Enumeration of standard Young tableaux of certain truncated shapes

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Dedicated to Doron Zeilberger on the occasion of his 60*th birthday.*
Mazal Tov!

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Abstract

Unexpected product formulas for the number of standard Young tableaux of certain truncated shapes are found and proved. These include shifted staircase shapes minus a square in the NE corner, rectangular shapes minus a square in the NE corner, and some variations.

1 Introduction

A truncated shape is obtained from a Ferrers diagram (in the English notation, where parts decrease from top to bottom) by deleting cells from the NE corner. Interest in the enumeration of standard Young tableaux of truncated shapes is enhanced by a recent result [1, Prop. 9.7]: the number of geodesics between distinguished pairs of antipodes in

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the flip graph of triangle-free triangulations is equal to twice the number of Young tableaux of a truncated shifted staircase shape. Motivated by this result, extensive computations were carried out for the number of standard Young tableaux of these and other truncated shapes. It was found that, in certain distinguished cases, all prime factors of these numbers are relatively small, hinting at the existence of product formulas. In this paper, such product formulas are proved for rectangular and shifted staircase shapes truncated by a square, or nearly a square.

A different method to derive product formulas, for other families of truncated shapes, has independently been developed by G. Panova [7].

The rest of the paper is organized as follows. Basic concepts are described in Section 2. The general idea of pivoting is presented in Section 3. Section 4 contains detailed proofs for truncated shifted staircase shapes, with Theorem 4.6 as the main result; while Section 5 contains an analogous development for truncated rectangular shapes, with Theorem 5.5 as the main result. Section 6 contains final remarks and open problems.

2 Preliminaries and Basic Concepts

A *partition* λ of a positive integer N is a sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ and the total size $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_m$ is N. The *Ferrers diagram* $[\lambda]$ of shape λ is a left-justified array of N cells, with row i (from top to bottom) containing λ_i cells. A *standard Young tableau* (SYT) T of shape λ is a labeling by $\{1, 2, \ldots, N\}$ of the cells in the diagram $[\lambda]$ such that every row is increasing from left to right, and every column is increasing from top to bottom. The number of SYT of shape λ is denoted by f^λ.

Proposition 2.1 (The Frobenius-Young Formula) *[4, 10] The number of SYT of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ is*

$$f^\lambda = \frac{(|\lambda|)!}{\prod_{i=1}^{m}(\lambda_i + m - i)!} \cdot \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j - i + j).$$

Note that adding trailing zeros to λ (with m appropriately increased) does not affect the right-hand side of the above formula.

A partition $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of N is *strict* if $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$. The corresponding diagram of *shifted shape* λ is the array of N cells with row i containing λ_i cells and indented $i - 1$ places. A standard Young tableau (SYT) T of shifted shape λ is a labeling by $\{1, 2, \ldots, N\}$ of the cells in the diagram $[\lambda]$ such that each row and column is increasing. The number of SYT of shifted shape λ is denoted by g^λ.

Proposition 2.2 (Schur's Formula)*[8][6, p. 267 (2)] The number of SYT of shifted shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$ is*

$$g^\lambda = \frac{(|\lambda|)!}{\prod_{i=1}^{m} \lambda_i!} \cdot \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

It is well known that both f^λ and g^λ also have hook length formulas, but the equivalent formulas above will be more convenient for our calculations.

We shall frequently use the following two basic operations on partitions. The *union* $\lambda \cup \mu$ of two partitions λ and μ is simply their multiset union. We shall usually assume that each part of λ is greater or equal than each part of μ. The *sum* of two partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ (with trailing zeros added in order to get the same number of parts) is

$$\lambda + \mu := (\lambda_1 + \mu_1, \ldots, \lambda_m + \mu_m).$$

For any nonnegative integer m, let $[m] := (m, m-1, \ldots, 1)$ be the corresponding shifted staircase shape. Consider the *truncated* shifted staircase shape $[m] \setminus \kappa$, where a partition $\kappa = (\kappa_1, ..., \kappa_k)$, with $\kappa_i \leq m - i$ for all $1 \leq i \leq k < m$, is deleted from the NE corner; namely, κ_1 cells are deleted from the (right) end of the first row, κ_2 cells are deleted from the end of the second row, etc. Let N be the size of $[m] \setminus \kappa$. A standard Young tableau (SYT) of truncated shifted staircase shape $[m] \setminus \kappa$ is a labeling of the cells of this shape by $\{1, \ldots, N\}$ such that labels increase across rows (from left to right), down columns (from top to bottom) and down the main diagonal (from top left to bottom right).

Example 2.3 There are four SYT of shape $[4] \setminus (1)$:

$$\begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{array}\ ,\quad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{array}\ ,\quad \begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{array}\ ,\quad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{array}\ .$$

Similarly, for nonnegative integers m and n, let $(n^m) = (n, \ldots, n)$ (m parts) be the corresponding rectangular shape. Consider the truncated rectangular shape $(n^m) \setminus \kappa$, where $\kappa \subseteq (n^m)$ is deleted from the NE corner; namely, κ_1 cells are deleted from the end of the first row, κ_2 cells are deleted from the end of the second row, etc. Letting N be the size of $(n^m) \setminus \kappa$, a SYT of truncated shape $(n^m) \setminus \kappa$ is a labeling of the cells of this shape by $\{1, \ldots, N\}$, such that labels increase along rows (from left to right) and down columns (from top to bottom).

Preliminary computer experiments hinted that a remarkable phenomenon occurs when a *square* is truncated from a staircase shape: while the number of SYT of truncated staircase shape $[m] \setminus (k^k)$ increases exponentially as a function of the size $N = \binom{m+1}{2} - k^2$ of the shape, all the prime factors of this number are actually smaller than the size. A similar phenomenon occurs for squares truncated from rectangular shapes [1]. In this paper, product formulas for these (and related) truncated shapes, explaining the above factorization phenomenon, will be proved; see Corollaries 4.7, 4.8, 5.6 and 5.7 below.

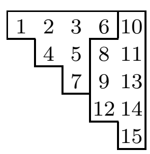
[1] It should be noted that this factorization phenomenon does not hold in the general case. For example, the number of SYT of truncated rectangular shape $(7^6) \setminus (2)$, of size 40, has 5333 as a prime factor.

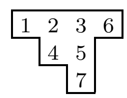
3 Pivoting

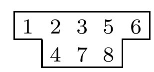
The basic idea of the proofs in the following sections is to enumerate the SYT T of a given shape ζ by mapping them bijectively to pairs (T_1, T_2) of SYT of some other shapes. This will be done in two distinct (but superficially similar) ways, which will complement each other and lead to the desired results.

The first bijection will be applied only to non-truncated shapes of two types: a rectangle and a shifted staircase. Let N be the size of the shape ζ (of either of the aforementioned types), and fix an integer $1 \leq t \leq N$. Subdivide the entries in a SYT T of shape ζ into those that are less than or equal to t and those that are greater than t. The entries $\leq t$ constitute T_1. To obtain T_2, replace each entry $i > t$ of T by $N - i + 1$, and suitably transpose (actually, reflect in the line $y = x$) the resulting array. It is easy to see that both T_1 and T_2 are SYT.

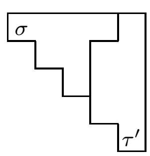
This is illustrated schematically in the case of a shifted staircase shape by the following shifted SYT, where $\zeta = (5, 4, 3, 2, 1)$, $N = 15$ and $t = 7$.

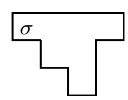
$T =$ 

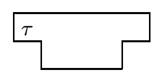
$\Leftrightarrow$ $T_1, T_2 =$ 

, 

In terms of shapes we have

$[\zeta] =$ 

$\Leftrightarrow$ $[\sigma], [\tau] =$ 

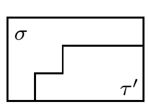
, 

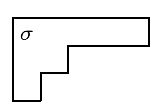
where τ' denotes the conjugate of the (strict) partition τ. It follows immediately from this argument that

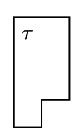
$$g^{\zeta} = \sum_{\substack{\sigma \subseteq \zeta \\ |\sigma| = t}} g^{\sigma} \, g^{(\zeta/\sigma)'} ,$$

where $\tau = (\zeta/\sigma)'$ is the conjugate of the shifted skew shape ζ/σ. The shape $[\tau]$ is formed by deleting the cells of $[\sigma]$ from those of $[\zeta]$ and then reflecting in the line $y = x$.

In the case of a rectangular shape we have

$[\zeta] =$ 

$\Leftrightarrow$ $[\sigma], [\tau] =$ 

, 

with

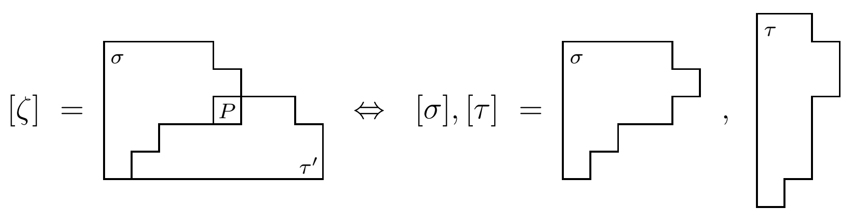
$$f^{\zeta} = \sum_{\substack{\sigma \subseteq \zeta \\ |\sigma| = t}} f^{\sigma} \, f^{(\zeta/\sigma)'} .$$

The second bijection will be applied to truncated shapes, where a partition is truncated from the NE corner of either a rectangle or a shifted staircase. Given such a truncated shape ζ, choose a *pivot cell* P. This is a cell of ζ which belongs to its NE boundary, namely such that there is no cell of ζ which is both strictly north and strictly east of the cell P. If T is a SYT of shape ζ, let t be the entry of T in the pivot cell P. Subdivide the other entries of T into those that are (strictly) less than t and those that are greater than t. The entries less than t constitute T_1. To obtain T_2, replace each entry $i > t$ of T by $N - i + 1$, where N is the total number of entries in T, and suitably transpose the resulting array. It is easy, once again, to see that both T_1 and T_2 are SYT.

By way of example, consider the truncated rectangular shape $\zeta = (4, 5, 7, 8, 8)$ and let P be the cell in position $(3, 5)$. Then for $t = 17$ the map from a SYT T of truncated shape ζ to a corresponding pair (T_1, T_2) is illustrated by

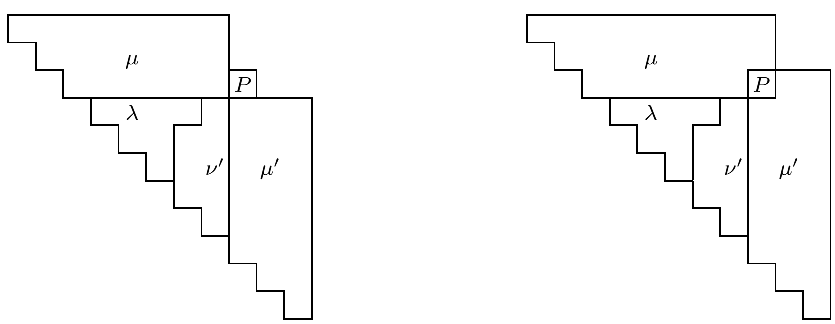
$$T = \begin{array}{cccccccc} 1 & 2 & 4 & 9 & & & & \\ 3 & 5 & 11 & 12 & 13 & & & \\ 6 & 8 & 14 & 15 & 17 & 21 & 24 & \\ 7 & 16 & 18 & 20 & 25 & 26 & 27 & 30 \\ 10 & 19 & 22 & 23 & 28 & 29 & 31 & 32 \end{array} \quad \Leftrightarrow \quad T_1, T_2 = \begin{array}{ccccc} 1 & 2 & 4 & 9 & \\ 3 & 5 & 11 & 12 & 13 \\ 6 & 8 & 14 & 15 & \\ 7 & 16 & & & \\ 10 & & & & \end{array} \;,\; \begin{array}{ccc} 1 & 3 & \\ 2 & 6 & 9 \\ 4 & 7 & 12 \\ 5 & 8 & \\ 10 & 13 & \\ 11 & 15 & \\ 14 & & \end{array}$$

In terms of shapes we have



where τ' is the conjugate of τ. Similarly for a truncated shifted staircase shape.

The particular shapes required in the sequel are the following:



(1)

and

(2)

A crucial property of these particular shapes is that their subdivision gives rise to shapes $\mu \cup \lambda$ and $\mu \cup \nu$ (or $(\mu + \alpha) \cup \lambda$ and $(\mu + \beta) \cup \nu$) which are *not* truncated.

A more explicit relation between λ and ν will be given later. For the time being it suffices to observe that

$$g^{\zeta} = \sum_{\lambda \subseteq [m]} g^{\mu \cup \lambda} \, g^{\mu \cup ([m]/\lambda)'} \tag{3}$$

and

$$f^{\zeta} = \sum_{\lambda \subseteq (n^m)} f^{(\mu+\alpha) \cup \lambda} \, f^{(\mu+\beta) \cup ((m^n)/\lambda)'} \,. \tag{4}$$

Since ζ is a truncated shape, the notation on the LHS of the above equalities is to be taken in a generalized sense.

4 Truncated Shifted Staircase Shapes

In this section, $\lambda = (\lambda_1, \ldots, \lambda_m)$ (with $\lambda_1 > \ldots > \lambda_m > 0$ integers) will be a strict partition, with g^{λ} denoting the number of SYT of shifted shape λ. We shall use the union operation on partitions, defined in Section 2.

For any nonnegative integer m, let $[m] := (m, m-1, \ldots, 1)$ be the corresponding shifted staircase shape. Schur's formula (Proposition 2.2) implies the following.

Observation 4.1 *The number of SYT of shifted staircase shape $[m]$ is*

$$g^{[m]} = M! \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!},$$

where $M := |[m]| = \binom{m+1}{2}$.

We shall now use the first bijection outlined in Section 3.

Lemma 4.2 *Let m and t be nonnegative integers, with $t \leq \binom{m+1}{2}$. Let T be a SYT of shifted staircase shape $[m]$, let T_1 be the set of all cells in T with values at most t, and let T_2 be obtained from $T \setminus T_1$ by transposing the shape and replacing each entry i by $M-i+1$. Then:*

- *(i) T_1 and T_2 are shifted SYT.*
- *(ii) Treating strict partitions as sets, $[m]$ is the disjoint union of the shape of T_1 and the the shape of T_2.*

Proof. (i) is clear. In order to prove (ii), denote the shifted shapes of T_1 and T_2 by λ^1 and λ^2, respectively. The borderline between T_1 and $T \setminus T_1$ is a lattice path of length exactly m, starting at the NE corner of the staircase shape $[m]$ and using only S and W steps. If the first step is S then the first part of λ^1 is m, and the rest (of both λ^1 and λ^2) corresponds to a lattice path in $[m-1]$. Similarly, if the first step is W then the first part of λ^2 is m, and the rest corresponds to a lattice path in $[m-1]$. Thus exactly one of λ^1, λ^2 has a part equal to m, and the whole result follows by induction on m.

$\square$

Corollary 4.3 *For any nonnegative integers m and t with $t \le \binom{m+1}{2}$,*

$$\sum_{\substack{\lambda \subseteq [m] \\ |\lambda| = t}} g^{\lambda} g^{\lambda^c} = g^{[m]}.$$

Here summation is over all strict partitions λ with the prescribed restrictions, and λ^c is the complement of λ in $[m]$ (where strict partitions are treated as sets). In particular, the LHS is independent of t.

Lemma 4.4 *Let λ and λ^c be strict partitions whose disjoint union (as sets) is $[m]$, and let $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_1 > \ldots > \mu_k > m$. Let $\hat{g}^{\lambda} := g^{\lambda}/|\lambda|!$. Then*

$$\hat{g}^{\mu\cup\lambda} \hat{g}^{\mu\cup\lambda^c} \hat{g}^{[m]} = \hat{g}^{\lambda} \hat{g}^{\lambda^c} \hat{g}^{\mu\cup[m]} \hat{g}^{\mu}.$$

Equivalently,

$$g^{\mu\cup\lambda} g^{\mu\cup\lambda^c} = c(\mu, |\lambda|, |\lambda^c|) \cdot g^{\lambda} g^{\lambda^c},$$

where

$$c(\mu, |\lambda|, |\lambda^c|) = \frac{g^{\mu\cup[m]} g^{\mu}}{g^{[m]}} \cdot \frac{|[m]|!(|\mu| + |\lambda|)!(|\mu| + |\lambda^c|)!}{(|\mu| + |[m]|)!|\mu|!|\lambda|!|\lambda^c|!}$$

depends only on the sizes $|\lambda|$ and $|\lambda^c|$ and not on the actual partitions λ and λ^c.

Proof. By Proposition 2.2,

$$\frac{\hat{g}^{\mu\cup\lambda} \hat{g}^{\mu\cup\lambda^c}}{\hat{g}^{\lambda} \hat{g}^{\lambda^c}} = \left(\prod_i \frac{1}{\mu_i!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2 \cdot \prod_{i,j} \frac{\mu_i - \lambda_j}{\mu_i + \lambda_j} \prod_{i,j} \frac{\mu_i - \lambda^c_j}{\mu_i + \lambda^c_j}. \tag{5}$$

By the assumption on λ and λ^c,

$$\prod_j \frac{\mu_i - \lambda_j}{\mu_i + \lambda_j} \prod_j \frac{\mu_i - \lambda^c_j}{\mu_i + \lambda^c_j} = \prod_{j=1}^{m} \frac{\mu_i - j}{\mu_i + j} \qquad (\forall i).$$

Thus the RHS of (5) is independent of λ and λ^c. Substituting $\lambda = [m]$ (and $\lambda^c = [0]$, the empty partition) yields

$$\frac{\hat{g}^{\mu\cup\lambda} \hat{g}^{\mu\cup\lambda^c}}{\hat{g}^{\lambda} \hat{g}^{\lambda^c}} = \frac{\hat{g}^{\mu\cup[m]} \hat{g}^{\mu}}{\hat{g}^{[m]}},$$

which is the desired identity. The other equivalent formulation follows readily.

□

A technical lemma, which will be used to prove Theorems 4.6 and 5.5, is the following.

Lemma 4.5 *Let t_1, t_2 and N be nonnegative integers. Then*

$$\sum_{i=0}^{N} \binom{t_1 + i}{t_1} \binom{t_2 + N - i}{t_2} = \binom{t_1 + t_2 + N + 1}{t_1 + t_2 + 1}.$$

Proof. This is a classical binomial identity, which follows for example from computation of the coefficients of x^N on both sides of the identity

$$(1-x)^{-(1+t_1)} \cdot (1-x)^{-(1+t_2)} = (1-x)^{-(2+t_1+t_2)}.$$

$\square$

Theorem 4.6 *Let m be a nonnegative integer, denote $M := \binom{m+1}{2}$, and let $\mu = (\mu_1, \ldots, \mu_k)$ be a strict partition with $\mu_1 > \ldots > \mu_k > m$. Then*

$$\sum_{\lambda \subseteq [m]} g^{\mu\cup\lambda} g^{\mu\cup\lambda^c} = g^{\mu\cup[m]} g^{\mu} \cdot \frac{(M+2|\mu|+1)!|\mu|!}{(M+|\mu|)!(2|\mu|+1)!}.$$

Proof. Restrict the summation on the LHS to strict partitions λ of a fixed size $|\lambda| = t$. By Lemma 4.4 and Corollary 4.3,

$$\sum_{\substack{\lambda \subseteq [m] \\ |\lambda|=t}} g^{\mu\cup\lambda} g^{\mu\cup\lambda^c} = c(\mu, t, M-t) \cdot \sum_{\substack{\lambda \subseteq [m] \\ |\lambda|=t}} g^{\lambda} g^{\lambda^c} = c(\mu, t, M-t) \cdot g^{[m]}.$$

Now sum over all t and use the explicit formula for $c(\mu, t, M-t)$ (from Lemma 4.4) together with Lemma 4.5:

$$\begin{aligned}
\sum_{\lambda \subseteq [m]} g^{\mu\cup\lambda} g^{\mu\cup\lambda^c} &= g^{\mu\cup[m]} g^{\mu} \cdot \frac{M!}{(|\mu|+M)!|\mu|!} \cdot \sum_{t=0}^{M} \frac{(|\mu|+t)!(|\mu|+M-t)!}{t!(M-t)!} \\
&= g^{\mu\cup[m]} g^{\mu} \cdot \binom{|\mu|+M}{|\mu|}^{-1} \cdot \sum_{t=0}^{M} \binom{|\mu|+t}{|\mu|} \binom{|\mu|+M-t}{|\mu|} \\
&= g^{\mu\cup[m]} g^{\mu} \cdot \binom{|\mu|+M}{|\mu|}^{-1} \cdot \binom{2|\mu|+M+1}{2|\mu|+1}.
\end{aligned}$$

$\square$

We shall apply this theorem to several special cases. In each case the result will follow from an application of equation (3) to one or the other of the shapes illustrated in diagram (1), where the union of λ and ν is the shifted staircase shape $[m]$. Using Lemma 4.2, we can now state explicitly the relation between these partitions, mentioned before equation (3): $\nu = ([m]/\lambda)' = \lambda^c$.

First, take $\mu = (m+k, \ldots, m+1)$ (k parts), for $k \geq 1$. This corresponds to truncating a $k \times k$ square from the NE corner of a shifted staircase shape $[m+2k]$, but adding back the SW corner of this square; see the first shape in diagram (1).

Corollary 4.7 *The number of SYT of truncated shifted staircase shape $[m+2k] \setminus (k^{k-1}, k-1)$ is*

$$g^{[m+k]} g^{(m+k,\ldots,m+1)} \cdot \frac{N!|\mu|!}{(N-|\mu|-1)!(2|\mu|+1)!},$$

where $N = \binom{m+2k+1}{2} - k^2 + 1$ is the size of the shape and $|\mu| = k(2m+k+1)/2$.

The special case $k = 1$ (with $\mu = (m+1)$) gives back the number $g^{[m+2]}$ of SYT of shifted staircase shape $[m+2]$:

$$g^{[m+1]} \cdot \frac{N!(m+1)!}{(N-m-2)!(2m+3)!} = N! \cdot \prod_{i=0}^{m+1} \frac{i!}{(2i+1)!} = g^{[m+2]},$$

where $N = (m+2)(m+3)/2$ is the size of the shape. This agrees, of course, with Observation 4.1.

The special case $k = 2$ (with $\mu = (m+2, m+1)$) corresponds to truncating a small shifted staircase shape $[2] = (2,1)$ from the shifted staircase shape $[m+4]$. Thus, the number of SYT of truncated shifted staircase shape $[m+4] \setminus [2]$ is

$$\begin{aligned} & g^{[m+2]} g^{(m+2,m+1)} \cdot \frac{N!(2m+3)!}{(N-2m-4)!(4m+7)!} \\ = & \prod_{i=0}^{m+1} \frac{i!}{(2i+1)!} \cdot \frac{(2m+3)!}{(m+2)!(m+1)!(2m+3)} \cdot \frac{N!(2m+3)!}{(4m+7)!} \\ = & N! \cdot \frac{2}{(4m+7)!(m+2)} \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!}, \end{aligned}$$

where $N = \binom{m+5}{2} - 3 = (m+2)(m+7)/2$ is the size of the shape.

Now take $\mu = (m+k+1, \ldots, m+3, m+1)$ (k parts), for $k \geq 2$. This corresponds to truncating a $(k-1) \times (k-1)$ square from the NE corner of a shifted staircase shape $[m+2k]$; see the second shape in diagram (1).

Corollary 4.8 *The number of SYT of truncated shifted staircase shape $[m+2k] \setminus ((k-1)^{k-1})$ is*

$$g^{(m+k+1,\ldots,m+3,m+1,\ldots,1)} g^{(m+k+1,\ldots,m+3,m+1)} \cdot \frac{N!|\mu|!}{(N-|\mu|-1)!(2|\mu|+1)!},$$

where $N = \binom{m+2k+1}{2} - (k-1)^2$ is the size of the shape and $|\mu| = k(2m+k+3)/2 - 1$.

In particular, take $k = 2$ and $\mu = (m+3, m+1)$. This corresponds to truncating the NE corner cell of a shifted staircase shape $[m+4]$. The corresponding enumeration problem was actually the original motivation for the current work, because of its combinatorial interpretation, as explained in [1]. Thus, the number of SYT of truncated shifted staircase shape $[m+4] \setminus (1)$ is

$$\begin{aligned} & g^{(m+3,m+1,\ldots,1)} g^{(m+3,m+1)} \cdot \frac{N!(2m+4)!}{(N-2m-5)!(4m+9)!} \\ = & \frac{N! \cdot 4(2m+3)}{(4m+9)! \cdot (m+3)} \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!}, \end{aligned}$$

where $N = \binom{m+5}{2} - 1 = (m+3)(m+6)/2$ is the size of the shape. For $m = 0$ ($N = 9$) this number is 4, as shown in Example 2.3.

5 Truncated Rectangular Shapes

In this section, $\lambda = (\lambda_1, \ldots, \lambda_m)$ (with $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$ integers) will be a partition with (at most) m parts. Two partitions which differ only in trailing zeros will be considered equal. Denote by f^λ the number of SYT of regular (non-shifted) shape λ.

For any nonnegative integers m and n, let $(n^m) := (n, \ldots, n)$ (m times) be the corresponding rectangular shape. The Frobenius-Young formula (Proposition 2.1) implies the following.

Observation 5.1 *The number of SYT of rectangular shape (n^m) is*

$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}},$$

where

$$F_m := \prod_{i=0}^{m-1} i!.$$

Recall from Section 2 the definition of the sum $\lambda + \mu$ of two partitions λ and μ. Note that if either λ or μ is a strict partition then so is $\lambda + \mu$.

Lemma 5.2 *Let m, n and t be nonnegative integers, with $t \leq mn$. Let T be a SYT of rectangular shape (n^m), let T_1 be the set of all cells in T with values at most t, and let T_2 be obtained from $T \setminus T_1$ by transposing the shape and replacing each entry i by $mn - i + 1$. Then:*

(i) T_1 and T_2 are SYT.

(ii) Denote by λ^1 and λ^2 the shapes of T_1 and T_2, respectively, and treat strict partitions as sets. Then the strict partition $[m+n]$ is the disjoint union of the strict partitions $\lambda^1 + [m]$ and $\lambda^2 + [n]$.

Proof. (i) is clear; let us prove (ii). The borderline between T_1 and $T \setminus T_1$ is a lattice path of length exactly $m+n$, starting at the NE corner of the rectangular shape (n^m), using only S and W steps, and ending at the SW corner of this shape. If the first step is S then the first part of $\lambda^1 + [m]$ is $m+n$, and the rest (of both $\lambda^1 + [m]$ and $\lambda^2 + [n]$) corresponds to a lattice path in n^{m-1}. Similarly, if the first step is W then the first part of $\lambda^2 + [n]$ is $m+n$, and the rest corresponds to a lattice path in $(n-1)^m$. Thus exactly one of the strict partitions $\lambda^1 + [m]$ and $\lambda^2 + [n]$ has a part equal to $m+n$, and the whole result follows by induction on $m+n$.

□

Corollary 5.3 *For any nonnegative integers m, n and t with $t \leq mn$,*

$$\sum_{\substack{\lambda \subseteq (n^m) \\ |\lambda| = t}} f^\lambda f^{\lambda^c} = f^{(n^m)}.$$

Here summation is over all partitions λ with the prescribed restrictions, and λ^c is such that $\lambda^c + [n]$ is the complement of $\lambda + [m]$ in $[m+n]$ (where strict partitions are treated as sets). In particular, the LHS is independent of t.

Lemma 5.4 *Let λ and λ^c be partitions such that $[m+n]$ is the disjoint union of $\lambda + [m]$ and $\lambda^c + [n]$, and let $\mu = (\mu_1, \ldots, \mu_k)$ be an arbitrary partition ($\mu_1 \geq \ldots \geq \mu_k \geq 0$). Let $\hat{f}^\lambda := f^\lambda / |\lambda|!$. Then*

$$\hat{f}^{(\mu+(n^k))\cup\lambda} \hat{f}^{(\mu+(m^k))\cup\lambda^c} = \hat{f}^\lambda \hat{f}^{\lambda^c} \hat{f}^{\mu+(m+n)^k} \hat{f}^\mu.$$

Equivalently,

$$f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} = d(\mu, |\lambda|, |\lambda^c|) \cdot f^\lambda f^{\lambda^c},$$

where

$$d(\mu, |\lambda|, |\lambda^c|) = f^{\mu+((m+n)^k)} f^\mu \cdot \frac{(|\mu| + nk + |\lambda|)!(|\mu| + mk + |\lambda^c|)!}{(|\mu| + (m+n)k)!(|\mu|)!(|\lambda|)!(|\lambda^c|)!}.$$

Proof. From the assumptions it follows that λ is contained in (n^m). We may thus assume that it has m (nonnegative) parts. Similarly, λ^c is contained in (m^n) and we may assume that it has n (nonnegative) parts. Thus $((\mu+(n^k))\cup\lambda$ has $k+m$ parts and $(\mu+(m^k))\cup\lambda^c$ has $k+n$ parts. By Proposition 2.1,

$$\frac{\hat{f}^{(\mu+(n^k))\cup\lambda} \hat{f}^{(\mu+(m^k))\cup\lambda^c}}{\hat{f}^\lambda \hat{f}^{\lambda^c}} =$$
$$\cdot \left(\prod_{i=1}^{k} \frac{1}{(\mu_i + m + n + k - i)!} \prod_{i<j} (\mu_i - \mu_j - i + j) \right)^2 \cdot$$
$$\cdot \prod_{i,j} (\mu_i + n - \lambda_j - i + k + j) \prod_{i,j} (\mu_i + m - \lambda^c_j - i + k + j).$$

By the assumption on λ and λ^c,

$$\prod_j (\mu_i + n - \lambda_j - i + k + j) \prod_j (\mu_i + m - \lambda^c_j - i + k + j) = \prod_{j=1}^{m+n} (\mu_i - i + k + j) \qquad (\forall i).$$

Since

$$\frac{1}{(\mu_i + m + n + k - i)!} \cdot \prod_{j=1}^{m+n} (\mu_i - i + k + j) = \frac{1}{(\mu_i + k - i)!} \qquad (\forall i),$$

an application of Proposition 2.1 to f^μ and to $f^{\mu+((m+n)^k)}$ gives the desired result.

□

Theorem 5.5 *Let m and n be nonnegative integers, and let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition ($\mu_1 \geq \ldots \geq \mu_k \geq 0$). Then*

$$\sum_{\lambda \subseteq (n^m)} f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} = f^{\mu+((m+n)^k)} f^\mu f^{(n^m)} \cdot$$
$$\cdot \binom{mn + 2|\mu| + mk + nk + 1}{mn} \cdot \frac{(|\mu| + mk)!(|\mu| + nk)!}{(|\mu| + mk + nk)!(|\mu|)!}.$$

Proof. Restrict the summation to partitions λ of a fixed size $|\lambda| = t$. By Lemma 5.4 and Corollary 5.3,

$$\sum_{\substack{\lambda\subseteq(n^m)\\|\lambda|=t}} f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} = d(\mu, t, M-t)\cdot \sum_{\substack{\lambda\subseteq(n^m)\\|\lambda|=t}} f^\lambda f^{\lambda^c} = d(\mu, t, M-t)\cdot f^{(n^m)}.$$

Now sum over all t and use the explicit formula for $d(\mu, t, M-t)$ (from Lemma 5.4) together with Lemma 4.5, to obtain the explicit formula above.

□

Again, we shall apply this theorem in several special cases. In each case the result is a special case of equation (4) with appropriately chosen ζ of one or the other of the shapes illustrated in diagram (2). Note that, by Lemma 5.2, $\nu = ((n^m)/\lambda)' = \lambda^c$.

First, let $\alpha = (n^k)$, $\beta = (m^k)$ and $\mu = (0^k)$ (the empty partition with k "parts"). This corresponds to truncating a $k \times k$ square from the NE corner of a rectangular shape $((n+k)^{m+k})$, but adding back the SW corner of this square; see the first shape in diagram (2).

Corollary 5.6 *The number of SYT of truncated rectangular shape* $((n+k)^{m+k})\setminus(k^{k-1}, k-1)$ *is*

$$f^{((m+n)^k)} f^{(n^m)} \cdot \binom{mn+mk+nk+1}{mn} \cdot \frac{(mk)!(nk)!}{(mk+nk)!} = \frac{N!(mk)!(nk)!}{(mk+nk+1)!} \cdot \frac{F_m F_n F_k}{F_{m+n+k}},$$

where $N = (m+k)(n+k) - k^2 + 1 = mn + mk + nk + 1$ *is the size of the shape and* F_n *is as in Observation 5.1.*

For $k = 1$ we obtain

$$f^{(n+1)^{m+1}} = \frac{N!m!n!}{(m+n+1)!} \cdot \frac{F_m F_n}{F_{m+n+1}} = N! \cdot \frac{F_{m+1} F_{n+1}}{F_{m+n+2}},$$

in accordance with Observation 5.1.

For $k = 2$ we obtain that the number of SYT of truncated rectangular shape $((n+2)^{m+2}) \setminus (2, 1)$ is

$$\frac{N!(2m)!(2n)!}{(2m+2n+1)!} \cdot \frac{F_m F_n}{F_{m+n+2}},$$

where $N = (m+2)(n+2) - 3 = mn + 2m + 2n + 1$ is the size of the shape.

Now take $\alpha = (n^k)$, $\beta = (m^k)$ and $\mu = (1^{k-1}, 0)$, for $k \geq 2$. This corresponds to truncating a $(k-1) \times (k-1)$ square from the NE corner of a rectangular shape $((n+k)^{m+k})$; see the second shape in diagram (2).

Corollary 5.7 *The number of SYT of truncated rectangular shape* $((n+k)^{m+k}) \setminus ((k-1)^{k-1})$ *is*

$$f^{((m+n+1)^{k-1}, m+n)} f^{(n^m)} \cdot \binom{mn+mk+nk+2k-1}{mn} \cdot \frac{(mk+k-1)!(nk+k-1)!}{(mk+nk+k-1)!(k-1)!} =$$

$$= \frac{N!(mk+k-1)!(nk+k-1)!(m+n+1)!k}{(mk+nk+2k-1)!} \cdot \frac{F_m F_n F_{k-1}}{F_{m+n+k+1}},$$

where $N = (m+k)(n+k) - (k-1)^2 = mn + mk + nk + 2k - 1$ *is the size of the shape and* F_n *is as in Observation 5.1.*

In particular, letting $k = 2$ and $\mu = (1, 0)$ implies that the number of SYT of truncated rectangular shape $((n+2)^{m+2}) \setminus (1)$ is

$$\frac{N!(2m+1)!(2n+1)! \cdot 2}{(2m+2n+3)!(m+n+2)} \cdot \frac{F_m F_n}{F_{m+n+2}},$$

where $N = (m+2)(n+2) - 1 = mn + 2m + 2n + 3$ is the size of the shape and F_n is as in Observation 5.1.

6 Final Remarks and Open Problems

Most recently, the numbers of SYT of some other truncated shapes have been shown to have nice product formulas. Using complex analysis and volume computations, G. Panova proved such a formula for a rectangular shape minus a staircase in the NE corner [7]. Computer experiments indicate that there are other shapes with a similar property.

The number of SYT of a rectangular shape (n^m) $(n \neq m)$, with *two* cells truncated from the NE corner, seems in general to have large prime factors. Nevertheless, the situation is different for $n = m$.

Conjecture 6.1 *For* $n \geq 2$

$$f^{(n^n)\setminus(2)} = \frac{(n^2-2)!(3n-4)!^2 \cdot 6}{(6n-8)!(2n-2)!(n-2)!^2} \cdot \frac{F_{n-2}^2}{F_{2n-4}},$$

where

$$F_m := \prod_{i=0}^{m-1} i!.$$

This empirical formula has been verified for $n \leq 13$.

On the basis of further computations we conjecture that, if either $\kappa = (k^{k-1})$ or $\kappa = (k^{k-2}, k-1)$ then, for n large enough, both $f^{(n^n)\setminus\kappa}$ and $g^{[n]\setminus\kappa}$ have no prime factor larger than the size of the shape.

Question 6.2

(i) Find and characterize the truncated (and other "unusual") shapes for which the number of SYT has no prime factor larger than the size of the shape.

(ii) Give explicit product formulas for the number of SYT of such shapes.

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