department of Mathematics
7602 Stellenbosch, South Africa

hproding@sun.ac.za

Submitted: May 16, 2011; Accepted: Aug 3, 2011; Published: Aug 19, 2011
Mathematics Subject Classifications: 05A30, 05A10

To Doron Zeilberger who turned me into an addict of creative guessing

Abstract

For the q-tangent function introduced by Foata and Han (this volume) we provide the continued fraction expansion, by creative guessing and a routine verification. Then an even more recent q-tangent function due to Cieslinski is also expanded. Lastly, a general version is considered that contains both versions as special cases.

1 Foata and Han's tangent function

Foata and Han [3] defined

$$\begin{aligned}
\sin_q^{(r)}(u) &= \sum_{n\ge 0}(-1)^n\frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}}u^{2n+1},\\
\cos_q^{(r)}(u) &= \sum_{n\ge 0}(-1)^n\frac{(q^r;q)_{2n}}{(q;q)_{2n}}u^{2n},\\
\tan_q^{(r)}(u) &= \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)}.
\end{aligned}$$

Here we use the (classic) notation, where we assume $|q| < 1$:

$$(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Note that for $r \to \infty$, we obtain the classic q-tangent function of Jackson's [5].

In this paper, we compute the continued fraction expansion of this new q-tangent function. In the spirit of Zeilberger, the coefficients in it (a_k in the sequel) were obtained first by *guessing them*. After that, some additional power series $s_k(z)$ were also guessed (using the recursion that later will be proved). Once one has them, the proof of a recursion for $s_k(z)$ is routine, and turns immediately into the continued fraction expansion. In a sense, this is the most elementary approach possible.

Set, for $k \geq -1$,

$$s_k(z) := q^{\binom{k+1}{2}} \sum_{n \geq 0} \frac{(q^{r-k};q^2)_{k+n+1}(q^{r+k+1};q^2)_n}{(q;q^2)_{k+n+1}(q^2;q^2)_n} z^n,$$

and for $k \geq 0$

$$a_k = \frac{(q^{r+1-k};q^2)_k(1-q^{2k+1})}{(q^{r-k};q^2)_{k+1}q^k}.$$

Note that for $r \to \infty$, we obtain

$$a_k = \frac{1-q^{2k+1}}{q^k},$$

which are the well-known coefficients for the classic q-tangent function.

Now we compute

$$\begin{aligned}
[z^n]\Big(s_{k-1}(z) - a_k s_k(z)\Big) &= q^{\binom{k}{2}} \frac{(q^{r+1-k};q^2)_{k+n}(q^{r+k};q^2)_n}{(q;q^2)_{k+n}(q^2;q^2)_n} \\
&\quad - \frac{(q^{r+1-k};q^2)_k(1-q^{2k+1})}{(q^{r-k};q^2)_{k+1}q^k} q^{\binom{k+1}{2}} \frac{(q^{r-k};q^2)_{k+n+1}(q^{r+k+1};q^2)_n}{(q;q^2)_{k+n+1}(q^2;q^2)_n} \\
&= q^{\binom{k}{2}} \frac{(q^{r-k};q^2)_{k+n+1}(q^{r+k+1};q^2)_n}{(q;q^2)_{k+n+1}(q^2;q^2)_n} \\
&\quad \times \left[\frac{(q^{r+1-k};q^2)_k(1-q^{2k+2n+1})}{(q^{r-k};q^2)_k(1-q^{r+k+2n})} - \frac{(q^{r+1-k};q^2)_k(1-q^{2k+1})}{(q^{r-k};q^2)_{k+1}} \right] \\
&= q^{\binom{k}{2}} \frac{(q^{r-k};q^2)_{k+n+1}(q^{r+k+1};q^2)_n(q^{r+1-k};q^2)_k}{(q;q^2)_{k+n+1}(q^2;q^2)_n(q^{r-k};q^2)_k} \left[\frac{1-q^{2k+2n+1}}{1-q^{r+k+2n}} - \frac{1-q^{2k+1}}{1-q^{r+k}} \right] \\
&= q^{\binom{k}{2}} \frac{(q^{r-k};q^2)_{k+n+1}(q^{r+k+1};q^2)_n(q^{r+1-k};q^2)_k}{(q;q^2)_{k+n+1}(q^2;q^2)_n(q^{r-k};q^2)_k} \frac{q^{2k+1}(1-q^{2n})(1-q^{r-k-1})}{(1-q^{r+k+2n})(1-q^{r+k})} \\
&= q^{\binom{k+1}{2}} \frac{(q^{r-k};q^2)_{k+n}(q^{r+k+1};q^2)_n(q^{r-1-k};q^2)_{k+1}}{(q;q^2)_{k+n+1}(q^2;q^2)_{n-1}(q^{r-k};q^2)_{k+1}} \\
&= q^{\binom{k+1}{2}} \frac{(q^{r-1-k};q^2)_{k+n+1}(q^{r+k+2};q^2)_{n-1}}{(q;q^2)_{k+n+1}(q^2;q^2)_{n-1}} \\
&= [z^{n-1}]s_{k+1}(z).
\end{aligned}$$

Since the constant term in this difference cancels out, we found the recurrence

$$s_{k-1}(z) - a_k s_k(z) = z s_{k+1}(z).$$

Therefore we have

$$\frac{zs_0}{s_{-1}} = \frac{zs_0}{a_0 s_0 + z s_1} = \frac{z}{a_0 + \dfrac{zs_1}{s_0}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}}.$$

If r is a positive integer, this continued fraction expansions stops, since $s_r(z) = 0$.

Replacing z by $-z$ we get

$$\frac{zs_0(-z)}{s_{-1}(-z)} = \cfrac{z}{a_0 - \cfrac{z}{a_1 - \cfrac{z}{a_2 - \cfrac{z}{\cdots}}}}.$$

This translates into a continued fraction of $\tan_q^{(r)}(u)$:

$$\tan_q^{(r)}(u) = \cfrac{u}{a_0 - \cfrac{u^2}{a_1 - \cfrac{u^2}{a_2 - \cfrac{u^2}{\cdots}}}}.$$

2 Cieslinski's new q-tangent

After a first draft about the Foata and Han q-tangent was produced, a further q-tangent function was introduced by Cieslinski [1]. Recall that Jackson's [5] classical q-trigonometric functions are defined as

$$\sin_q z = \sum_{n\ge 0} \frac{(-1)^n z^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q z = \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{(q;q)_{2n}}.$$

Sometimes, instead of $(q;q)_n$, the term $(q;q)_n/(1-q)^n$ is used, but that is clearly just a change of variable. The corresponding tangent function is defined by $\tan_q z = \sin_q z/\cos_q z$.

Cieslinski [1] introduced new ("improved"?) q-trigonometric functions:

$$\mathcal{S}in_q(2z) = \frac{2\tan_q z}{1+\tan_q^2 z},$$

$$\mathcal{C}os_q(2z) = \frac{1-\tan_q^2 z}{1+\tan_q^2 z}.$$

Of course, this also introduces a (new) q-tangent function: $\mathcal{T}an_q(z) = \mathcal{S}in_q(z)/\mathcal{C}os_q(z)$.

As we know, q-tangents are good candidates for beautiful continued fraction expansions [6, 4, 7, 8]; and this is confirmed by the results of the previous section. This new version is no exception; we are going to prove that

$$z\,\mathcal{T}an(2z) = \cfrac{z^2}{a_0 + \cfrac{z^2}{a_1 + \cfrac{z^2}{\cdots}}}$$

with

$$a_{2k} = \frac{(1-q^{4k+1})(-q;q^2)_k^2}{2q^k(-q^2;q^2)_k^2},$$
$$a_{2k+1} = -\frac{2(1-q^{4k+3})(-q^2;q^2)_k^2}{q^k(-q;q^2)_{k+1}^2}.$$

As before, we obtain all the relevant quantities first by guessing them.

First, we need the power series expansions of sine and cosine:

$$\mathcal{S}in_q(2z) = \sum_{n\ge0} z^{2n+1}\frac{(-1)^n(-1;q)_{2n+1}}{(q;q)_{2n+1}},$$
$$\mathcal{C}os_q(2z) = \sum_{n\ge0} z^{2n}\frac{(-1)^n(-1;q)_{2n}}{(q;q)_{2n}}.$$

Cieslinski [1] has given the representations

$$\mathcal{S}in_q(2z) = \frac{e_q^{iz}E_q^{iz} - e_q^{-iz}E_q^{-iz}}{2i},$$
$$\mathcal{C}os_q(2z) = \frac{e_q^{iz}E_q^{iz} + e_q^{-iz}E_q^{-iz}}{2},$$

with

$$e_q^z = \sum_{n\ge0}\frac{z^n}{(q;q)_n}, \qquad E_q^z = \sum_{n\ge0}\frac{z^n q^{\binom{n}{2}}}{(q;q)_n}.$$

From this, the desired expansions follow from comparing coefficients and simple q-identities.

Now define

$$\sigma_0 := \sum_{n\ge0} z^n\frac{(-1)^n(-1;q)_{2n+1}}{(q;q)_{2n+1}},$$
$$\sigma_{-1} := \sum_{n\ge0} z^n\frac{(-1)^n(-1;q)_{2n}}{(q;q)_{2n}}$$

and, more generally,

$$\sigma_{2k} = \frac{q^{k^2}(-1)^k(-q^2;q^2)_k}{(-q;q^2)_k}\sum_{n\geq 0} z^n \frac{(-1)^n(-1;q)_{2k+2n+1}}{(q;q^2)_{2k+n+1}(q^2;q^2)_n},$$

$$\sigma_{2k+1} = \frac{q^{k^2+k}(-1)^{k+1}(-q^2;q^2)_{k+1}}{(-1;q^2)_{k+1}}\sum_{n\geq 0} z^n \frac{(-1)^n(-1;q)_{2k+2n+1}}{(q;q^2)_{2k+n+2}(q^2;q^2)_n}.$$

As in the previous section, we obtain the recursion

$$\sigma_{i+1} = \frac{\sigma_{i-1} - a_i\sigma_i}{z}$$

by a routine computation.

Consequently, we can write

$$\frac{z\sigma_0}{\sigma_{-1}} = \frac{z\sigma_0}{a_0\sigma_0 + z\sigma_1} = \cfrac{z}{a_0 + \cfrac{z\sigma_1}{\sigma_0}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z\sigma_2}{\sigma_1}}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}}.$$

The claimed continued fraction expansion of $z\,\mathcal{T}an(2z)$ follows from this by substituting z by z^2.

I was informed that this expansion could also be derived using results of Denis [2]. The present elementary approach should, however, not be without merits.

3 A uniform approach to the two q-tangents

It is apparent that

$$\begin{aligned}
\sin_q(u) &= \sum_{n\geq 0}(-1)^n\frac{(w;q)_{2n+1}}{(q;q)_{2n+1}}u^{2n+1},\\
\cos_q(u) &= \sum_{n\geq 0}(-1)^n\frac{(w;q)_{2n}}{(q;q)_{2n}}u^{2n},\\
\tan_q(u) &= \frac{\sin_q(u)}{\cos_q(u)}
\end{aligned}$$

generalises for $w = q^r$ the Foata and Han version, and for $w = -1$ the Cieslinski version. Our elementary approach can handle this situation as well:

Set

$$\sigma_0(z) = \sum_{n\geq 0}\frac{(w;q)_{2n+1}}{(q;q)_{2n+1}}z^n,$$

$$\sigma_{-1}(z) = \sum_{n\geq 0}\frac{(w;q)_{2n}}{(q;q)_{2n}}z^n,$$

then

$$a_k = \frac{(wq^{1-k};q^2)_k(1-q^{2k+1})}{(wq^{-k};q^2)_{k+1}q^k}$$

and

$$\sigma_k(z) = q^{\frac{k(k+1)}{2}} \sum_{n\geq 0} z^n \frac{(wq^{-k};q^2)_{n+k+1}(wq^{k+1};q^2)_n}{(q;q^2)_{n+k+1}(q^2;q^2)_n}.$$

As before, we get

$$\sigma_{k+1} = \frac{\sigma_{k-1} - a_k\sigma_k}{z}$$

and

$$\frac{z\sigma_0(z)}{\sigma_{-1}(z)} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}}.$$

This gives the expansion of the q-tangent:

$$\frac{z\sigma_0(-z^2)}{\sigma_{-1}(-z^2)} = \cfrac{z}{a_0 - \cfrac{z^2}{a_1 - \cfrac{z^2}{a_2 - \cfrac{z^2}{\cdots}}}}.$$

References

[1] J. L. Cieslinski. Improved q-exponential and q-trigonometric functions. *Appl. Math. Lett.*, to appear, 2011.

[2] R. Y. Denis. On (a) generalization of (a) continued fraction of Gauss. *Int. J. Math. Math. Sci.*, 4:741–746, 1990.

[3] D. Foata and G.-N. Han. The (t, q)-analogs of secant and tangent numbers. *Electronic Journal of Combinatorics*, 18(2):#*P*7, 2011.

[4] M. Fulmek. A continued fraction expansion for a q-tangent function. *Sém. Lothar. Combin.*, 45:Art. B45b, 5 pp. (electronic), 2000/01.

[5] F. H. Jackson. A basic-sine and cosine with symbolic solutions of certain differential equations. *Proc. Edinburg Math. Soc.*, 22:28–39, 1904.

[6] H. Prodinger. Combinatorics of geometrically distributed random variables: new q-tangent and q-secant numbers. *Int. J. Math. Math. Sci.*, 24(12):825–838, 2000.

[7] H. Prodinger. A continued fraction expansion for a q-tangent function: an elementary proof. *Sém. Lothar. Combin.*, 60:Art. B60b, 3, 2008/09.

[8] H. Prodinger. Continued fraction expansions for q-tangent and q-cotangent functions. *Discrete Math. Theor. Comput. Sci.*, 12(2):47–64, 2010.