Continued fractions related to 
(t, q)-tangents and variants

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To Doron Zeilberger who turned me into an addict of creative guessing

Abstract

For the q-tangent function introduced by Foata and Han (this volume) we provide
the continued fraction expansion, by creative guessing and a routine verification.
Then an even more recent q-tangent function due to Cieslinski is also expanded.
Lastly, a general version is considered that contains both versions as special cases.

1 Foata and Han’s tangent function

Foata and Han [3] defined

\[
\sin_q^{(r)}(u) = \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1},
\]

\[
\cos_q^{(r)}(u) = \sum_{n \geq 0} (-1)^n \frac{(q^r; q)_{2n}}{(q; q)_{2n}} u^{2n},
\]

\[
\tan_q^{(r)}(u) = \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)}.
\]

Here we use the (classic) notation, where we assume |q| < 1:

\[(x; q)_n := (1 - x)(1 - xq) \ldots (1 - xq^{n-1}).\]

Note that for \(r \to \infty\), we obtain the classic q-tangent function of Jackson’s [5].
In this paper, we compute the continued fraction expansion of this new \( q \)-tangent function. In the spirit of Zeilberger, the coefficients in it (\( a_k \) in the sequel) were obtained first by guessing them. After that, some additional power series \( s_k(z) \) were also guessed (using the recursion that later will be proved). Once one has them, the proof of a recursion for \( s_k(z) \) is routine, and turns immediately into the continued fraction expansion. In a sense, this is the most elementary approach possible.

Set, for \( k \geq -1 \),

\[
s_k(z) := q^{\frac{k+1}{2}} \sum_{n \geq 0} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n}{(q; q^2)_{k+n+1} (q^2; q^2)_n} z^n,
\]

and for \( k \geq 0 \)

\[
a_k = \frac{(q^{r+1-k}; q^2)_k (1 - q^{2k+1})}{(q^{-k}; q^2)_{k+1} q^k}.
\]

Note that for \( r \to \infty \), we obtain

\[
a_k = \frac{1 - q^{2k+1}}{q^k},
\]

which are the well-known coefficients for the classic \( q \)-tangent function.

Now we compute

\[
[z^n] \left( s_{k-1}(z) - a_k s_k(z) \right) = q^{\frac{k}{2}} \frac{(q^{r-1-k}; q^2)_{k+n} (q^{r+k}; q^2)_n}{(q; q^2)_{k+n} (q^2; q^2)_n}
\]

\[
- \frac{(q^{r+1-k}; q^2)_k (1 - q^{2k+1})}{(q^{-k}; q^2)_{k+1} q^k} \frac{(q^{-k}; q^2)_{k+1} q^k}{(q; q^2)_{k+n+1} (q^2; q^2)_n}
\]

\[
= q^{\frac{k}{2}} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n}{(q; q^2)_{k+n+1} (q^2; q^2)_n}
\]

\[
\times \left[ \frac{(q^{r+1-k}; q^2)_k (1 - q^{2k+2n+1})}{(q^{-k}; q^2)_{k} (1 - q^{r+k+2n})} - \frac{(q^{r+1-k}; q^2)_k (1 - q^{2k+1})}{(q^{-k}; q^2)_{k+1}} \right]
\]

\[
= q^{\frac{k}{2}} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n (q^{r+k+1}; q^2)_n q^{2k+1} (1 - q^{2n})(1 - q^{r-k-1})}{(1 - q^{r+k+2n})(1 - q^{r+k})}
\]

\[
= q^{\frac{k+1}{2}} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n (q^{r+k+1}; q^2)_n q^{2k+1} (1 - q^{2n})(1 - q^{r+k})}{(1 - q^{r+k+2n})(1 - q^{r+k})}
\]

\[
= q^{\frac{k+1}{2}} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n (q^{r+k+1}; q^2)_n q^{2k+1} (1 - q^{2n})(1 - q^{r+k})}{(1 - q^{r+k+2n})(1 - q^{r+k})}
\]

\[
= q^{\frac{k+1}{2}} \frac{(q^{r-k}; q^2)_{k+n+1} (q^{r+k+1}; q^2)_n (q^{r+k+1}; q^2)_n q^{2k+1} (1 - q^{2n})(1 - q^{r+k})}{(1 - q^{r+k+2n})(1 - q^{r+k})}
\]

\[
= [z^{n-1}] s_{k+1}(z).
\]

Since the constant term in this difference cancels out, we found the recurrence

\[
s_{k-1}(z) - a_k s_k(z) = z s_{k+1}(z).
\]
Therefore we have
\[
\frac{zs_0}{s_{-1}} = \frac{zs_0}{a_0s_0 + zs_1} = \frac{z}{a_0 + \frac{zs_1}{s_0}} = \frac{z}{a_0 + \frac{zs_1}{a_1 + \frac{z}{a_2 + \frac{z}{\ldots}}}}.
\]

If \( r \) is a positive integer, this continued fraction expansions stops, since \( s_r(z) = 0 \).

Replacing \( z \) by \(-z\) we get
\[
\frac{zs_0(-z)}{s_{-1}(-z)} = \frac{z}{a_0 - \frac{z}{a_1 - \frac{z}{a_2 - \frac{z}{\ldots}}}}.
\]

This translates into a continued fraction of \( \tan_q^{(r)}(u) \):
\[
\tan_q^{(r)}(u) = \frac{u}{a_0 - \frac{u^2}{a_1 - \frac{u^2}{a_2 - \frac{u^2}{\ldots}}}}.
\]

2 Cieslinski’s new \( q \)-tangent

After a first draft about the Foata and Han \( q \)-tangent was produced, a further \( q \)-tangent function was introduced by Cieslinski [1]. Recall that Jackson’s [5] classical \( q \)-trigonometric functions are defined as
\[
\sin_q z = \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{(q; q)_{2n+1}},
\]
\[
\cos_q z = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{(q; q)_{2n}}.
\]

Sometimes, instead of \( (q; q)_n \), the term \( (q; q)_n/(1 - q)^n \) is used, but that is clearly just a change of variable. The corresponding tangent function is defined by \( \tan_q z = \sin_q z/\cos_q z \).

Cieslinski [1] introduced new (“improved”? \( q \)-trigonometric functions:
\[
\sin_q(2z) = \frac{2\tan_q z}{1 + \tan_q^2 z},
\]
\[
\cos_q(2z) = \frac{1 - \tan_q^2 z}{1 + \tan_q^2 z}.
\]
\[ \cos_q(2z) = \frac{1 - \tan_q^2 z}{1 + \tan_q^2 z}. \]

Of course, this also introduces a (new) \( q \)-tangent function: \( \tan_q(z) = \sin_q(z)/\cos_q(z) \).

As we know, \( q \)-tangents are good candidates for beautiful continued fraction expansions [6, 4, 7, 8]; and this is confirmed by the results of the previous section. This new version is no exception; we are going to prove that

\[ z \tan(2z) = \frac{z^2}{a_0 + \frac{z^2}{a_1 + \frac{z^2}{\ldots}}}, \]

with

\[ a_{2k} = \frac{(1 - q^{4k+1})(-q^2)^2_k}{2q^k(-q^2)^2_k^k}, \]
\[ a_{2k+1} = -\frac{2(1 - q^{4k+3})(-q^2)^2_k}{q^k(-q^2)^2_{k+1}}. \]

As before, we obtain all the relevant quantities first by guessing them. First, we need the power series expansions of sine and cosine:

\[ \sin_q(2z) = \sum_{n \geq 0} z^{2n+1} \frac{(-1)^n(-1; q)_{2n+1}}{(q; q)_{2n+1}}, \]
\[ \cos_q(2z) = \sum_{n \geq 0} z^{2n} \frac{(-1)^n(-1; q)_{2n}}{(q; q)_{2n}}. \]

Cieslinski [1] has given the representations

\[ \sin_q(2z) = e_q^{iz} E_q^{iz} - e_q^{-iz} E_q^{-iz}, \]
\[ \cos_q(2z) = e_q^{iz} E_q^{iz} + e_q^{-iz} E_q^{-iz}, \]

with

\[ e_q^z = \sum_{n \geq 0} \frac{z^n}{(q; q)_n}, \quad E_q^z = \sum_{n \geq 0} \frac{z^n q^{(n)}_q}{(q; q)_n}. \]

From this, the desired expansions follow from comparing coefficients and simple \( q \)-identities.

Now define

\[ \sigma_0 := \sum_{n \geq 0} z^n \frac{(-1)^n(-1; q)_{2n+1}}{(q; q)_{2n+1}}, \]
\[ \sigma_{-1} := \sum_{n \geq 0} z^n \frac{(-1)^n(-1; q)_{2n}}{(q; q)_{2n}}. \]
and, more generally,

\[
\sigma_{2k} = q^{k^2} \frac{(-1)^k (-q^2; q^2)_k}{(-q; q^2)_k} \sum_{n \geq 0} z^n \frac{(-1)^n (-1; q)_{2k+2n+1}}{(q; q^2)_{2k+n+1} (q^2; q^2)_n},
\]

\[
\sigma_{2k+1} = q^{k^2+k} \frac{(-1)^k+1 (-q^2; q^2)_{k+1}}{(-1; q^2)_{k+1}} \sum_{n \geq 0} z^n \frac{(-1)^n (-1; q)_{2k+2n+1}}{(q; q^2)_{2k+n+2} (q^2; q^2)_n}.
\]

As in the previous section, we obtain the recursion

\[
\sigma_{i+1} = \frac{\sigma_{i-1} - a_i \sigma_i}{z}
\]

by a routine computation.

Consequently, we can write

\[
\frac{z \sigma_0}{\sigma_{-1}} = \frac{z \sigma_0}{a_0 \sigma_0 + z \sigma_1} = \frac{z}{a_0 + \frac{z \sigma_1}{\sigma_0}} = \frac{z}{a_0 + \frac{z}{a_1 + \frac{z \sigma_2}{\sigma_1}}} = \frac{z}{a_0 + \frac{z}{a_1 + \frac{z}{a_2 + \frac{z}{\ldots}}}}.
\]

The claimed continued fraction expansion of \(z \tan(2z)\) follows from this by substituting \(z\) by \(z^2\).

I was informed that this expansion could also be derived using results of Denis [2]. The present elementary approach should, however, not be without merits.

3 A uniform approach to the two \(q\)-tangents

It is apparent that

\[
\sin_q(u) = \sum_{n \geq 0} (-1)^n \frac{(w; q)_{2n+1}}{(q; q)_{2n+1}} u^{2n+1},
\]

\[
\cos_q(u) = \sum_{n \geq 0} (-1)^n \frac{(w; q)_{2n}}{(q; q)_{2n}} u^{2n},
\]

\[
\tan_q(u) = \frac{\sin_q(u)}{\cos_q(u)}
\]

generalises for \(w = q^r\) the Foata and Han version, and for \(w = -1\) the Cieslinski version. Our elementary approach can handle this situation as well:

Set

\[
\sigma_0(z) = \sum_{n \geq 0} \frac{(w; q)_{2n+1}}{(q; q)_{2n+1}} z^n,
\]

\[
\sigma_{-1}(z) = \sum_{n \geq 0} \frac{(w; q)_{2n}}{(q; q)_{2n}} z^n,
\]
then
\[ a_k = \frac{(wq^{1-k}; q^2)_k(1 - q^{2k+1})}{(wq^{-k}; q^2)_{k+1}q^k} \]
and
\[ \sigma_k(z) = q^{\frac{k(k+1)}{2}} \sum_{n \geq 0} z^n \frac{(wq^{-k}; q^2)_{n+k+1}(wq^{k+1}; q^2)_n}{(q; q^2)_{n+k+1}(q^2; q^2)_n}. \]

As before, we get
\[ \sigma_{k+1} = \frac{\sigma_{k-1} - a_k \sigma_k}{z} \]
and
\[ \frac{z \sigma_0(z)}{\sigma_{-1}(z)} = \frac{z}{a_0 + \frac{z}{a_1 + \frac{z}{a_2 + \ldots}}} \]

This gives the expansion of the \( q \)-tangent:
\[ \frac{z \sigma_0(-z^2)}{\sigma_{-1}(-z^2)} = \frac{z}{a_0 - \frac{z^2}{a_1 - \frac{z^2}{a_2 - \ldots}}} \]

References


