Aperiodic Subtraction Games

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To Doron Zeilberger at 60, Ekhad and Maple,
...and a threefold cord is not quickly broken (Ecclesiastes 4, 12).

Abstract

Periodicity is a fundamental property of many combinatorial games. It is sought vigorously, yet remains elusive in important cases, such as for some octal games, notably Grundy's game. Periodicity is important, because it provides poly-time winning strategies for many games. In particular, subtraction games, impartial and partizan, have been proved to be periodic. Our main purpose here is to exhibit constructively a class of subtraction games which is demonstratively aperiodic and yet is shown to have linear-time winning strategies.

Keywords: Combinatorial games, Complementary sequences, Numeration systems, Complexity, Aperiodicity

1 Prologue

Throughout we deal with two-player impartial games where the two players move alternately. We are mainly concerned with *normal* play, but we consider *misère* play in Section 4. Normal play means that the player first unable to move loses and the opponent wins. In misère play the outcome is reversed: the player making the last move loses, and the opponent wins.

In the theory of impartial combinatorial games, the notion of *periodicity* or its extension is central. Thus *octal games* have a poly-time winning strategy if the *Sprague-Grundy* function – to be discussed in Section 3 – is periodic [2]. The question whether certain octal games are periodic is still open. The most famous among them is Grundy's game: given a pile of tokens, divide it into two unequal parts. The player first unable to play (because

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all piles have size $\leqslant 2$) loses and the opponent wins. In many other games periodicity also plays an important role. Zeilberger showed that this is the case for chomp [6], [7]. See also [3]. Similarly, *subtraction games* have been proved to be periodic, both impartial [2] and partizan subtraction games [5].

The main purpose of this paper is to produce a class of *aperiodic* subtraction games.

In Section 2 we introduce the game Mark and provide a linear-time winning strategy for it, in normal play. In Section 3 we provide a linear-time winning strategy for the sum of Mark games, by characterizing the structure of the Sprague-Grundy function for the game. In Section 4 we provide a linear-time winning strategy for Mark played in misère. The variation UpMark of Mark is analyzed in Section 5. In Section 6 we discuss briefly the generalization Mark-t of Mark, and prove that it is aperiodic, lending justification to the title of this paper. We wrap up with an Epilogue in Section 7.

2 The game Mark

Given a nonnegative integer n. In the game Mark, two players alternate in moving from n. Either $n \to n-1$ or $n \to \lfloor n/2 \rfloor$. In other words, we can either reduce n by 1, or halve it, rounding down. We use *normal* play, as defined at the beginning of the Prologue. In particular, if $n = 0$, then the first player loses and the second wins.

Let $S \subsetneq \mathbb{Z}_{\geqslant 0}$, $\operatorname{mex} S = \min(\mathbb{Z}_{\geqslant 0} \setminus S)$ (the least nonnegative integer not in S). Notice that if S is the empty set, then $\operatorname{mex} S = 0$. Define two infinite sequences of integers $A = \cup_{n \geqslant 1} a_n$, $B = \cup_{n \geqslant 0} b_n$ recursively by

$$a_n = \operatorname{mex}\{a_i, b_i : 0 \leqslant i < n\} \ (n \geqslant 0), \quad (1)$$
$$b_n = 2a_n \ (n \geqslant 0). \quad (2)$$

The first few terms of the sequences A and B are depicted in the following table. They are the sequences A003159 and A036554 respectively in the useful and helpful OEIS – "Online Encyclopedia of Integer Sequences", created and maintained by the famous guru Neil Sloane.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23	25	27	28	29	31	33	35
b_n	0	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46	50	54	56	58	62	66	70

Proposition 1. *The sequences A, B are complementary: $A \cup B = \mathbb{Z}_{\geqslant 0}$, $A \cap B = \emptyset$.*

Proof. In view of (1), no nonnegative integer can be missing from the union. Suppose that $a_n = b_m$ for some $m, n \in \mathbb{Z}_{\geqslant 1}$. Then $n > m$ implies that a_n is the mex of a set containing $b_m = a_n$, a contradiction; and $n \leqslant m$ implies $b_m = 2a_m \geqslant 2a_n = 2b_m$ so $m = 0$, a contradiction. ■

A position u in a game is an N-position if the player moving from u (the **N**ext player) has a winning strategy. It's a P-position if the opponent of the player moving from u (the **P**revious player) can win. Notice that for normal play, the empty game is a P-position. The set of all N-positions of a game is denoted by $\mathcal{N}$, and the set of all its P-positions is denoted by $\mathcal{P}$. A moment's reflection will convince one that a position u is in $\mathcal{N}$ if it has an option in $\mathcal{P}$, whereas u is in $\mathcal{P}$ only if all its options are in $\mathcal{N}$.

Theorem 1. *For the game* Mark*, $\mathcal{P} = B$, $\mathcal{N} = A$.*

Proof. Since the game is acyclic, it suffices to show two properties: I. A player moving from any $b_n \in B$ always lands in a position in A; II. Given any position $a_n \in A$, there exists a move into B.

I. A move of type 1 from $b_n \in B$ results in $b_n - 1$, which is odd, hence not in B, since, by definition, B consists of even integers only. Since A and B are complementary, $b_n - 1 \in A$. A move of type 2 from $b_n \in B$ results in $b_n/2 = a_n \in A$.

II. Let $c \in \mathbb{Z}_{\geqslant 1}$. Observe that by (2), $c \in A$ if and only if $2c \in B$. Further, $c \in B$ if and only if $2c \in A$. Indeed, let $c \in B$. If $2c \in B$, then $c \in A$ by (2), a contradiction. Let $2c \in A$. If $c \in A$, then $2c \in B$ by (2), a contradiction. Now let $a_n \in A$. If a_n is odd, then $\lfloor a_n/2 \rfloor = (a_n - 1)/2$. Then either $a_n - 1 \in B$, or else $(a_n - 1)/2 \in B$ by the preceding observation. If a_n is even, then $a_n/2$ is in B by this observation. ■

Notation 1. The *vile* numbers are those whose binary representations end in an even number of 0s, and the *dopey* numbers are those that end in an odd number of 0s. Their names are inspired by the *evil* and *odious* numbers, those that have an even and an odd number of 1's in their binary representation respectively. To indicate that we count 0s rather than 1s, and only at the tail end, the "ev" and "od" are reversed to "ve" and "do" in "vile" and "dopey". "Evil" and "odious" were coined by Elwyn Berlekamp, John Conway and Richard Guy [2]. Let V be the set of all vile numbers, D the set of all dopey numbers.

Notation 2. For any positive integer n we use the notation $R(n)$ to denote the usual binary representation of n.

Theorem 2. *$\mathcal{P} = D$, $\mathcal{N} = V$.*

Proof. It suffices to show: I. A player moving from any $d \in D$ always lands in a position in V; II. Given any position $v \in V$, there exists a move into D.

I. We may assume $d > 0$. Then $R(d) = b_n \dots b_m 10^{2t-1}$, where $b_n \dots b_m$ is a (possibly empty) binary word, and $t \in \mathbb{Z}_{\geqslant 1}$. A move of type 1 transforms the suffix 10^{2t-1} of $R(d)$ into 01^{2t-1}, so $d - 1$ is vile. A move of type 2 deletes the trailing 0, so $d/2$ is vile.

II. The suffix of $R(v)$ has the form 10^{2t} for some $t \geqslant 0$. If $t > 0$, then a move of type 2 deletes the trailing 0, so $v/2$ is dopey. So assume $t = 0$. Then the suffix of $R(v)$ has the form 0^r1^s for some $r \geqslant 0$, $s \geqslant 1$. If $s > 1$, then a move of type 1 results in the suffix $0^r1^{s-1}0$, so $v-1$ is dopey. So assume $s = 1$. Since the result is clear if $v = 1$, we may assume that the suffix of $R(v)$ has the form 10^r1, $r > 0$. Then the suffix of $R(v-1)$ is 10^{r+1}. Thus $v-1$ is dopey if r is even. If r is odd, then a move of type 2 transforms v into $\lfloor v/2 \rfloor = (v-1)/2$. The latter is dopey, since $R(\lfloor v/2 \rfloor)$ has a suffix of the form 10^r. ■

Note 1. The recursive construction of (1), (2) seems to be computationally inefficient. The importance of Theorem 2 lies in the fact that it provides an easy linear-time winning strategy for MARK.

Note 2. Since $b_n = 2a_n$, $R(b_n)$ is just a *left shift* of $R(a_n)$; that is, $R(b_n) = R(a_n)0$. Notation: $\mathcal{L}R(a_n) = R(b_n)$ for every $n \geqslant 0$. This holds also for some results in the sequel, but we don't comment on it further.

The *spite* of a positive integer n is a mapping $\mathbb{Z}_{\geqslant 1} \to \{\text{odious, evil}\}$; we have spite$(n) =$ odious if n is odious, spite$(n) =$ evil if n is evil.

The following theorem is of independent interest, but it is also conducive to providing a linear-time algorithm for computing the g-function introduced in the next section.

Theorem 3. *The elements of A alternate in spite: a_{2n-1} odious, a_{2n} evil for all $(n \geqslant 1)$; the same holds for B: b_{2n-1} odious, b_{2n} evil for all $(n \geqslant 1)$.*

Proof. Clearly $a_1 = 1$ and $b_1 = 2$ are odious. Let $a \in A$, $a \geqslant 1$. Since a is vile (Theorem 2), the suffix of $R(a)$ has the form 10^{2t}, $t \geqslant 0$. We consider two cases.

(i) $t > 0$. Then $R(a+1)$ has the suffix $10^{2t-1}1$, so $a+1$ is vile, $a+1 \in A$, and $R(a+1)$ has precisely one 1-bit more then $R(a)$, so spite$(a+1) \neq$ spite(a).

(ii) $t = 0$. Then the suffix of $R(a)$ has the form 01^s, $s > 0$. This case subdivides into the following two subcases.

(ii1) $s = 2r$ is even $(r > 0)$, so the suffix of $R(a)$ has the form 01^{2r}. Then $R(a+1)$ has the suffix 10^{2r}, $a+1$ is vile so $a+1 \in A$, and $R(a+1)$ has precisely $2r-1$ less 1-bits than $R(a)$. Thus spite$(a+1) \neq$ spite(a).

(ii2) $s = 2r-1$ is odd $(r > 0)$, so the suffix of $R(a)$ has the form 01^{2r-1}. Then the suffix of $R(a+1)$ has the form 10^{2r-1}, so $a+1$ is dopey, and it has $2r-2$ less 1-bits than a. Then $R(a+2)$ has the suffix $10^{2r-2}1$, hence $a+2$ is vile so $a+2 \in A$ is the successor of $a \in A$, and $R(a+2)$ has $2r-3$ less 1 bits than $R(a)$. Hence also in this case spite$(a+2) \neq$ spite(a).

Since $R(b_n) = R(a_n)0$ for all $n \geqslant 1$, the result for B follows immediately from that of A. ■

3 Sums of Games, Including MARK

We begin with some definitions and background material.

If in a game there is a move $u \to v$, we say that position v is a *follower* or *option* of position u.

The *sum* of games is a collection of games such that a move consists of selecting one of the component games and making a legal move in it. In normal play, which we consider here, the player first unable to move (no component game has any move left) loses and the opponent wins. It is easy to see that the P, N tool of the component games is too weak to compute the P, N positions of the sum. The *Sprague-Grundy* function, g-function for short, enables us to compute the P, N positions of the sum. The notion of sum of games is fundamental in the theory of combinatorial game theory.

If u is any position in a game Γ, then $g(u) = \text{mex}\, g(F(u))$, where $F(u)$ denotes the set of all options of u. In particular, $g(\emptyset) = 0$. Now $g(u) = 0$ if $u \in \mathcal{P}$ of Γ, $g(u) > 0$ if $u \in \mathcal{N}$ of Γ. Given component games $\Gamma_1, \dots, \Gamma_m$ and positions $u_i \in \Gamma_i$, $i = 1, \dots, m$ then the position $u = (u_1, \dots, u_m)$ of the sum game Γ has g-function $\sigma(u) = \sum'^{m}_{i=1} g(u_i)$, where $\sum'$ denotes *Nim-sum* (sum over GF(2), also known as Xor). In particular, $u \in \mathcal{P}$ if $\sigma(u) = 0$, $u \in \mathcal{N}$ if $\sigma(u) > 0$. See [2].

For studying the structure of the g-function on Mark, we resort to the language of combinatorics on words. We view $g(0)g(1)g(2)\dots$ as an infinite ternary word W. Since every position u in Mark has at most two options, $g(u) \leqslant 2$. The first few g-values of Mark are shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
g	0	1	0	2	1	2	0	1	0	2	0	1	2	1	0	2	1	2	0	1	2	1	0	2

The structure of W can be revealed by demonstrating the structure of its *subwords* (also called *factors*). We preface the structure theorem by an auxiliary result.

Proposition 2. *Let $k, t \in \mathbb{Z}_{\geqslant 1}$, and let $R(k) = b_n \dots b_m 01^t$, where $b_n \dots b_m$ is a (possibly empty) binary word. Then*

$$\text{spite}(k)\text{=spite}(k+1) \textit{ if } t \textit{ is odd, and } \text{spite}(k) \neq \text{spite}(k+1) \textit{ if } t \textit{ is even.}$$

Proof. We have $R(k+1) = b_n \dots b_{n-i}10^t$, so $R(k+1)$ has $t-1$ less 1-bits than $R(k)$. Thus the spite is preserved if and only if $t = 2r+1$ is odd. ■

When we talk about the 0s of W, we mean the set $\{n : g(n) = 0\}$. Similarly about the 1s and 2s.

Theorem 4. (i) *The* 0*s of W are dopey and alternate in spite.*
(ii) *Every odious* 0 *is preceded by* 1 *and followed by* 2*; every evil* 0 *is preceded by* 2 *and followed by* 1*. The subwords* 102 *and* 201 *alternate, where possibly the trailing character of one subword coincides with the leading character of its neighbor:* 10201, 20102.
(iii) *All* 1*s of W are odious and all* 2*s are evil. Both are vile.*
(iv) *The only subwords between two consecutive* 0*s are* 010, 01210 (*if the leading* 0 *is evil*)*; and* 020, 02120 (*if the leading* 0 *is odious*).
(iv) *Each of the subwords listed in* (*ii*) *and* (*iii*) *appear infinitely often in W.*

Note 3. Sequence A091855 of the OEIS seems to confirm that the 1s of W comprise all vile-odious numbers, and sequence A091785 seems to confirm that the 2s comprise all vile-evil numbers, so A091855 $\cup$ A091785=A003159.

Note 4. Items (i) and (iii) provide a linear-time strategy for computing the g-function of every positive integer. This gives a linear-time winning strategy for sums of MARK games.

Proof. (i) The 0s are the P-positions. By Theorems 2 and 3, they are dopey and alternate in spite.

Items (ii), (iii) are proved simultaneously by induction on n, where n is the size of the prefix $g(0)\cdots g(n-1)$ of W. The statements can be verified directly for small n. Suppose they hold for all $m < n$. We consider three cases.

(a) $g(n) = 0$.

(a1) n odious. Since n is even, also $n/2$ is odious, so by induction, $g(n/2) = 1$. Now $g(\lfloor (n+1)/2 \rfloor) = g(n/2) = 1$. Hence $g(n+1) = 2$. Since n is dopey, $n-1$ is odious by Proposition 2, so by induction, $g(n-1) = 1$. Further, Theorem 3 implies that $n+1$ is evil.

(a2) n evil. Since n is even, also $n/2$ is evil, so by induction, $g(n/2) = 2$. Now $g(\lfloor (n+1)/2 \rfloor) = g(n/2) = 2$. Hence $g(n+1) = 1$. Since n is dopey, $n-1$ is evil by Proposition 2, so by induction, $g(n-1) = 2$. Further, Theorem 3 implies that $n+1$ is odious.

(b) $g(n) = 1$. Then $g(n-1) \in \{0, 2\}$.

(b1) $g(n-1) = 0$. By induction, $n-1$ is evil. Also by induction $g(n-2) = 2$ and $n-2$ is evil. Therefore Theorem 3 implies that n is odious.

(b2) $g(n-1) = 2$. By induction, $n-1$ is evil, so by Theorem 3, n is odious.

(c) $g(n) = 2$. Then $g(n-1) \in \{0, 1\}$.

(c1) $g(n-1) = 0$. By induction, $n-1$ is odious, $g(n-2) = 1$ and $n-2$ is odious. Therefore Theorem 3 implies that n is evil.

(c2) $g(n-1) = 1$. By induction, $n-1$ is odious, so by Theorem 3, n is evil.

This completes the induction proof of (ii), (iii), noting that all 1s and 2s are vile by Theorem 2, and 102 and 201 alternate since the odious and evil 0s alternate by (i).

(iv) The only possible chains between two consecutive 0s have the form $01212\ldots210$, and $02121\ldots120$. We first show that there cannot be more than three nonzero characters in any of these chains. Suppose $g(n) = 0$. Then n is even. If there are at least four nonzero characters following n, then we must have $g((n+2)/2) = g(1 + n/2) = 0$, and also $g((n+4)/2) = g(2 + n/2) = 0$. Thus two adjacent numbers, one being a follower of the other have g-value 0, a contradiction. We have shown that there can be at most three consecutive nonzero characters in W. The fact that the listed subwords are the only possibilities may be left to the reader.

(v) The subwords 01210 and 02120 appear infinitely often by (ii), since there are infinitely many odious 0s and infinitely many evil 0s. It is straightforward to see that at every $n = 1^{2t-1}0$ begins a subword of the form 01210 or 02120, and at every $n = 1^{2t}0$ begins a subword of the form 010 or 020. ■

4 Misère Mark

Mark in misère play is dubbed MiMark (Misère Mark). For MiMark define two infinite sequences of integers $A = \cup_{n\geqslant 1} a_n$ by (1), and

$$B = \cup_{n\geqslant 0} b_n \quad \text{where } b_0 = 1 \quad \text{and } b_n = 2a_n \ (n \geqslant 1), \tag{3}$$

which is the same as (2), except for specifying the value of b_0 and replacing $n \geqslant 0$ by $n \geqslant 1$. The initial entries are displayed in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	2	3	5	7	8	9	11	12	13	15	17	19	20	21	23	25	27	28	29	31	32	33	35
b_n	1	4	6	10	14	16	18	22	24	26	30	34	38	40	42	46	50	54	56	58	62	64	66	70

The sequence A is A053661 in OEIS, which is identical to A003159 except for the replacement of vile by dopey powers of 2. The sequence B is A171945, which is identical with A036554 except for an opposite replacement of powers of 2. Incidentally, the two sequences A053661 and A171944 of OEIS are identical (except for the first term of each).

It is easy to see that the sequences A, B are complementary. The proof is as in Proposition 1. Moreover,

Theorem 5. *For* MiMark, $\mathcal{P} = B$, $\mathcal{N} = A$.

The proof is the same as that of Theorem 1, except that (2) is replaced by (3).

However, Theorem 2 undergoes a metamorphosis. Let

$$\begin{aligned} V' &:= (V \setminus \{2^{2k} : k \geqslant 0\}) \cup \{2^{2k+1} : k \geqslant 0\}, \\ D' &:= (D \setminus \{2^{2k+1} : k \geqslant 0\}) \cup \{2^{2k} : k \geqslant 0\}. \end{aligned}$$

In other words, V' consists of all vile numbers, except that the powers of 2 are swapped: all odd powers replace all even powers. Similarly, D' consists of all dopey numbers, except that all even powers of 2 replace all odd powers of 2.

Theorem 6. *For* MiMark, $\mathcal{P} = D'$, $\mathcal{N} = V'$.

Proof. It suffices to show: I. A player moving from any $d \in D'$ always lands in a position in V'; II. Given any position $v \in V'$, there exists a move into D'.

I. We may assume $d > 0$. We consider two cases.

(i) $R(d) = b_n \dots b_m 10^{2t-1}$, where $b_n \dots b_m$ is a nonempty binary word, and $t \in \mathbb{Z}_{\geqslant 1}$. A move of type 1 transforms the suffix 10^{2t-1} of $R(d)$ into 01^{2t-1}, so $d-1$ is vile. A move of type 2 deletes the trailing 0, so $d/2$ is vile.

(ii) $R(d) = 10^{2t}$. Since $1 \in \mathcal{P}$, we may assume $t > 0$. A move of type 1 transforms $R(d)$

into 01^{2t-1}, so $d-1$ is vile. A move of type 2 deletes the trailing 0, so $d/2$ is dopey. Both options are in V'.

II. We consider again two cases.
(i) $R(v) = b_n \dots b_m 10^{2t}$ for some $t \geqslant 0$, where $b_n \dots b_m$ is a nonempty binary word. If $t > 0$, then a move of type 2 deletes the trailing 0, so $v/2$ is dopey. So assume $t = 0$. Then the suffix of $R(v)$ has the form 0^r1^s for some $r \geqslant 0$, $s \geqslant 1$. If $s > 1$, then a move of type 1 results in the suffix $0^r1^{s-1}0$, so $v-1$ is dopey. So assume $s = 1$. Since $v > 1$, we may assume that the suffix of $R(v)$ has the form 10^r1, $r > 0$. Then the suffix of $R(v-1)$ is 10^{r+1}. Thus $v-1$ is dopey if r is even. If r is odd, then a move of type 2 transforms v into $\lfloor v/2 \rfloor = (v-1)/2$. The latter is dopey, since $R(\lfloor v/2 \rfloor)$ has a suffix of the form 10^r.
(ii) $R(v) = 10^{2t+1}$ for some $t \geqslant 0$. A move of type 2 deletes the trailing 0, so $v/2$ is vile and in D'. ■

Again, Theorem 6 provides a linear-time winning strategy for MiMark.

5 UpMark

UpMark is the same as Mark, in normal play, except that halving is rounded up rather than down. Since the followers of 1 are 0 and 1, the game is loopy. To avoid loops, we define 1 to be the end position.

For UpMark define two infinite sequences of integers $A = \cup_{n \geqslant 1} a_n$ by (1), and $B = \cup_{n \geqslant 0} b_n$ by $b_0 = 1$ and

$$b_n = 2a_n - 1 \ (n \geqslant 1). \tag{4}$$

An initial segment of the sequences is depicted in the following table. The sequence A is A171945 of OEIS, and B is A171947.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	2	4	5	6	8	10	12	13	14	16	17	18	20	21	22	24	26	28	29	30	32	34	36
b_n	1	3	7	9	11	15	19	23	25	27	31	33	35	39	41	43	47	51	55	57	59	63	67	71

It is easy to see that the sequences A, B are complementary. The proof is as in Proposition 1.

Theorem 7. *For the game* UpMark, $\mathcal{P} = B$, $\mathcal{N} = A$.

Proof. As in he proof of Theorem 1, it suffices to show: I. A player moving from any $b_n \in B$ always lands in a position in A; II. Given any position $a_n \in A$, there exists a move into B.

I. A move of type 1 from $b_n \in B$ results in $b_n - 1$, which is even, hence not in B, since, by definition, B consists of odd integers only. Since A and B are complementary, $b_n - 1 \in A$. A move of type 2 from $b_n \in B$ results in $\lceil b_n/2 \rceil = (b_n + 1)/2 = a_n \in A$.

II. Let $c \in \mathbb{Z}_{>1}$. Observe that by (4),

$$c \in A \iff 2c - 1 \in B. \tag{5}$$

Further,

$$c \in B \iff 2c - 1 \in A. \tag{6}$$

Indeed, let $c \in B$. If $2c - 1 \in B$, then $c \in A$ by (5), a contradiction. Let $2c - 1 \in A$. If $c \in A$, then $2c - 1 \in B$ by (5), a contradiction. Now let $a_n \in A$. If a_n is odd, $a_n = 2d - 1 \in A$, then $d \in B$ by (6), and $d = (a_n + 1)/2 = \lceil a_n/2 \rceil$. So assume that $a_n = 2d$ is even. If $a_n - 1 = 2d - 1 \in B$, we are done. Otherwise, $a_n - 1 = 2d - 1 \in A$, so $d = a_n/2 \in B$ by (6). ■

Theorem 8. *For* UpMark, *A consists of all numbers a_n for which $R(a_n)$ ends in* 0 (*even numbers*) *or in $10^{2k-1}1$, $k \geqslant 1$; B consists of* 1 *and all numbers b_n for which $R(b_n)$ ends in $10^{2k}1$, $k \geqslant 1$, or in 101^k, $k > 1$. The members of B alternate in spite: b_{2n} is odious, b_{2n-1} is evil for all $n \geqslant 1$.*

Proof. Let A' be the set of all numbers a_n such that $R(a_n)$ ends in 0 or in $10^{2k-1}1$, $k \geqslant 1$; B' the set of all numbers b_n such that $R(b_n)$ ends in $10^{2k}1$, $k \geqslant 1$, or in 1^k, $k > 1$. By Theorem 7, it suffices to show:
I. Every move from any $b_n \in B'$, $n > 1$, leads to a position in A'; II. for every position $a_n \in A'$ there exists a move to a position in B'.

I. We obviously have $b_n \to b_n - 1 \in A'$, since $b_n - 1$ is even. If $R(b_n)$ ends in $10^{2k}1$, $k \geqslant 1$, then $R(b_n + 1)$ ends in $10^{2k-1}10$, so $R((b_n + 1)/2) = R(\lceil b_n/2 \rceil)$ ends in $10^{2k-1}1$, hence $\lceil b_n/2 \rceil \in A'$. If $R(b_n)$ ends in 1^k, $k > 1$, then $R(b_n+1)$ ends in 10^k, and $R((b_n+1)/2)$ ends in 0, so again $\lceil b_n/2 \rceil \in A'$.

II. (a) If $R(a_n)$ ends in 10^k, $k > 1$, then $R(a_n - 1)$ ends in 1^k, $k > 1$, so $a_n - 1 \in B'$.

(b) Suppose $R(a_n)$ ends in 1^k0, $k \geqslant 1$. If $k > 1$, then $R(a_n/2)$ ends in 1^k, so $a_n/2 \in B'$. We may thus assume that $k = 1$. Then $R(a_n)$ ends in 10^c10, $c \geqslant 1$. If c is even, then $R(a_n/2)$ ends in 10^c1, so $a_n/2 \in B'$. If c is odd, then $R(a_n - 1)$ ends in $10^{c+1}1$, so $a_n - 1 \in B'$.

(c) Suppose $R(a_n)$ ends in $10^{2k-1}1$, $k \geqslant 1$. If $k = 1$, then $R((a_n + 1)/2)$ ends in 11, so $(a_n + 1)/2 \in B'$. For $k > 1$, it is straightforward to verify that then $R((a_n + 1)/2)$ ends in $10^{2c}1$, $c \in \mathbb{Z}_{\geqslant 1}$, so again $(a_n + 1)/2 \in B'$.

Thus $A' = A$, $B' = B$.

Since spite(b_1) = evil, it suffices to show that spite $(b_{n+1}) \neq$ spite(b_n) for all $n \geqslant 1$. We consider two cases.

(i) $R(b_n)$ ends in 101^k, $k > 1$. Then $R(b_n + 1)$ ends in 10^k, and $b_n + 1$ has $k - 1$ less 1-bits than b_n. Further, $R(b_n + 2)$ ends in $10^{k-1}1$, and $b_n + 2$ has $k - 2$ less 1-bits than b_n.

Moreover, $b_n+2=b_{n+1}\in B$ if k is odd, and then spite $(b_{n+1})\neq$ spite(b_n). Now $R(b_n+3)$ ends in $10^{k-2}10$ and b_n+3 has $k-2$ less 1-bits than b_n. Then $R(b_n+4)$ ends in $10^{k-2}11$, so for k even, $b_n+4=b_{n+1}\in B$, and spite $(b_{n+1})\neq$ spite(b_n).

(ii) $R(b_n)$ ends in $10^{2k}1$, $k\geqslant 1$. Then $R(b_n+1)$ ends in $10^{2k-1}10$, and $R(b_n+2)$ ends in $10^{2k-1}11$. Thus $b_n+2=b_{n+1}$ and spite $(b_{n+1})\neq$ spite(b_n). ■

Theorem 8 also provides a linear-time winning strategy for UpMark. The proof about the alternating spite immediately implies the following result.

Corollary 1. (i) *If* $2t-1\in\mathcal{P}$, *then* $2t+1\in\mathcal{P}$ *if and only if* spite$(2t+1)\neq$ spite$(2t-1)$. (ii) spite$(2t+1)=$ spite$(2t-1)$ *if and only if* $R(2t-1)$ *ends in* 01^{2k}, $k\geqslant 1$. *If* $R(2t-1)$ *ends in* 01^{2k}, $k\geqslant 1$, *then* $R(2t+3)$ *ends in* 11, *and* $2t+3\in\mathcal{P}$.

6 Mark-t

For every $t\geqslant 2$ we define Mark-t as the game of removing one of $1,2,\ldots,t-1$ from a given positive integer n, or moving n to $\lfloor n/t\rfloor$, where Mark-2 = Mark. Letting a_n be given by (1) and $b_n=ta_n$, $n\geqslant 0$, we then have $\mathcal{P}=B$, $\mathcal{N}=A$, where $A=\cup_{n\geqslant 1}a_n$, $B=\cup_{n\geqslant 0}b_n$. Moreover, $\mathcal{P}=D$, $\mathcal{N}=V$, where now D is the set of all dopey numbers in the t-ary numeration system, and V is the set of all vile numbers in the t-ary numeration system. The proofs are very similar to the above for the case $t=2$ and are therefore omitted. In particular, there is a linear-time winning strategy for Mark-t for every $t\geqslant 2$. The following table depicts the first few N-positions (in the a_n row) and P-positions (the b_n row) for $t=4$. The sequence A is A171948 of OEIS and B is A171949.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
a_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16	17	18	19	21	22	23	25	26	27	29
b_n	0	4	8	12	20	24	28	36	40	44	52	56	60	64	68	72	76	84	88	92	100	104	108	116

We conclude our results with a theorem that justifies the title of this paper.

Theorem 9. *For every* $t\geqslant 2$, *the game* Mark-t *is aperiodic.*

Proof. We use the notation $P(n)$ for the statement: $n\in\mathcal{P}$. If the game is periodic, so are its P-positions, in particular. Suppose that there are constants $r,n_0\in\mathbb{Z}_{\geqslant 1}$ such that $P(n)=P(n+r)$ for all $n\geqslant n_0$. Then also $P(n)=P(n+kr)$ for all $n\geqslant n_0$ and all $k\geqslant 1$. Let $k\geqslant n_0$. We may assume that $tkr\in\mathcal{P}$, since if $tkr\in\mathcal{N}$, then $t^2kr\in\mathcal{P}$, so we replace k by tk. We have, $P(tkr)=P(tkr+(t-1)tkr)=P(t^2kr)$ by the assumed periodicity. Now one of the followers of t^2kr is tkr. Thus both t^2kr and its follower tkr are in $\mathcal{P}$, a contradiction. ■

Since the P-positions are aperiodic, so is, a fortiori, the g-function: the Ps are but the 0s of g. Theorem 9 shows that there are aperiodic subtraction games. This of course does not contradict the theorems that subtraction games, impartial and partizan, are periodic, since in the latter case the amount subtracted is restricted to a few constants, whereas here the amount subtracted is a function of the size of the pile.

7 Epilogue

The genesis of this paper reverts back to [1], where the following puzzle was proposed: "Nathan and Peter are playing a game. Nathan always goes first. The players take turns changing a positive integer to a smaller one and then passing the smaller number back to their opponent. On each move, a player may either subtract one from the integer or halve it, rounding down if necessary. Thus, from 28 the legal moves are to 27 or to 14; from 27, the legal moves are to 26 or to 13. The game ends when the integer reaches 0. The player who makes the last move wins. For example, if the starting integer is 15, a legal sequence of moves might be to 7, then 6, then 3, then 2, then 1, and then to 0. (In this sample game one of the players could have played better!) Assuming both Nathan and Peter play according to the best possible strategy, who will win if the starting integer is 1000? 2000?"

The names Nathan and Peter presumably derive from N- and P-positions respectively. We dubbed the game MARK because it is due to Mark Krusemeyer according to [1].

Since $15 \in \mathcal{N}$ which has the follower $14 \in \mathcal{P}$, we indeed see, as hinted in [1], that Nathan could have played better by moving $15 \to 14$ rather than $15 \to 7$, thus securing his win. Now $R(1000) = 1111101000$ is dopey, and $R(2000) = 11111010000$ is vile, since it is but the left shift of $R(1000)$. Therefore Peter, who moves second, can win 1000 and Nathan, who moves first, can win 2000.

In addition to analyzing the game MARK, and providing a linear-time winning strategy for it, we also determined the structure of its Sprague-Grundy function, and gave a linear-time algorithm for computing it. Further, we gave a linear-time winning strategy for MIMARK, which is MARK played in misère. We also analyzed the variation UPMARK of MARK, where rounding down was replaced by rounding up, again providing a linear-time winning strategy. We further sketched a linear-time winning strategy for the generalization MARK-t of MARK, and proved that this generalization is aperiodic.

The results of the present paper appeared in [4] without proof. There are some further directions to be pursued, such as computing the g-function for MARK-t for every $t \geqslant 2$, a strategy for playing MARK-t in misère play, a sum of MARK-t games in misère play, and UPMARK-t for every $t \geqslant 2$, permitting both rounding down and up, and, more generally, permitting moves that depend on the pile sizes. One can also investigate partizan versions, such as where one player can remove 1 or $\lfloor n/2 \rfloor$, and the other 2 or $\lfloor n/3 \rfloor$.

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