On Kadell's two Conjectures for the q-Dyson Product

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Abstract

By extending Lv-Xin-Zhou's first layer formulas of the q-Dyson product, we prove Kadell's conjecture for the Dyson product and show the error of his q-analogous conjecture. With the extended formulas we establish a q-analog of Kadell's conjecture for the Dyson product.

1 Introduction

In 1962, Freeman Dyson [3] conjectured the following constant term identity.

Theorem 1.1 (Dyson's Conjecture)**.** *For nonnegative integers $a_0, a_1, \ldots, a_n$,*

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0\leqslant i\neq j\leqslant n} \left(1-\frac{x_i}{x_j}\right)^{a_i} = \frac{a!}{a_0!\,a_1!\cdots a_n!},$$

where $a := a_0 + a_1 + \cdots + a_n$ and $\mathrm{CT}_{\mathbf{x}}\, f(\mathbf{x})$ means to take constant term in the x's of the series $f(\mathbf{x})$.

The conjecture was quickly proved independently by Gunson [6] and by Wilson [15]. An elegant recursive proof was published by Good [5], and a combinatorial proof was given by Zeilberger [16]. In 1975, George Andrews [1] came up with a q-analog of the Dyson conjecture.

Theorem 1.2. (Zeilberger-Bressoud)*. For nonnegative integers $a_0, a_1, \ldots, a_n$,*

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0\leqslant i<j\leqslant n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j} = \frac{(q)_a}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}},$$

where $(z)_m := (1-z)(1-zq)\cdots(1-zq^{m-1})$.

The Laurent polynomials in the above two theorems are respectively called the *Dyson product* and the *q-Dyson product* and denoted by $D_n(\mathbf{x}, \mathbf{a})$ and $D_n(\mathbf{x}, \mathbf{a}, q)$ respectively, where $\mathbf{x} := (x_0, \ldots, x_n)$ and $\mathbf{a} := (a_0, \ldots, a_n)$.

The Zeilberger-Bressoud q-Dyson Theorem was first proved, combinatorially, by Zeilberger and Bressoud [17] in 1985. Recently, Gessel and Xin [4] gave a very different proof by using the properties of formal Laurent series and of polynomials. The coefficients of the Dyson and the q-Dyson product were researched in [2, 7, 8, 9, 11, 12, 13]. In the equal parameter case, the identity reduces to Macdonald's constant term conjecture [10] for root systems of type A. In 1988 Stembridge [14] gave the first layer formulas of the q-Dyson product in the equal parameter case.

Condition 1. Let $I = \{i_1, \ldots, i_m\}$ be a proper subset of $\{0, 1, \ldots, n\}$ and $J = \{j_1, \ldots, j_m\}$ be a multi-subset of $\{0, 1, \ldots, n\} \setminus I$, where $0 \leqslant i_1 < \cdots < i_m \leqslant n$ and $0 \leqslant j_1 \leqslant \cdots \leqslant j_m \leqslant n$.

Our first objective in this paper is to prove the following conjecture of Kadell [7].

Conjecture 1.3. *For nonnegative integers $a_0, a_1, \ldots, a_n$ we have*

$$\Big(1 + a - \sum_{k\in I} a_k\Big) \operatorname*{CT}_{\mathbf{x}} \prod_{k=1}^{m} \Big(1 - \frac{x_{j_k}}{x_{i_k}}\Big) \prod_{0\leqslant i\neq j\leqslant n} \Big(1 - \frac{x_i}{x_j}\Big)^{a_i} = \Big(1 + a\Big) \frac{a!}{a_0! a_1! \cdots a_n!}. \tag{1.1}$$

In the same paper, Kadell also gave a q-analogous conjecture, we restate it as follows.

Conjecture 1.4. *Let $P = \{(i_k, j_k) \mid i_k \in I, j_k \in J, k = 1, 2, \ldots, m\}$. Then for nonnegative integers $a_0, a_1, \ldots, a_n$ we have*

$$\begin{aligned}\Big(1-q^{1+a-\sum_{k\in I} a_k}\Big) \operatorname*{CT}_{\mathbf{x}} \prod_{0\leqslant s<t\leqslant n} \left(\frac{x_s}{x_t}\right)_{a_i+\chi((t,s)\in P)} \left(\frac{x_t}{x_s} q\right)_{a_j+\chi((s,t)\in P)} \\ = \Big(1 - q^{1+a}\Big) \frac{(q)_a}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}},\end{aligned} \tag{1.2}$$

where the expression $\chi(S)$ is 1 if the statement S is true, and 0 otherwise.

In trying to prove Conjecture 1.4, we find that the conjectured formula is incorrect. One way to modify the conjecture is to evaluate the left-hand side of (1.2). This can be done by writing it as a linear combination of some first layer coefficients of the q-Dyson product, and then applying the formulas of [8]. Unfortunately, we are not able to derive a nice formula.

Our second objective is to contribute a q-analogous formula of (1.1), which is motivated by the proof of (1.1), and is stated in Theorem 4.1.

This paper is organized as follows. In Section 2 we reformulate the main result in [8] and give an extended form of it. In Section 3 we prove Conjecture 1.3 and give an example to show the error of Conjecture 1.4. In Section 4 we give our main theorem.

2 Basic results

Let $T = \{t_1, \ldots, t_d\}$ be a d-element subset of I with $t_1 < \cdots < t_d$. Define

$$w_i(T) = \begin{cases} a_i, & for \quad i \notin T; \\ 0, & for \quad i \in T. \end{cases} \tag{2.1}$$

Let S be a set and k be an element in $\{0, 1, \ldots, n\}$. Define $N(k, S)$ to be the number of elements in S no larger than k, i.e.,

$$N(k, S) = \big|\{i \leqslant k \mid i \in S\}\big|. \tag{2.2}$$

In particular, $N(k, \varnothing) = 0$.

The *first layer formulas* of the q-Dyson product can be restated as follows.

Theorem 2.1. [8] *Let I, J with $i_1 = 0$ satisfying Condition 1. Then for nonnegative integers $a_0, a_1, \ldots, a_n$ we have*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1} x_{j_2} \cdots x_{j_m}}{x_{i_1} x_{i_2} \cdots x_{i_m}} D_n(\mathbf{x}, \mathbf{a}, q) = \frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \sum_{\varnothing \neq T \subseteq I} (-1)^d q^{L(T|I)} \frac{1 - q^{\sum_{k \in T} a_k}}{1 - q^{1+a-\sum_{k \in T} a_k}}, \tag{2.3}$$

where

$$L(T \mid I) = \sum_{k=0}^{n} \big[N(k, I) - N(k, J)\big] w_k(T). \tag{2.4}$$

We need the explicit formula for the case $i_1 \neq 0$ for our calculation. As stated in [8], the formula for this case can be derived using an action π on Laurent polynomials:

$$\pi\big(F(x_0, x_1, \ldots, x_n)\big) = F(x_1, x_2, \ldots, x_n, x_0/q).$$

By iterating, if $F(x_0, x_1, x_2, \ldots, x_n)$ is homogeneous of degree 0, then

$$\pi^{n+1}\big(F(x_0, x_1, \ldots, x_n)\big) = F(x_0/q, x_1/q, x_2/q, \ldots, x_n/q) = F(x_0, x_1, x_2, \ldots, x_n),$$

so that in particular π is a cyclic action on $D_n(\mathbf{x}, \mathbf{a}, q)$. We use the following lemma to derive an extended form of Theorem 2.1.

Lemma 2.2. [8] *Let $L(\mathbf{x})$ be a Laurent polynomial in the x's. Then*

$$\operatorname*{CT}_{\mathbf{x}} L(\mathbf{x})\, D_n(\mathbf{x}, \mathbf{a}, q) = \operatorname*{CT}_{\mathbf{x}} \pi\big(L(\mathbf{x})\big) D_n\big(\mathbf{x}, (a_n, a_0, \ldots, a_{n-1}), q\big). \tag{2.5}$$

By iterating (2.5) *and renaming the parameters, evaluating $\operatorname{CT}_{\mathbf{x}} L(\mathbf{x})\, D_n(\mathbf{x}, \mathbf{a}, q)$ is equivalent to evaluating $\operatorname{CT}_{\mathbf{x}} \pi^k(L(\mathbf{x}))\, D_n(\mathbf{x}, \mathbf{a}, q)$ for any integer k.*

For I, J satisfying condition 1, let t be such that $j_t < i_1$ and $j_{t+1} > i_1$, where we treat $j_0 = -\infty$ and $j_{m+1} = \infty$. Denote by $J^- = \{j_1, \ldots, j_t\}$ and $J^+ = \{j_{t+1}, \ldots, j_m\}$.

Theorem 2.3. *For nonnegative integers $a_0, a_1, \ldots, a_n$ we have*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}x_{j_2}\cdots x_{j_m}}{x_{i_1}x_{i_2}\cdots x_{i_m}} D_n(\mathbf{x}, \mathbf{a}, q) = \frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}} \sum_{\varnothing \neq T \subseteq I} (-1)^d q^{L^*(T|I)} \frac{1 - q^{\sum_{k\in T} a_k}}{1 - q^{1+a-\sum_{k\in T} a_k}}, \tag{2.6}$$

where

$$L^*(T \mid I) = t + \sum_{k=i_1}^{n} \big[N(k, I) - N(k, J^+)\big] w_k(T) + \sum_{k=0}^{i_1-1} \big[t - N(k, J^-)\big] a_k. \tag{2.7}$$

The idea to prove this theorem is by iterating Lemma 2.2 to transform the random i_1 in (2.6) to zero and then applying Theorem 2.1. But in the proof there are many tedious transformations of the parameters, so we put the proof to the appendix for those who are interested in.

Letting $q \to 1^-$ in Theorem 2.3 we get

Corollary 2.4. [8] *For nonnegative integers $a_0, \ldots, a_n$ we have*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}\cdots x_{j_m}}{x_{i_1}\cdots x_{i_m}} \prod_{0\leqslant i\neq j\leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \frac{a!}{a_0!\cdots a_n!} \sum_{\varnothing \neq T \subseteq I} (-1)^d \frac{\sum_{k\in T} a_k}{1 + a - \sum_{k\in T} a_k}. \tag{2.8}$$

This result also follows from [8, Theorem 1.7] by permuting the variables. Note that the right-hand side of (2.8) is independent of the j's.

3 Proof of Conjecture 1.3

Now we are ready to prove Conjecture 1.3.

Proof of Conjecture **1.3**. If $I = \varnothing$ then Conjecture 1.3 reduces to the Dyson Theorem, which is also the case when $m = 0$ in Corollary 2.4. So we assume that $I \neq \varnothing$. Expanding the first product of (1.1) gives

$$\operatorname*{CT}_{\mathbf{x}} \prod_{i=1}^{m} \Big(1 - \frac{x_{j_k}}{x_{i_k}}\Big) \prod_{0\leqslant i\neq j\leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \operatorname*{CT}_{\mathbf{x}} \left[1 + \sum_{l=1}^{m} (-1)^l \sum_{\varnothing \neq I_l \subseteq I} \frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}}\right] \prod_{0\leqslant i\neq j\leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}$$

where $I_l = \{u_1, \ldots, u_l\}$ ranges over all nonempty subsets of I and $\{v_1, \ldots, v_l\}$ is the corresponding subset of J. Denote the left constant term in the above equation by LC. By applying Corollary 2.4, we get

$$LC = \left[1 + \sum_{l=1}^{m} (-1)^l \sum_{\varnothing \neq I_l \subseteq I} \sum_{\varnothing \neq T \subseteq I_l} (-1)^d \frac{\sum_{k\in T} a_k}{1 + a - \sum_{k\in T} a_k}\right] \frac{a!}{a_0!\cdots a_n!}, \tag{3.1}$$

where $d=|T|$. Changing the order of the summations, and observing that for any fixed set T the number of I_l satisfying $T\subseteq I_l\subseteq I$ is $\binom{m-d}{l-d}$, we obtain

$$\begin{aligned} LC &= \left[1+\sum_{\varnothing\neq T\subseteq I}\sum_{l=d}^{m}(-1)^{l+d}\binom{m-d}{l-d}\frac{\sum_{k\in T}a_k}{1+a-\sum_{k\in T}a_k}\right]\frac{a!}{a_0!\cdots a_n!}\\ &= \left(1+\frac{\sum_{k\in I}a_k}{1+a-\sum_{k\in I}a_k}\right)\frac{a!}{a_0!\cdots a_n!}, \end{aligned} \tag{3.2}$$

where we used the easy fact that for $d\neq m$

$$\sum_{l=d}^{m}(-1)^{l+d}\binom{m-d}{l-d}=\sum_{l=0}^{m-d}(-1)^{l}\binom{m-d}{l}=(1-x)^{m-d}\big|_{x=1}=0.$$

The conjecture then follows by multiplying both sides of (3.2) by $1+a-\sum_{k\in I}a_k$. □

For the q-case, Conjecture 1.4 does not hold even for $m=1$. To see this take $n=2, I=\{0\}, J=\{1\}$ and $a_0=a_1=a_2=1$. For these values the left-hand side of (1.2) is

$$\begin{aligned}(1-q^3)\operatorname*{CT}_{\mathbf{x}}(1-\frac{x_0}{x_1})(1-q\frac{x_1}{x_0})(1-q^2\frac{x_1}{x_0})(1-\frac{x_0}{x_2})(1-q\frac{x_2}{x_0})(1-\frac{x_1}{x_2})(1-q\frac{x_2}{x_1})\\ =(1-q^3)(1+2q+3q^2+2q^3),\end{aligned}$$

while the right-hand side of (1.2) equals $(1-q^4)(1+q)(1+q+q^2)$.

4 A q-analog of Kadell's conjecture

4.1 Motivation and presentation of the main theorem

In this section we will construct a q-analog of Conjecture 1.3. The new identity is motivated by the proof of Conjecture 1.3 in the last section, where massive cancelations happen. We hope for similar cancelations in the q-case.

Our first hope is to modify Conjecture 1.4 to obtain a formula of the form:

$$\left(1-q^{1+a-\sum_{k\in I}a_k}\right)\operatorname*{CT}_{\mathbf{x}}\prod_{k=1}^{m}\left(1-q^{L_k}\frac{x_{j_k}}{x_{i_k}}\right)D_n(\mathbf{x},\mathbf{a},q)=\left(1-q^{1+a}\right)\frac{(q)_a}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}, \tag{4.1}$$

where L_k is an integer depending on i_k, j_k and $\mathbf{a}$.

It is intuitive to consider the $m=2$ case, so take $I=\{i_1,i_2\}$. We need to choose appropriate L_1 and L_2 such that

$$\left(1-q^{1+a-a_{i_1}-a_{i_2}}\right)\operatorname*{CT}_{\mathbf{x}}\left(1-q^{L_1}\frac{x_{j_1}}{x_{i_1}}\right)\left(1-q^{L_2}\frac{x_{j_2}}{x_{i_2}}\right)D_n(\mathbf{x},\mathbf{a},q)=\left(1-q^{1+a}\right)\frac{(q)_a}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}. \tag{4.2}$$

By applying Theorem 2.3, the left-hand side of (4.2) becomes

$$\Big(1-q^{1+a-a_{i_1}-a_{i_2}}\Big)\Big(1+q^{L_1+L^*(\{i_1\}|\{i_1\})}\frac{1-q^{a_{i_1}}}{1-q^{1+a-a_{i_1}}}+q^{L_2+L^*(\{i_2\}|\{i_2\})}\frac{1-q^{a_{i_2}}}{1-q^{1+a-a_{i_2}}}$$
$$-q^{L_1+L_2+L^*(\{i_1\}|\{i_1,i_2\})}\frac{1-q^{a_{i_1}}}{1-q^{1+a-a_{i_1}}}-q^{L_1+L_2+L^*(\{i_2\}|\{i_1,i_2\})}\frac{1-q^{a_{i_2}}}{1-q^{1+a-a_{i_2}}}$$
$$+q^{L_1+L_2+L^*(\{i_1,i_2\}|\{i_1,i_2\})}\frac{1-q^{a_{i_1}+a_{i_2}}}{1-q^{1+a-a_{i_1}-a_{i_2}}}\Big)\frac{(q)_a}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}. \tag{4.3}$$

It is natural to have the following requirements to get (4.2).

$$\begin{aligned} q^{L_1+L^*(\{i_1\}|\{i_1\})}-q^{L_1+L_2+L^*(\{i_1\}|\{i_1,i_2\})}&=0,\\ q^{L_2+L^*(\{i_2\}|\{i_2\})}-q^{L_1+L_2+L^*(\{i_2\}|\{i_1,i_2\})}&=0,\\ q^{L_1+L_2+L^*(\{i_1,i_2\}|\{i_1,i_2\})}&=q^{1+a-a_{i_1}-a_{i_2}}. \end{aligned} \tag{4.4}$$

This is actually a linear system having no solutions, so our first hope broke.

Looking closer at (4.4), we see that the first two equalities must be satisfied to have a nice formula. Agreeing with this, for general I with $|I|=m$ we will need 2^m-2 restrictions for massive cancelations as in the proof of Conjecture 1.3. More precisely, by applying Theorem 2.3, the left-hand side of (4.1) will be written as

$$\Big(1-q^{1+a-\sum_{k\in I}a_k}\Big)\Big(1+\sum_T B_T\frac{1-q^{\sum_{k\in T}a_k}}{1-q^{1+a-\sum_{k\in T}a_k}}\Big)\frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}},$$

where T ranges over all nonempty subsets of I. We need to have $B_T=0$ for all T except for $T=I$. This is why using only m unknowns dooms to fail.

We hope for some nice A_T such that the constant term of

$$\sum_T A_T\frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}}D_n(\mathbf{x},\mathbf{a},q)$$

has the desired cancelations. We are optimistical because from the view of linear algebra, such A_T exists but is difficult to solve and might only be rational in q. Amazingly, it turns out that in many situations, the A_T may be chosen to be $\pm q^{integer}$. Our formula for A_T is inspired by the proof of Conjecture 1.3. To present our result, we need some notations.

Let I,J satisfy Condition 1. Given an l-element subset $I_l=\{u_1,\ldots,u_l\}$ of I, we say $J_l=\{v_1,\ldots,v_l\}$ is the *pairing set* of I_l if $u_k=i_t$ $(1\leqslant k\leqslant l)$ for some t implies that $v_k=j_t$. Write $I\setminus I_l=\{i_{r_1},\ldots,i_{r_{m-l}}\}$, $r_1<\cdots<r_{m-l}$. We use $A\xrightarrow{i}B$ to denote $B=A\cup\{i\}$, and define a sequence of sets:

$$I_l=\mathbb{I}_{m-l+1}\xrightarrow{i_{r_{m-l}}}\mathbb{I}_{m-l}\xrightarrow{i_{r_{m-l-1}}}\mathbb{I}_{m-l-1}\xrightarrow{i_{r_{m-l-2}}}\cdots\xrightarrow{i_{r_1}}\mathbb{I}_1=I. \tag{4.5}$$

For a set S of integers, we denote by $\min S$ the smallest element of S. Define $J_k^*(J_l)$ to be the set $\{j_s>\min\mathbb{I}_k\mid j_s\in J_l\cup\{j_{r_k}\}\}$, we use J_k^* as an abbreviation for $J_k^*(J_l)$.

Our q-analog of Conjecture 1.3 can be stated as follows.

Theorem 4.1. (Main Theorem) *For nonnegative integers $a_0, a_1, \ldots, a_n$, if there is no s, t, u such that $1 \leqslant s < t < u \leqslant m$ and $j_t < i_s < j_u < i_t$, then*

$$\Big(1-q^{1+a-\sum_{k\in I} a_k}\Big) \operatorname*{CT}_{\mathbf{x}} \left[\left(1+\sum_{\varnothing\neq I_l\subseteq I} (-1)^l q^{C(I_l)} \frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}}\right) D_n(\mathbf{x},\mathbf{a},q)\right]$$
$$= \Big(1-q^{1+a}\Big) \frac{(q)_a}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}, \tag{4.6}$$

where, with $L^*(I_l \mid I_l)$ defined as in (2.7),

$$C(I_l) = 1 + a - \sum_{k\in I_l} a_k + \sum_{k=1}^{m-l} \big[N(i_{r_k}, I_l) - N(i_{r_k}, J_k^*)\big] a_{i_{r_k}} - L^*(I_l \mid I_l). \tag{4.7}$$

We remark that there is no analogous simple formula if the u's and the v's are not paired up, and that the sum $1+\sum_{\varnothing\neq I_l\subseteq I}(-1)^l q^{C(I_l)} \frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}}$ in (4.6) does not factor.

4.2 Factorization and cancelation lemma

To prove the main theorem, we need some lemmas.

Let U be a subset of I_l, $|U| = d$ and $I \setminus U = \{i_{t_1}, \ldots, i_{t_{m-d}}\}$, $t_1 < \cdots < t_{m-d}$. For fixed I_l, suppose that $\min I_l = i_v$. By tedious calculation we can get the following lemma.

Lemma 4.2. *Let $U, C(I_l), L^*(U \mid I_l)$ be as described. Then for $i_{t_s} \in I_l$ but $i_{t_s} \notin U \cup \{i_v\}$ we have*

$$C(I_l) + L^*(U \mid I_l) - C(I_l \setminus \{i_{t_s}\}) - L^*(U \mid I_l \setminus \{i_{t_s}\})$$
$$= -\sum_{k=v}^{s-1} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}} + \sum_{k=s+1}^{m-d} \chi(\overline{i_{t_k} > j_{t_s} > i_v}) a_{i_{t_k}}, \tag{4.8}$$

where $\chi(\overline{i_{t_k} > j_{t_s} > i_v}) := 1 - \chi(i_{t_k} > j_{t_s} > i_v)$.

We denote $-\sum_{k=v}^{s-1} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}} + \sum_{k=s+1}^{m-d} \chi(\overline{i_{t_k} > j_{t_s} > i_v}) a_{i_{t_k}}$ by $g(i_{t_s})$.

Lemma 4.3. *For $n \geq 2$, every term in the expansion of $\prod_{s=1}^n \sum_{k\neq s} a(s,k)$ has $a(k,r)a(s,l)$ as a factor for some k, r, s, l satisfying $1 \leqslant r \leqslant s < k \leqslant l \leqslant n$.*

Proof. Construct a matrix A with 0's in the main diagonal as follows.

$$A = \begin{pmatrix} 0 & a(1,2) & \cdots & a(1,n) \\ a(2,1) & 0 & \cdots & a(2,n) \\ \vdots & \vdots & \vdots & \vdots \\ a(n,1) & a(n,2) & \cdots & 0 \end{pmatrix}.$$

Then each term in the expansion of $\prod_{s=1}^{n}\sum_{k\neq s}a(s,k)$ corresponds to picking out one entry except for the 0's from each row of A. We prove by contradiction.

Suppose we choose $a(1,k_1)$ ($k_1\geqslant 2$) from the first row. Then we can not choose $a(2,1)$, for otherwise $a(2,1)a(1,k_1)$ forms the desired factor. Now from the second row, we have to choose $a(2,k_2)$ ($k_2\geqslant 3$). It then follows that $a(3,1)$ and $a(3,2)$ can not be chosen, for otherwise $a(3,e)a(2,k_2), e=1,2$ forms the desired factor. Repeat this discussion until the $n-1$st row, where we have to choose $a(n-1,n)$. But then our nth row element $a(n,e)$ (with $1\leqslant e\leqslant n-1$) together with $a(n-1,n)$ forms the desired factor, a contradiction. □

The following factorization and cancelation lemma plays an important role and it is our main discovery in this paper.

Lemma 4.4. *For fixed set $U\neq I$ and integer $i_v\leqslant \min U$ we have the following factorization*

$$\sum_{I_l}(-1)^{l+d}q^{C(I_l)+L^*(U|I_l)}=(-1)^{\chi(\min U\neq i_v)}q^{C(U\cup\{i_v\})+L^*(U|U\cup\{i_v\})}\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right),\tag{4.9}$$

where I_l ranges over all supersets of U with the restriction $\min I_l=i_v$. Furthermore, if there is no s,t,u such that $1\leqslant s<t<u\leqslant m$ and $j_t<i_s<j_u<i_t$, then

$$\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right)=0,\tag{4.10}$$

with the only exceptional case when $I\setminus U\setminus\{i_1,\ldots,i_v\}=\varnothing$.

Proof. We prove this lemma in two parts.

1. Proof of (4.9).

Notice that $I_l=U\cup\{i_v\}$ is the smallest set which satisfies $\min I_l=i_v$ and $U\subseteq I_l$. So first we extract the common factor $q^{C(U\cup\{i_v\})+L^*(U|U\cup\{i_v\})}$ from the summation of (4.9). Thus we need to calculate

$$C(I_l)+L^*(U\mid I_l)-C(U\cup\{i_v\})-L^*(U\mid U\cup\{i_v\}).$$

By Lemma 4.2 we have

$$C(I_l)+L^*(U\mid I_l)-C(I_l\setminus\{i_{t_s}\})-L^*(U\mid I_l\setminus\{i_{t_s}\})=g(i_{t_s}),\tag{4.11}$$

where $i_{t_s}\in I_l$ but $i_{t_s}\notin U\cup\{i_v\}$. Thus iterating (4.11) we get

$$C(I_l)+L^*(U\mid I_l)-C(U\cup\{i_v\})-L^*(U\mid U\cup\{i_v\})=\sum_{i_{t_s}\in I_l\setminus U\setminus\{i_v\}}g(i_{t_s}).\tag{4.12}$$

So extracting the common factor $q^{C(U\cup\{i_v\})+L^*(U|U\cup\{i_v\})}$ from the left-hand side of (4.9) and by (4.12) we have

$$\sum_{I_l}(-1)^{l+d}q^{C(I_l)+L^*(U|I_l)} =q^{C(U\cup\{i_v\})+L^*(U|U\cup\{i_v\})}\sum_{I_l}(-1)^{l+d}q^{\sum_{i_{t_s}\in I_l\setminus U\setminus\{i_v\}}g(i_{t_s})}, \tag{4.13}$$

where I_l ranges over all supersets of U with the restriction $\min I_l = i_v$.

Next we prove the following factorization.

$$\sum_{I_l}(-1)^{l+d}q^{\sum_{i_{t_s}\in I_l\setminus U\setminus\{i_v\}}g(i_{t_s})} = (-1)^{\chi(\min U\neq i_v)}\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right), \tag{4.14}$$

where I_l ranges over all supersets of U and we restrict $\min I_l = i_v$.

If $\min U = i_v$, then the sign in the right-hand side of (4.14) is positive. Every term in the expansion of the right-hand side of (4.14) is of the form $(-1)^{|G|}\prod_{i_{t_s}\in G}q^{g(i_{t_s})} = (-1)^{|G|}q^{\sum_{i_{t_s}\in G}g(i_{t_s})}$, where G is a subset of $I\setminus U\setminus\{i_1,\ldots,i_v\}$. Thus expanding the product of (4.14) we get

$$\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right) = \sum_{G\subseteq I\setminus U\setminus\{i_1,\ldots,i_v\}}(-1)^{|G|}q^{\sum_{i_{t_s}\in G}g(i_{t_s})}. \tag{4.15}$$

Notice that $I_l\setminus U\setminus\{i_v\}$ reduces to $I_l\setminus U$ when $\min U = i_v$. Substitute $I_l\setminus U$ by G' in the left-hand side of (4.14). Then G' ranges over all subsets of $I\setminus U\setminus\{i_1,\ldots,i_v\}$ if I_l ranges over all supersets of U with the restriction $\min I_l = i_v$. Notice that $(-1)^{|G'|} = (-1)^{l-d} = (-1)^{l+d}$, thus the left-hand side of (4.14) can also be written as the right hand side of (4.15). Hence (4.14) holds when $\min U = i_v$. The case $\min U \neq i_v$ is similar.

Therefore (4.9) follows from (4.13) and (4.14).

2. Under the assumption that there is no s,t,u such that $1\leqslant s<t<u\leqslant m$ and $j_t<i_s<j_u<i_t$ we need to prove (4.10).

If $\min I_l = \min U = i_v$, recall that $I\setminus U = \{i_{t_1},\ldots,i_{t_{m-d}}\}$ and $t_1<\cdots<t_{m-d}$, then $t_k = k$ for $k = 1,\ldots,v-1$ and $t_v > v$. Thus $t_v \in I\setminus U\setminus\{i_1,\ldots,i_v\}$. It follows that $\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right) = \prod_{s=v}^{m-d}\left(1-q^{g(i_{t_s})}\right)$.

If $\min I_l \neq \min U$, then $t_v = v$. It follows that $t_v \notin I\setminus U\setminus\{i_1,\ldots,i_v\}$. Thus we have $\prod_{i_{t_s}\in I\setminus U\setminus\{i_1,\ldots,i_v\}}\left(1-q^{g(i_{t_s})}\right) = \prod_{s=v+1}^{m-d}\left(1-q^{g(i_{t_s})}\right)$ and $\chi(i_{t_v}>j_{t_s}>i_v) = \chi(i_v>j_{t_s}>i_v) = 0$. In this case $g(i_{t_s})$ reduces to

$$g(i_{t_s}) = -\sum_{k=v+1}^{s-1}\chi(i_{t_k}>j_{t_s}>i_v)a_{i_{t_k}} + \sum_{k=s+1}^{m-d}\chi(\overline{i_{t_k}>j_{t_s}>i_v})a_{i_{t_k}}.$$

We only prove (4.10) when $\min I_l = \min U$, the case $\min I_l \neq \min U$ is similar.

We can write the left-hand side of (4.10) as $\prod_{s=v}^{m-d}\left(1-q^{g(i_{t_s})}\right)$ when $\min I_l = \min U$. To prove $\prod_{s=v}^{m-d}\left(1-q^{g(i_{t_s})}\right) = 0$, it is sufficient to prove $\prod_{s=v}^{m-d}g(i_{t_s}) = 0$.

Taking $a(s,k) = -\chi(i_{t_k} > j_{t_s} > i_v)a_{i_{t_k}}$ for $s > k$ and $a(s,k) = \chi(\overline{i_{t_k} > j_{t_s} > i_v})a_{i_{t_k}}$ for $s < k$, by the definition of $g(i_{t_s})$ we can write $\prod_{s=v}^{m-d} g(i_{t_s})$ as $\prod_{s=v}^{m-d}\sum_{k\neq s} a(s,k)$. By Lemma 4.3 each term in the expansion of $\prod_{s=v}^{m-d} g(i_{t_s})$ has a factor of the form $-\chi(i_{t_r} > j_{t_k} > i_v)\chi(\overline{i_{t_l} > j_{t_s} > i_v})a_{i_{t_r}}a_{i_{t_l}}$, where $v \leqslant r \leqslant s < k \leqslant l \leqslant m-d$. Thus

$$\prod_{s=v}^{m-d} g(i_{t_s}) = \sum_{v\leqslant r\leqslant s<k\leqslant l\leqslant m-d} -\chi(i_{t_r} > j_{t_k} > i_v)\chi(\overline{i_{t_l} > j_{t_s} > i_v})a_{i_{t_r}}a_{i_{t_l}} \cdot \Delta, \tag{4.16}$$

where Δ is the product of some $a(s,k)$'s.

Next we prove each $\chi(i_{t_r} > j_{t_k} > i_v)\chi(\overline{i_{t_l} > j_{t_s} > i_v}) = 0$ by contradiction under the assumption that there is no s,t,u such that $1 \leqslant s < t < u \leqslant m$ and $j_t < i_s < j_u < i_t$.

Suppose $\chi(i_{t_r} > j_{t_k} > i_v)\chi(\overline{i_{t_l} > j_{t_s} > i_v}) = 1$ for some $v \leqslant r \leqslant s < k \leqslant l \leqslant m-d$. Then $\chi(i_{t_r} > j_{t_k} > i_v) = \chi(\overline{i_{t_l} > j_{t_s} > i_v}) = 1$. By $\chi(i_{t_r} > j_{t_k} > i_v) = 1$ we have

$$i_{t_r} > j_{t_k} > i_v. \tag{4.17}$$

By $\chi(\overline{i_{t_l} > j_{t_s} > i_v}) = 1$ we obtain

$$i_{t_l} < j_{t_s} \quad \text{or} \quad j_{t_s} < i_v \quad \text{or} \quad i_{t_l} < i_v. \tag{4.18}$$

Since $l > v$, we have $t_l \geqslant l > v$ and $i_{t_l} > i_v$. Thus the last inequality of (4.18) can not hold. Because $l > r$, $k > s$ and $i_{t_r} > j_{t_k}$ in (4.17), we have $i_{t_l} > i_{t_r} > j_{t_k} \geqslant j_{t_s}$. So the first inequality of (4.18) can not hold too. Thus by (4.17) and the middle inequality of (4.18) we obtain that if $\chi(i_{t_r} > j_{t_k} > i_v)\chi(\overline{i_{t_l} > j_{t_s} > i_v}) = 1$ then $j_{t_s} < i_v < j_{t_k} < i_{t_r}$. It follows that $j_{t_s} < i_v < j_{t_k} < i_{t_s}$ since $r \leqslant s$. Because $v \leqslant s < k$, we have $v < t_v \leqslant t_s < t_k$. Thus for $v < t_s < t_k$ the fact $j_{t_s} < i_v < j_{t_k} < i_{t_s}$ conflicts with our assumption. □

Lemma 4.5. *If U is of the form $\{i_h, i_{h+1}, \ldots, i_m\}$, then*

$$q^{C(U)+L^*(U|U)} - q^{C(U\cup\{i_{h-1}\})+L^*(U|U\cup\{i_{h-1}\})} = 0. \tag{4.19}$$

Proof. By the formula of $C(I_l)$ in (4.7) we have

$$C(U) + L^*(U \mid U) = 1 + a - \sum_{k\in U} a_k + \sum_{k=1}^{h-1} \big[N(i_{r_k}, U) - N(i_{r_k}, V_k^*)\big]a_{i_{r_k}},$$

where $V_k^* = \{j_s > i_k \mid j_s \in V_1 \cup \{j_{r_k}\}\}$ and $V_1 = \{j_h, \ldots, j_m\}$ is the pairing set of U. Since U is of the form $\{i_h, i_{h+1}, \ldots, i_m\}$, we have $i_{r_k} = i_k$ for $k = 1, \ldots, h-1$. Hence $N(i_{r_k}, U) = N(i_{r_k}, V_k^*) = 0$ for $k = 1, \ldots, h-1$. It follows that $C(U) + L^*(U \mid U) = 1 + a - \sum_{k\in U} a_k$.

Meanwhile

$$\begin{aligned}
&C(U\cup\{i_{h-1}\}) + L^*(U \mid U\cup\{i_{h-1}\})\\
=&1 + a - \sum_{k\in U} a_k - a_{i_{h-1}} + \sum_{k=1}^{h-2} \big[N(i_{r'_k}, U\cup\{i_{h-1}\}) - N(i_{r'_k}, \overline{V_k^*})\big]a_{i_{r'_k}}\\
&- L^*(U\cup\{i_{h-1}\} \mid U\cup\{i_{h-1}\}) + L^*(U \mid U\cup\{i_{h-1}\}),
\end{aligned}$$

where $\overline{V_k^*} = \{j_s > i_k \mid j_s \in V_2 \cup \{j_{r_k'}\}\}$ and $V_2 = \{j_{h-1}, \ldots, j_m\}$. Since $U \cup \{i_{h-1}\}$ is of the form $\{i_{h-1}, i_h, \ldots, i_m\}$, we have $i_{r_k'} = i_k$ for $k = 1, \ldots, h-2$. Hence $N(i_{r_k'}, U \cup \{i_{h-1}\}) = N(i_{r_k'}, \overline{V_k^*}) = 0$ for $k = 1, \ldots, h-2$. And by the definition of $L^*(T \mid I)$ in (2.7) we have $-L^*(U \cup \{i_{h-1}\} \mid U \cup \{i_{h-1}\}) + L^*(U \mid U \cup \{i_{h-1}\}) = a_{i_{h-1}}$. Therefore $C(U \cup \{i_{h-1}\}) + L^*(U \mid U \cup \{i_{h-1}\})$ has the same value as $C(U) + L^*(U \mid U)$. □

4.3 Proof of the main theorem

With Lemma 4.4 and Lemma 4.5, we are ready to prove the main theorem.

Proof of Theorem **4.1.** If $m = 0$, then the theorem reduces to the q-Dyson Theorem. So we assume that $m \geqslant 1$.

Applying Theorem 2.3 to the constant term in the left-hand side of (4.6) yields

$$\begin{aligned}&\operatorname*{CT}_{\mathbf{x}}\left[\left(1+\sum_{\varnothing\neq I_l\subseteq I}(-1)^l q^{C(I_l)}\frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}}\right)D_n(\mathbf{x},\mathbf{a},q)\right]\\ &\quad=\frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}}\left(1+\sum_{\varnothing\neq I_l\subseteq I}\sum_{\varnothing\neq U\subseteq I_l}(-1)^{d+l}q^{C(I_l)+L^*(U|I_l)}\frac{1-q^{\sum_{k\in U}a_k}}{1-q^{1+a-\sum_{k\in U}a_k}}\right),\end{aligned}\tag{4.20}$$

where $l = |I_l|$ and $d = |U|$.

Because U is a subset of I_l, we have $\min I_l = i_v \leqslant \min U$. By changing the summation order, the right-hand side of (4.20) can be rewritten as

$$\frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}}\left(1+\sum_{\varnothing\neq U\subseteq I}\sum_{i_v=i_1}^{\min U}\sum_{I_l}(-1)^{d+l}q^{C(I_l)+L^*(U|I_l)}\frac{1-q^{\sum_{k\in U}a_k}}{1-q^{1+a-\sum_{k\in U}a_k}}\right),\tag{4.21}$$

where I_l ranges over all supersets of U with the restriction $\min I_l = i_v$.

If $U \neq I$, then by Lemma 4.4, under the assumption that there is no s, t, u such that $1 \leqslant s < t < u \leqslant m$ and $j_t < i_s < j_u < i_t$ we have

$$\sum_{I_l}(-1)^{l+d}q^{C(I_l)+L^*(U|I_l)} = 0,\tag{4.22}$$

with the only exceptional case when $I \setminus U \setminus \{i_1, \ldots, i_v\} = \varnothing$, where I_l ranges over all supersets of U and we restrict $\min I_l = i_v$.

If $I \setminus U \setminus \{i_1, \ldots, i_v\} = \varnothing$, then U is of the form $\{i_h, i_{h+1}, \ldots, i_m\}$ and i_v is either i_h or i_{h-1} corresponding to $I_l = U$ or $I_l = U \cup \{i_{h-1}\}$ respectively. Thus by Lemma 4.5 we have

$$q^{C(U)+L^*(U|U)} - q^{C(U\cup\{i_{h-1}\})+L^*(U|U\cup\{i_{h-1}\})} = 0.\tag{4.23}$$

By (4.22) and (4.23) the summands in (4.21) cancel with each other except for the summand when $U = I_l = I$. It follows that (4.21) reduces to

$$\frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}}\Big(1+q^{C(I)+L^*(I|I)}\frac{1-q^{\sum_{k\in I}a_k}}{1-q^{1+a-\sum_{k\in I}a_k}}\Big). \tag{4.24}$$

By the formula of $C(I_l)$ in (4.7) we get $C(I) = 1 + a - \sum_{k\in I} a_k - L^*(I \mid I)$. Substituting $C(I)$ into (4.24) and multiplying the equation by $1 - q^{1+a-\sum_{k\in I} a_k}$ we can obtain the right-hand side of (4.6). □

5 Remark

If there exist some s, t, u such that $s < t < u$ and $j_t < i_s < j_u < i_t$, then our main theorem does not lead to the desired cancelations. As stated in Section 4.1, we can solve for A_T such that the constant term of $\sum_T A_T \frac{x_{v_1}\cdots x_{v_l}}{x_{u_1}\cdots x_{u_l}} D_n(\mathbf{x}, \mathbf{a}, q)$ has the desired cancelations. However, experiments show that there is no nice form for A_T in this situation.

Another possibility to let the u's and the v's be not paired up. Some of the cases can be established by applying the operator π defined in Section 2 to our main theorem. But not all the un-paired up cases can be obtained in this way.

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6 Appendix: Proof of Theorem 2.3

Proof. By the definition of π, it is easy to deduce that

$$\pi^k x_i = \begin{cases} x_{i+k}, & for \quad i+k \leqslant n; \\ x_{i+k-n-1}/q, & for \quad i+k > n. \end{cases} \tag{6.1}$$

Iterating Lemma 2.2 $n - i_1 + 1$ times, i.e., acting with π^{n-i_1+1}, we obtain

$$\begin{aligned} &\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}\cdots x_{j_m}}{x_{i_1}\cdots x_{i_m}} D_n(\mathbf{x}, \mathbf{a}, q) \\ &= \operatorname*{CT}_{\mathbf{x}} \frac{\prod_{l=1}^{t} x_{j_l+n-i_1+1} \prod_{l=t+1}^{m} x_{j_l-i_1} q^{-(m-t)}}{x_0 x_{i_2-i_1}\cdots x_{i_m-i_1} q^{-m}} D_n(\mathbf{x}, (b_0, \ldots, b_n), q), \end{aligned} \tag{6.2}$$

where

$$b_k = \begin{cases} a_{k+i_1}, & for \quad k = 0, \ldots, n-i_1; \\ a_{k-(n-i_1+1)}, & for \quad k = n-i_1+1, \ldots, n. \end{cases} \tag{6.3}$$

To apply Theorem 2.1, we define $\widetilde{I}=\{0,i_2-i_1,\ldots,i_m-i_1\}$, and $\widetilde{J^-}=\{j_1+n-i_1+1,\cdots,j_t+n-i_1+1\}$, $\widetilde{J^+}=\{j_{t+1}-i_1,\ldots,j_m-i_1\}$, $\widetilde{J}=\widetilde{J^-}\cup\widetilde{J^+}$. Then by Theorem 2.1 we have

$$\operatorname*{CT}_{\mathbf{x}}\frac{x_{j_1}\cdots x_{j_m}}{x_{i_1}\cdots x_{i_m}}D_n(\mathbf{x},\mathbf{a},q)=q^t\frac{(q)_a}{(q)_{a_0}\cdots(q)_{a_n}}\sum_{\varnothing\neq\widetilde{T}\subseteq\widetilde{I}}(-1)^d q^{L(\widetilde{T}|\widetilde{I})}\frac{1-q^{\sum_{k\in\widetilde{T}}b_k}}{1-q^{1+a-\sum_{k\in\widetilde{T}}b_k}},$$

where $|\widetilde{T}|=d$ and

$$L(\widetilde{T}\mid\widetilde{I})=\sum_{k=0}^{n}\big[N(k,\widetilde{I})-N(k,\widetilde{J})\big]\widetilde{w}_k(\widetilde{T}),\tag{6.4}$$

in which $\widetilde{w}_k(\widetilde{T})$ is b_k if $k\notin\widetilde{T}$ and 0 otherwise.

There is a natural one-to-one correspondence between I and $\widetilde{I}$: $I\xrightarrow{f}\widetilde{I}$, $f(a)=a-i_1$, $a\in I$. This correspondence clearly applies between their subsets T and $\widetilde{T}$.

Since the largest element in $\widetilde{T}$ is not larger than i_m-i_1 and $i_m-i_1\leqslant n-i_1$, by the definition of b_k we have

$$\sum_{k\in\widetilde{T}}b_k=\sum_{k\in\widetilde{T}}a_{k+i_1}=\sum_{k\in T}a_k.$$

Next we have to rewrite (6.4) in terms of $w_k(T)$, $N(k,I)$ and $N(k,J)$ to get $L^*(T\mid I)$.

Because the largest element in $\widetilde{I}$ is $i_m-i_1\leqslant n-i_1$, so if $k>n-i_1$ then $k\notin\widetilde{T}$. It follows that

$$\widetilde{w}_k(\widetilde{T})=b_k=a_{k-(n-i_1+1)}.\tag{6.5}$$

If $k\leqslant n-i_1$, then

$$\widetilde{w}_k(\widetilde{T})=\begin{cases}b_k=a_{k+i_1}, & \text{if}\quad k\notin\widetilde{T};\\ 0, & \text{if}\quad k\in\widetilde{T},\end{cases}\tag{6.6}$$

which is in fact $w_{k+i_1}(T)$.

It is straightforward to check that

$$N(k,\widetilde{I})=N(k+i_1,I),\tag{6.7}$$

$$N(k,\widetilde{J^-})=N(k-(n-i_1+1),J^-),\quad N(k,\widetilde{J^+})=N(k+i_1,J^+),\tag{6.8}$$

$$N(k,\widetilde{J})=N(k,\widetilde{J^-})+N(k,\widetilde{J^+}).\tag{6.9}$$

Substituting (6.5) and (6.6) into (6.4) we have

$$L(\widetilde{T}\mid\widetilde{I})=\sum_{k=0}^{n-i_1}\big[N(k,\widetilde{I})-N(k,\widetilde{J})\big]w_{k+i_1}(T)+\sum_{k=n-i_1+1}^{n}\big[N(k,\widetilde{I})-N(k,\widetilde{J})\big]a_{k-(n-i_1+1)}.$$

By (6.7)–(6.9) the above equation becomes

$$L(\widetilde{T} \mid \widetilde{I}) = \sum_{k=0}^{n-i_1} \big[N(k+i_1, I) - N(k-(n-i_1+1), J^-) - N(k+i_1, J^+)\big] w_{k+i_1}(T) + \sum_{k=n-i_1+1}^{n} \big[N(k+i_1, I) - N(k-(n-i_1+1), J^-) - N(k+i_1, J^+)\big] a_{k-(n-i_1+1)}. \tag{6.10}$$

If $k \in [0, n-i_1]$ then $k-(n-i_1+1) < 0$. Thus $N(k-(n-i_1+1), J^-) = 0$.

If $k \in [n-i_1+1, n]$ then $k+i_1 > n$. Thus $N(k+i_1, I) = m$ and $N(k+i_1, J^+) = m-t$.

Therefore (6.10) reduces to

$$\begin{aligned} L(\widetilde{T} \mid \widetilde{I}) &= \sum_{k=0}^{n-i_1} \big[N(k+i_1, I) - N(k+i_1, J^+)\big] w_{k+i_1}(T) \\ &\quad + \sum_{k=n-i_1+1}^{n} \big[t - N(k-(n-i_1+1), J^-)\big] a_{k-(n-i_1+1)} \\ &= \sum_{k=i_1}^{n} \big[N(k, I) - N(k, J^+)\big] w_k(T) + \sum_{k=0}^{i_1-1} \big[t - N(k, J^-)\big] a_k. \end{aligned}$$

Then we obtain

$$L^*(T \mid I) = t + L(\widetilde{T} \mid \widetilde{I}) = t + \sum_{k=i_1}^{n} \big[N(k, I) - N(k, J^+)\big] w_k(T) + \sum_{k=0}^{i_1-1} \big[t - N(k, J^-)\big] a_k.$$

□

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