

Minimal overlapping patterns in colored permutations

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Dedicated to Doron Zeilberger on the occasion of his sixtieth birthday

Abstract

A pattern P of length j has the minimal overlapping property if two consecutive occurrences of the pattern can overlap in at most one place, namely, at the end of the first consecutive occurrence of the pattern and at the start of the second consecutive occurrence of the pattern. For patterns P which have the minimal overlapping property, we derive a general formula for the generating function for the number of consecutive occurrences of P in words, permutations and k -colored permutations in terms of the number of maximum packings of P which are patterns of minimal length which has n consecutive occurrences of the pattern P . Our results have as special cases several results which have appeared in the literature. Another consequence of our results is to prove a conjecture of Elizalde that two permutations α and β of size j which have the minimal overlapping property are strongly c -Wilf equivalent if α and β have the same first and last elements.

1 Introduction

For any alphabet A , we let A^* denote the set of all words over A and we let ϵ denote the empty word. Given any word $w = w_1 \dots w_n \in \{0, \dots, k-1\}^*$, we let $\sum w = w_1 + \dots + w_n$, $|w| = n$, $z(w) = \prod_{i=1}^n z_{w_i}$, and $\text{red}(w)$ be the word that results by replacing the i -th smallest integer that appears in w by $i-1$. For example, $\text{red}(13443551) = 01221330$. Given any word $u \in \{0, \dots, k-1\}^j$ such that $\text{red}(u) = u$, we say that a word $w = w_1 \dots w_n \in \{0, \dots, k-1\}^n$ has a u -match starting at position i if $\text{red}(w_i w_{i+1} \dots w_{i+j-1}) = u$. If $u \in \{0, \dots, k-1\}^j$, then we say that $w = w_1 \dots w_n \in \{0, \dots, k-1\}^n$ has an exact u -match starting at position i if $w_i w_{i+1} \dots w_{i+j-1} = u$. Let $u\text{-mch}(w)$ denote the number of u -matches in the word w and $Eu\text{-mch}(w)$ denote the number of exact u -matches in w . For example, if $u = 010$ and $w = 14121010330$, then w has u -matches starting at positions

1, 3, and 5 and has an exact u -match starting at position 5 so that $u\text{-mch}(w) = 3$ and $Eu\text{-mch}(w) = 1$.

Let S_n denote the symmetric group. Given any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, we let $\text{pred}(\sigma)$ be the permutation that results by replacing the i -th smallest integer that appears in the sequence σ by i . For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{pred}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j , we say that a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ has a τ -match at starting position i if $\text{pred}(\sigma_i \dots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ -matches in the permutation σ . We let

$$\begin{aligned} [n]_{p,q} &= p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}, \\ [n]_{p,q}! &= [1]_q [2]_q \cdots [n]_q, \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!} \end{aligned}$$

denote the usual p, q -analogues of n , $n!$, and $\binom{n}{k}$. We shall use the standard conventions that $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. Setting $p = 1$ in $[n]_{p,q}$, $[n]_{p,q}!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ yields $[n]_q$, $[n]_q!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$, respectively. For any permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we let $\text{inv}(\sigma)$ equal the number of $1 \leq i < j \leq n$ such that $\sigma_i > \sigma_j$ and $\text{coinv}(\sigma)$ equal the number of $1 \leq i < j \leq n$ such that $\sigma_i < \sigma_j$.

A k -colored permutation of size n is a pair $(\sigma, u) \in S_n \times \{0, 1, \dots, k-1\}^n$ and can be thought of as an element in the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n . Thus we will let $C_k \wr S_n$ denote the set of all k -colored permutations of size n . Given a pair $(\tau, u) \in C_k \wr S_j$ such that $\text{red}(u) = u$, we say that a k -colored permutation $(\sigma, w) \in C_k \wr S_n$ has a (τ, u) -match starting at position i if $\text{pred}(\sigma_i \dots \sigma_{i+j-1}) = \tau$ and $\text{red}(w_i w_{i+1} \dots w_{i+j-1}) = u$. If $(\tau, u) \in C_k \wr S_j$, we say that a k -colored permutation $(\sigma, w) \in C_k \wr S_n$ has an exact (τ, u) -match starting at position i if $\text{pred}(\sigma_i \dots \sigma_{i+j-1}) = \tau$ and $w_i w_{i+1} \dots w_{i+j-1} = u$. Let $(\tau, u)\text{-mch}((\sigma, w))$ denote the number of (τ, u) -matches in (σ, w) and $(\tau, Eu)\text{-mch}((\sigma, w))$ denote the number of exact (τ, u) -matches in (σ, w) . For example, if $\sigma = 132$ and $u = 010$ and $(\sigma, w) = (25316498, 14121010)$, then (σ, w) has (τ, u) -matches starting at positions 1 and 6 and has an exact (τ, u) -match starting at position 6 so that $(\tau, u)\text{-mch}((\sigma, w)) = 2$ and $(\tau, Eu)\text{-mch}((\sigma, w)) = 1$.

There are a number of papers on exact (τ, u) -pattern matching and exact (τ, u) -pattern avoidance in $C_k \wr S_n$, see [7, 16, 17, 18]. Our notions of (τ, u) -pattern matching was first introduced in [13] where the authors studied (τ, u) -matches for patterns of length 2.

Let u be a word in $\{0, 1, \dots, k-1\}^j$. If $\text{red}(u) = u$, then we say that u has the k -minimal overlapping property if the smallest i such that there exists a $w \in \{0, 1, \dots, k-1\}^i$ with $u\text{-mch}(w) = 2$ is $2j - 1$. This means that in a word $w \in \{0, 1, \dots, k-1\}^*$, two u -matches in w can share at most one letter which must occur at the end of the first u -match and at the start of the second u -match. Similarly, we say that u has the k -exact match minimal overlapping property if the smallest i such that there exists a $w \in \{0, 1, \dots, k-1\}^i$ with $Eu\text{-mch}(w) = 2$ is $2j - 1$. For example, it is easy to see that 010 has the both the k -minimal overlapping property and the k -exact match minimal overlapping property for

all $k \geq 2$. However, $u = 0011$ does not have the k -minimal overlapping property for $k \geq 3$ since $u\text{-mch}(001122) = 2$. Also $u = 0011$ does not have the k -exact match minimal overlapping property since no two exact u -matches can have a letter in common. However, $u = 01020$ has the k -exact match minimal overlapping property for all $k \geq 3$ and the k -minimal overlapping property for $k = 3$, but it does not have the k -minimal overlapping property for $k \geq 4$ since $u\text{-mch}(0102030) = 2$.

We say that a permutation $\tau \in S_j$ where $j \geq 3$ has the *minimal overlapping property* if the smallest i such that there is a permutation $\sigma \in S_i$ with $\tau\text{-mch}(\sigma) = 2$ is $2j - 1$. Again this means that in any permutation $\sigma = \sigma_1 \dots \sigma_n$, any two τ -matches in σ can share at most one letter which must be at the end of the first τ -match and the start of the second τ -match. For example $\tau = 123$ does not have the minimal overlapping property since the $\tau\text{-mch}(1234) = 2$ and the τ -match starting at position 1 and the τ -match starting at position 2 share two letters, namely, 2 and 3. However, it is easy to see that the permutation $\tau = 132$ does have the minimal overlapping property. That is, the fact that there is an ascent starting at position 1 and descent starting at position 2 means that there cannot be two τ -matches in a permutation $\sigma \in S_n$ which share 2 or more letters.

Suppose that we are given $u \in \{0, 1, \dots, k - 1\}^j$ and $\tau \in S_j$. If $\text{red}(u) = u$, then we say that u has the $C_k \wr S_n$ -*minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with $(\tau, u)\text{-mch}(w) = 2$ is $2j - 1$. This means that in a k -colored permutation (σ, w) , two (τ, u) -matches in (σ, w) can share at most one pair of letters which must occur at the end of the first (τ, u) -match and at the start of the second (τ, u) -match. Similarly, we say that (τ, u) has the $C_k \wr S_n$ -*exact match minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with $(\tau, Eu)\text{-mch}((\sigma, w)) = 2$ is $2j - 1$. For example, it is easy to see that $(132, 010)$ has the both the $C_k \wr S_n$ -minimal overlapping property and the $C_k \wr S_n$ -exact match minimal overlapping property for all $k \geq 2$. However, $(\tau, u) = (2143, 0011)$ does not have the $C_k \wr S_n$ -minimal overlapping property for $k \geq 3$ since $(\tau, u)\text{-mch}((214365, 001122)) = 2$. Note $(\tau, u) = (2143, 0011)$ does not have the $C_k \wr S_n$ -exact match minimal overlapping property since no two exact (τ, u) -matches can have a pair of letters in common. Also $(\tau, u) = (12345, 01020)$ has the $C_k \wr S_n$ -exact match minimal overlapping property for all $k \geq 3$ and the $C_k \wr S_n$ -minimal overlapping property for $k = 3$, but it does not have the $C_k \wr S_n$ -minimal overlapping property for $k \geq 4$ since $(\tau, u)\text{-mch}((1234567, 0102030)) = 2$.

The main goal of this paper is to find generating functions for the number of matches of minimal overlapping patterns in words, permutations, and k -colored permutations. To this end, suppose that $u \in \{0, 1, \dots, k - 1\}^j$. If $\text{red}(u) = u$ and u has the k -minimal overlapping property, then the shortest words $w \in \{0, 1, \dots, k - 1\}^*$ such that $u\text{-mch}(w) = n$ have length $j + (n - 1)(j - 1) = n(j - 1) + 1$ so we let $\mathcal{MP}_{u, n(j-1)+1}^k$ denote the set of words $w \in \{0, 1, \dots, k - 1\}^{n(j-1)+1}$ such that $u\text{-mch}(w) = n$. We will refer to elements of $\mathcal{MP}_{u, n(j-1)+1}^k$ as *maximum packings* for u . We let

$$mp_{u, n(j-1)+1}^k = |\mathcal{MP}_{u, n(j-1)+1}^k|,$$

$$mp_{u, n(j-1)+1}^k(r) = \sum_{w \in \mathcal{MP}_{u, n(j-1)+1}^k} r^{\Sigma w}, \text{ and}$$

$$mp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}) = \sum_{w \in \mathcal{MP}_{u,n(j-1)+1}^k} z(w).$$

If u has the k -exact match minimal overlapping property, then the shortest words $w \in \{0, 1, \dots, k-1\}^*$ such that $Eu\text{-mch}(w) = n$ have length $n(j-1)+1$ so we let $\mathcal{EMP}_{u,n(j-1)+1}^k$ denote the set of words $w \in \{0, 1, \dots, k-1\}^{n(j-1)+1}$ such that $Eu\text{-mch}(w) = n$. We will refer to elements of $\mathcal{EMP}_{u,n(j-1)+1}^k$ as *exact match maximum packings* for u . We let

$$\begin{aligned} emp_{u,n(j-1)+1}^k &= |\mathcal{EMP}_{u,n(j-1)+1}^k|, \\ emp_{u,n(j-1)+1}^k(r) &= \sum_{w \in \mathcal{EMP}_{u,n(j-1)+1}^k} r^{\Sigma w}, \text{ and} \\ emp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}) &= \sum_{w \in \mathcal{EMP}_{u,n(j-1)+1}^k} z(w). \end{aligned}$$

For example, suppose that $u = 010$ and $k \geq 2$. Then the only $w \in \{0, 1, \dots, k-1\}^{2n+1}$ such that $Eu\text{-mch}(w) = n$ is $0(10)^n$ so that $emp_{010,2n+1}^k = 1$, $emp_{010,2n+1}^k(r) = r^n$, and $emp_{010,n(j-1)+1}^k(z_0, \dots, z_{k-1}) = z_0^{n+1}z_1^n$ for all $n \geq 1$ and $k \geq 2$. However, if we are just considering u -matches instead of exact u -matches in $\{0, 1, \dots, k-1\}^{2n+1}$, then the only words $w \in \{0, 1, \dots, k-1\}^{2n+1}$ such that $u\text{-mch}(w) = n$ are of the form $si_1si_2j \dots si_ns$ where $s \in \{0, 1, \dots, k-2\}$ and $i_1, \dots, i_n \in \{s+1, \dots, k-1\}$. Thus

$$\begin{aligned} mp_{010,2n+1}^k &= \sum_{s=0}^{k-2} (k-1-s)^n = \sum_{i=1}^{k-1} i^n, \\ mp_{010,2n+1}^k(r) &= \sum_{s=0}^{k-2} r^{(n+1)s} (r^{s+1}[k-1-s]_r)^n = \sum_{i=1}^{k-1} r^{2n(k-i)-n-1} [i]_r^n, \text{ and} \\ mp_{010,2n+1}^k(z_0, \dots, z_{k-1}) &= \sum_{s=0}^{k-2} z_s^{n+1} \left(\sum_{t=s+1}^{k-1} z_t \right)^n. \end{aligned}$$

If $\tau \in S_j$ has the minimal overlapping property, then again the shortest permutations σ such that $\tau\text{-mch}(\sigma) = n$ have length $n(j-1)+1$. Thus we let $\mathcal{MP}_{\tau,n(j-1)+1}$ equal the set of permutations $\sigma \in S_{n(j-1)+1}$ such that $\tau\text{-mch}(\sigma) = n$. Again we refer to the permutations in $\mathcal{MP}_{\tau,n(j-1)+1}$ as *maximum packings* for τ . Then we let $mp_{\tau,n(j-1)+1} = |\mathcal{MP}_{\tau,n(j-1)+1}|$ and

$$mp_{\tau,n(j-1)+1}(p, q) = \sum_{\sigma \in \mathcal{MP}_{\tau,n(j-1)+1}} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}.$$

In general, it is a difficult problem to compute $mp_{\tau,n(j-1)+1}$ or $mp_{\tau,n(j-1)+1}(p, q)$, but we can compute these in the case that τ starts either ends or starts with 1 or ends or starts with j . For example, we shall prove the following theorem.

Theorem 1. *Suppose that $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = s$, then*

$$mp_{\tau,(n+1)(j-1)+1}(p, q) = p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)} p^{(s-1)n(j-1)} \left[\begin{matrix} (n+1)(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q} mp_{\tau,n(j-1)+1}(p, q)$$

so that

$$mp_{\tau,(n+1)(j-1)+1}(p, q) = (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^{n+1} p^{(s-1)(j-1)} \binom{n+1}{2} \prod_{i=1}^{n+1} \left[\begin{matrix} i(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q}. \quad (1)$$

Note that if $\tau = \tau_1 \dots \tau_j \in S_j$ has the minimal overlapping property, then the reverse of τ , $\tau^r = \tau_j \dots \tau_1$, and the complement of τ , $\tau^c = (j+1-\tau_1) \dots (j+1-\tau_j)$, also have the minimal overlapping property. Thus one can use Theorem 1 to compute $mp_{\tau,(n+1)(j-1)+1}(p, q)$ in the case where τ either ends or starts with j or ends with 1.

Now suppose that $u \in \{0, 1, \dots, k-1\}^j$ and $\tau \in S_j$. If $\text{red}(u) = u$ and (τ, u) has the $C_k \wr S_n$ -minimal overlapping property, then the shortest k -colored permutations (σ, w) such that $(\tau, u)\text{-mch}((\sigma, w)) = n$ have length $n(j-1)+1$. Thus we let $\mathcal{MP}_{(\tau,u),n(j-1)+1}^k$ equal the set of k -colored permutations $(\sigma, w) \in C_k \wr S_{n(j-1)+1}$ such that $(\tau, u)\text{-mch}((\sigma, w)) = n$. We refer to the k -colored permutations in $\mathcal{MP}_{(\tau,u),n(j-1)+1}^k$ as maximum packings for (τ, u) . We let

$$\begin{aligned} mp_{(\tau,u),n(j-1)+1}^k &= |\mathcal{MP}_{(\tau,u),n(j-1)+1}^k|, \\ mp_{(\tau,u),n(j-1)+1}^k(p, q, r) &= \sum_{(\sigma,w) \in \mathcal{MP}_{(\tau,u),n(j-1)+1}^k} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} r^{\sum w}, \text{ and} \\ mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) &= \sum_{(\sigma,w) \in \mathcal{MP}_{(\tau,u),n(j-1)+1}^k} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w). \end{aligned}$$

Similarly, we let $\mathcal{EMP}_{(\tau,u),n(j-1)+1}^k$ equal the set of $(\sigma, w) \in C_k \wr S_{n(j-1)+1}$ such that $(\tau, Eu)\text{-mch}((\sigma, w)) = n$. We refer to the k -colored permutations in $\mathcal{EMP}_{(\tau,u),n(j-1)+1}^k$ as exact match maximum packings for (τ, u) . We let

$$\begin{aligned} emp_{(\tau,u),n(j-1)+1}^k &= |\mathcal{EMP}_{(\tau,u),n(j-1)+1}^k|, \\ emp_{(\tau,u),n(j-1)+1}^k(p, q, r) &= \sum_{(\sigma,w) \in \mathcal{EMP}_{(\tau,u),n(j-1)+1}^k} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} r^{\sum w}, \text{ and} \\ emp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) &= \sum_{(\sigma,w) \in \mathcal{EMP}_{(\tau,u),n(j-1)+1}^k} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w). \end{aligned}$$

Then the main goal of this paper is to prove the following. Let $u \in \{0, 1, \dots, k-1\}^j$ and $\tau \in S_j$ where $j \geq 3$.

(I) If $\text{red}(u) = u$ and u has the k -minimal overlapping property, then

$$\sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n} x^{u\text{-mch}(w)} z(w) =$$

$$\frac{1}{1 - ((z_0 + \cdots + z_{k-1})t + \sum_{n \geq 1} t^{n(j-1)+1}(x-1)^n mp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}))}. \quad (2)$$

(II) If u has the k -exact match minimal overlapping property, then

$$\sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n} x^{Eu-\text{mch}(w)} z(w) = \frac{1}{1 - ((z_0 + \cdots + z_{k-1})t + \sum_{n \geq 1} t^{n(j-1)+1}(x-1)^n emp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}))}. \quad (3)$$

(III) If $\text{red}(u) = u$ and (τ, u) has the $C_k \wr S_n$ -minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, u)-\text{mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((z_0 + \cdots + z_{k-1})t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n mp_{(\tau, u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}))}. \quad (4)$$

(IV) If $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$ -exact match minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, Eu)-\text{mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((z_0 + \cdots + z_{k-1})t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n emp_{(\tau, u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}))}. \quad (5)$$

The special case of (3) or (4) when $k = 1$ proves the following result.

(V) If $\tau \in S_j$ has the minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau-\text{mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1}{1 - (t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n mp_{\tau, n(j-1)+1}(p, q))}. \quad (6)$$

We shall prove all of our generating functions by applying an appropriate ring homomorphism, defined on the ring Λ of symmetric functions over infinitely many variables $x_1, x_2, \dots$, to a simple symmetric function identity. There has been a long line of research, [2], [3], [1] [14], [15], [19], [20], [21], [23], [25], [22], which shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on Λ to simple symmetric function identities. For example, the n -th elementary symmetric function, e_n , and the n -th homogeneous symmetric function, h_n , are defined by the generating functions

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t) \quad (7)$$

and

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}. \quad (8)$$

Thus

$$H(t) = 1/E(-t). \quad (9)$$

It is well known that $\{e_0, e_1, \dots\}$ is an algebraically independent set of generators for Λ and hence we can define a ring homomorphism $\xi : \Lambda \rightarrow R$ where R is a ring by simply specifying $\xi(e_n)$ for all $n \geq 0$. We shall prove (1)-(5) by applying appropriate ring homomorphisms to (9).

We shall show that several theorems that have been proved in the literature are special cases of (I)-(V). For example, let

$$A_\tau(t) = \sum_{n \geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \quad (10)$$

and

$$P_\tau(t, x) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)}. \quad (11)$$

Elizalde and Noy [8] proved a number of results about $A_\tau(t)$ and $P_\tau(x, t)$. For example, they showed that

$$\begin{aligned} P_{132}(t, x) &= \frac{1}{1 - \int_0^t e^{\frac{(x-1)z^2}{2}} dz}, \text{ and} \\ P_{1342}(t, x) &= \frac{1}{1 - \int_0^t e^{\frac{(x-1)z^3}{6}} dz}. \end{aligned}$$

Later Kitaev [11] used an inclusion exclusion argument to prove the following result.

Theorem 2. *Let $\tau = 12 \cdots a\sigma(a+1)$, where σ is a permutation of $\{a+2, a+3, \dots, k+1\}$, then*

$$A_\tau(t) = \frac{1}{1 - t + \sum_{i \geq 1} \frac{(-1)^{i+1} t^{ki+1}}{(ki+1)!} \prod_{j=2}^i \binom{jk-a}{k-a}}. \quad (12)$$

Permutations of the form 132, 1342, and $\tau = 12 \cdots a\sigma(a+1)$ have the minimal overlapping property so that p, q -analogues of the results of Elizalde and Noy and Kitaev's result are special cases of our main results.

We shall see that our results have many applications to the consecutive Wilf equivalence problem. That is, given $\alpha, \beta \in S_n$, we say that α is **c-Wilf equivalent** to β if $A_\alpha(t) = A_\beta(t)$. Given permutations $\alpha, \beta \in S_n$, we say that α is **strongly c-Wilf equivalent** to β if $P_\alpha(t, x) = P_\beta(t, x)$. It is easy to see that c-Wilf equivalence classes and strong c-Wilf equivalence classes are closed under complement and reverse.

Elizalde [7] conjectured that if $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j which have the minimal overlapping property and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then α and

β are strongly c-Wilf equivalent. Note that (5) tells us that if α and β are elements of S_j which have the minimal overlapping property and $mp_{\alpha,s(j-1)+1} = mp_{\beta,s(j-1)+1}$ for all $s \geq 1$, then $P_\alpha(x, t) = P_\beta(x, t)$ and, hence, α and β are strongly c-Wilf equivalent. Thus to prove Elizalde's conjecture, we need only show that if $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are elements of S_j which have minimal overlapping property and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then $mp_{\alpha,s(j-1)+1} = mp_{\beta,s(j-1)+1}$ for all $s \geq 1$. We shall prove this fact in Section 3. We note that Elizalde's conjecture has been proved independently by Vladimir Dotsenko and Anton Khoroshkin [4] by a different method.

Our results will also allow us to find generating functions for the distribution for the number of non-overlapping matches of a pattern in permutations, words, and colored permutations. That is, if $u \in \{0, 1, \dots, k-1\}^j$ is such that $\text{red}(u) = u$ and $w \in \{0, 1, \dots, k-1\}^n$, then we let $u\text{-nlap}(w)$ denote the maximum number of non-overlapping u -matches in w where we say that two u -matches in w overlap that they have at least one position in common. Similarly, we let $Eu\text{-nlap}(w)$ denote the maximum number of non-overlapping exact u -matches in w . If $\tau \in S_j$ and $\sigma \in S_n$, let $\tau\text{-nlap}(\sigma)$ denote the maximum number of non-overlapping τ matches in σ . If $(\tau, u) \in C_k \wr S_j$ is such that $\text{red}(u) = u$ and $(\sigma, w) \in C_k \wr S_n$, then we let $(\tau, u)\text{-nlap}((\sigma, w))$ denote the maximum number of non-overlapping (τ, u) -matches in (σ, w) . Similarly, if $(\tau, u) \in C_k \wr S_j$, then we let $(\tau, Eu)\text{-nlap}((\sigma, w))$ denote the maximum number of non-overlapping exact (τ, u) -matches in (σ, w) .

Kitaev [10, 11] showed that if one can compute $A_\tau(t)$, then one can automatically compute the exponential generating function for the distribution of $\tau\text{-nlap}(\sigma)$. That is, Kitaev [10, 11] proved that

Theorem 3.

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\tau)\sigma} = \frac{A_\tau(t)}{(1-x) + x(1-t)A_\tau(t)} \quad (13)$$

where $A_\tau(t) = \sum_{n \geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\tau)\sigma = 0\}|$.

Mendes and Remmel [19] proved a q -analogue of Theorem 3. That is, they proved that

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} q^{\text{inv}(\sigma)} = \frac{A_\tau(q, t)}{(1-x) + x(1-t)A_\tau(q, t)} \quad (14)$$

where $A_\tau(q, t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n, \tau\text{-mch}(\sigma)=0} q^{\text{inv}(\sigma)}$. Kitaev and Mansour [12] proved an analogue of Theorem 3 for words. That is, they proved that if $u \in \{0, 1, \dots, k-1\}^*$, then

$$\sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n} x^{Eu\text{-nlap}(w)} = \frac{A_u(t)}{1 - x((kt-1)A_u(t))} \quad (15)$$

where $A_u(t) = \sum_{n \geq 0} t^n |\{w \in \{0, 1, \dots, k-1\}^n : Eu\text{-mch}(w) = 0\}|$. One can easily modify Kitaev and Mansour's proof to prove the following refinement of their results. That is, if

$$A_u(t, z_0, \dots, z_{k-1}) = \sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n, u\text{-mch}(w)=0} z(w),$$

$$\begin{aligned}
EA_u(t, z_0, \dots, z_{k-1}) &= \sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n, Eu\text{-mch}(w)=0} z(w), \\
N_u(t, x, z_0, \dots, z_{k-1}) &= \sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n} x^{u\text{-nlap}((\sigma,w))} z(w), \text{ and} \\
EN_u(t, x, z_0, \dots, z_{k-1}) &= \sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n} x^{Eu\text{-nlap}((\sigma,w))} z(w),
\end{aligned}$$

then

$$\begin{aligned}
N_u(t, x, z_0, \dots, z_{k-1}) &= \frac{A_u(t, z_0, \dots, z_{k-1})}{1 - x(1 + ((z_0 + \dots + z_{k-1})t - 1)A_u(t, z_0, \dots, z_{k-1}))} \text{ and} \\
EN_u(t, x, z_0, \dots, z_{k-1}) &= \frac{EA_u(t, z_0, \dots, z_{k-1})}{1 - x(1 + ((z_0 + \dots + z_{k-1})t - 1)EA_u(t, z_0, \dots, z_{k-1}))}.
\end{aligned}$$

Kitaev, Niedermaier, Remmel and Reihl [13] proved an analogue of Kitaev's theorem for k -colored permutations. That is, let

$$N_{(\tau,u)}(x, r, q, t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{(\sigma,w) \in C_k \wr S_n} q^{\text{inv}(\sigma)} r^{\sum w} x^{(\tau,u)\text{-nlap}((\sigma,w))}, \quad (16)$$

and

$$A_{(\tau,u)}(r, q, t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{(\sigma,w) \in C_k \wr S_n, (\tau,u)\text{-mch}((\sigma,w))=0} q^{\text{inv}(\sigma)} r^{\sum w}. \quad (17)$$

Then they showed that

$$N_{(\tau,u)}(x, r, q, t) = \frac{A_{(\tau,u)}(r, q, t)}{1 - x(1 + ([k]_r t - 1)A_{(\tau,u)}(r, q, t))}. \quad (18)$$

One can easily modify the proofs in [13] to obtain the following refinements of their results:

$$\begin{aligned}
N_{(\tau,u)}(t, x, p, q, z_0, \dots, z_{k-1}) &= \\
&= \frac{A_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1})}{1 - x(1 + ((z_0 + \dots + z_{k-1})t - 1)A_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1}))}
\end{aligned}$$

and

$$\begin{aligned}
EN_{(\tau,u)}(t, x, p, q, z_0, \dots, z_{k-1}) &= \\
&= \frac{A_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1})}{1 - x(1 + ((z_0 + \dots + z_{k-1})t - 1)EA_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1}))}
\end{aligned}$$

where

$$A_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1}) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n, (\tau,u)\text{-mch}((\sigma,w))=0} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w),$$

$$EA_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1}) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n, (\tau, Eu)\text{-mch}((\sigma,w))=0} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w),$$

$$N_{(\tau,u)}(t, x, p, q, z_0, \dots, z_{k-1}) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} x^{(\tau, u)\text{-nlap}((\sigma,w))} z(w),$$

and

$$EN_{(\tau,u)}(t, p, q, z_0, \dots, z_{k-1}) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} x^{(\tau, Eu)\text{-nlap}((\sigma,w))} z(w).$$

The outline of this paper is as follows. In section 2, we shall give the necessary background on symmetric functions and brick tabloids that we shall need for our results. In section 3, we shall prove results (I)-(V) listed above. We shall also show that we can obtain the generating functions for the number of consecutive occurrences in words and k -colored permutations of patterns which do not allow any overlaps as special cases of our proofs of (I)-(V). In section 4, we shall compute the number of maximum packings for various words, permutations, and k -colored permutations and show how Elizalde's conjecture follows from our results. Finally, in section 5, we shall briefly discuss some extensions of our results.

2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs of the generating functions (I)-(IV).

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \dots$ with coefficients in the field of complex numbers $\mathbb{C}$.

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an integer partition, that is, λ is a finite sequence of weakly increasing nonnegative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in λ . If the sum of these integers is n , we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ or, equivalently, that $\{e_0, e_1, \dots\}$ is an algebraically independent set of generators for Λ . Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ is also a basis for Λ . Since $\{e_0, e_1, \dots\}$ is an algebraically independent set of generators for Λ , we can specify a ring homomorphism θ on Λ by simply defining $\theta(e_n)$ for all $n \geq 0$.

A *brick tabloid* of shape (n) and type $\lambda = (\lambda_1, \dots, \lambda_k)$ is a filling of a row of n squares of cells with bricks of lengths $\lambda_1, \dots, \lambda_k$ such that bricks do not overlap. One brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$ is displayed below.

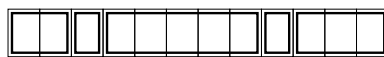


Figure 1: A brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$.

Let $\mathcal{B}_{\lambda,n}$ denote the set of all λ -brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Egecioglu and Remmel proved in [5] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_{\lambda}. \quad (19)$$

We end this section with a lemma from [22] that will be needed in later sections. Fix a brick tabloid $T = (b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}$. Let $IF(T)$ denote the set of all fillings of the cells of $T = (b_1, \dots, b_{\ell(\mu)})$ with the numbers $1, \dots, n$ so that the numbers increase within each brick reading from left to right. We then think of each such filling as a permutation of S_n by reading the numbers from left to right. For example, Figure 2 pictures an element of $IF(3, 6, 3)$ whose corresponding permutation is 4 6 12 1 5 7 8 10 11 2 3 9.

4	6	12	1	5	7	8	10	11	2	3	9
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Figure 2: An element of $IF(3, 6, 3)$.

Then the following lemma which is proved in [22] gives a combinatorial interpretation to $p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\mu)} \end{matrix} \right]_{p,q}$.

Lemma 4. *If $T = (b_1, \dots, b_{\ell(\mu)})$ is a brick tabloid in $\mathcal{B}_{\mu,n}$, then*

$$p^{\sum_i \binom{b_i}{2}} \left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\mu)} \end{matrix} \right]_{p,q} = \sum_{\sigma \in IF(T)} q^{inv(\sigma)} p^{coinv(\sigma)}.$$

3 Generating functions for minimal overlapping patterns

The main goal of this section is to prove results (I)-(V) described in the introduction. We shall start with results (I) and (II) since their proofs will illustrate the general method.

Theorem 5. *Let $u \in \{0, 1, \dots, k-1\}^j$ where $j \geq 3$.*

(I) *If u has the k -minimal overlapping property and $\text{red}(u) = u$, then*

$$\sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n} x^{u-\text{mch}(w)} z(w) = \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \sum_{n \geq 1} t^{n(j-1)+1} (x-1)^n m p_{u, n(j-1)+1}^k(z_0, \dots, z_{k-1}))}. \quad (20)$$

(II) *If u has the k -exact match minimal overlapping property, then*

$$\frac{\sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n} x^{Eu-\text{mch}(w)} z(w)}{1 - ((z_0 + \dots + z_{k-1})t + \sum_{n \geq 1} t^{n(j-1)+1} (x-1)^n \text{emp}_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}))}. \quad (21)$$

Proof. Suppose that $u \in \{0, \dots, k-1\}^j$ has the minimal overlapping property and $\text{red}(u) = u$. We define a ring homomorphism Θ on Λ by letting

- (1) $\Theta(e_0) = 1$,
- (2) $\Theta(e_1) = z_0 + \dots + z_{k-1}$,
- (3) $\Theta(e_{s(j-1)+1}) = (-1)^{s(j-1)}(x-1)^s \text{mp}_{u,s(j-1)+1}^k(z_0, \dots, z_{k-1})$ for all $s \geq 1$, and
- (4) $\Theta(e_n) = 0$ if $n \notin \{1\} \cup \{s(j-1)+1 : s \geq 1\}$.

Note that if $\Theta(e_n) \neq 0$ and $n \geq 1$, then the sign associated with $\Theta(e_n)$ is just $(-1)^{n-1}$.

We claim that for all $n \geq 1$,

$$\Theta(h_n) = \sum_{w \in \{0,1,\dots,k-1\}^n} x^{u-\text{mch}(w)} z(w). \quad (22)$$

That is, by (19), we have that

$$\Theta(h_n) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,n} \Theta(e_\mu). \quad (23)$$

Now if μ is not a partition whose parts come from $\{1\} \cup \{s(j-1)+1 : s \geq 1\}$, then $\Theta(e_\mu) = 0$. Thus let $P_{j,n}$ denote the set of all partitions of n whose parts come from $\{1\} \cup \{s(j-1)+1 : s \geq 1\}$. For any statement A , let $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. It follows that

$$\begin{aligned} \Theta(h_n) &= \sum_{\mu \in P_{j,n}} (-1)^{n-\ell(\mu)} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \Theta(e_{b_i}) = \\ &= \sum_{\mu \in P_{j,n}} (-1)^{n-\ell(\mu)} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} (-1)^{b_i-1} ((z_0 + \dots + z_{k-1}) \chi(b_i = 1) + \\ &\quad (x-1)^{(b_i-1)/(j-1)} \text{mp}_{u,b_i}^k(z_0, \dots, z_{k-1}) \chi(b_i > 1)) = \\ &= \sum_{\mu \in P_{j,n}} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} ((z_0 + \dots + z_{k-1}) \chi(b_i = 1) + \\ &\quad (x-1)^{(b_i-1)/(j-1)} \text{mp}_{u,b_i}^k(z_0, \dots, z_{k-1}) \chi(b_i > 1)). \quad (24) \end{aligned}$$

Next we want to give a combinatorial interpretation to the right hand side of (24). Suppose that we have a brick tabloid $B = (b_1, \dots, b_\ell)$ of size n such that $b_i \in \{1\} \cup \{s(j-1)+1 :$

$s \geq 1\}$ for all i . Then if $b_i = 1$, we shall interpret $z_0 + \cdots + z_{k-1}$ as allowing us to fill b_i with any letter from $\{0, 1, \dots, k-1\}$. If $b_i = s(j-1) + 1 > 1$, then we shall interpret the term $(x-1)^{(b_i-1)/(j-1)} mp_{u,b_i}^k(z_0, \dots, z_{k-1}) = (x-1)^s mp_{u,s(j-1)+1}^k(z_0, \dots, z_{k-1})$ as the number of ways of filling b_i with a word $v_i \in \mathcal{MP}_{u,s(j-1)+1}^k$ and then labeling each cell in b_i which is the start of a u -match in v_i with either x or -1 . Let $\mathcal{O}_{u,n}$ denote the set of all labeled brick tabloids that can be constructed in this way. Thus an $O \in \mathcal{O}_{u,n}$ will consist of a triple $T = (B, w, L)$ where

1. $B = (b_1, \dots, b_\ell)$ is a brick tabloid of shape (n) such that $b_i \in \{1\} \cup \{s(j-1) + 1 : s \geq 1\}$ for all i ,
2. $w = w_1 \dots w_n \in \{0, 1, \dots, k-1\}^n$ is a word such that w_i lies in i -th cell of B for $i = 1, \dots, n$,
3. if $b_i = 1$, then the letter in the cell corresponding to b_i can be any letter from $\{0, 1, \dots, k-1\}$, and
4. if $b_i = s(j-1) + 1 > 1$, then the cells of b_i are filled with a word v_i which is a maximum packing for u of size $s(j-1) + 1$ and each cell of b_i which corresponds to the start of a u -match in v_i is labeled with either -1 or x .

We then define the weight of T , $wt(T)$, to be $z(w)$ times the product of the x labels in T and the sign of T , $sign(T)$, to be the product of the -1 labels in T . For example, if $k = 5$ and $u = 010$, then in Figure 3, we have pictured four elements of $\mathcal{O}_{010,17}$, $T_i = (B^{(i)}, w^{(i)}, L^{(i)})$ for $i = 1, \dots, 4$. Then $wt(T^{(1)}) = xz_0^3z_1^5z_2^2z_3^4z_4^3$, $sign(T^{(1)}) = 1$, $wt(T^{(2)}) = x^2z_0^4z_1^4z_2^2z_3^3z_4^4$, $sign(T^{(2)}) = 1$, $wt(T^{(3)}) = x^2z_0^3z_1^5z_2^2z_3^4z_4^3$, $sign(T^{(3)}) = 1$, $wt(T^{(4)}) = x^2z_0^3z_1^5z_2^2z_3^4z_4^3$, and $sign(T^{(4)}) = -1$. It follows that

$$\Theta(h_n) = \sum_{T \in \mathcal{O}_{u,n}} sign(T)wt(T). \quad (25)$$

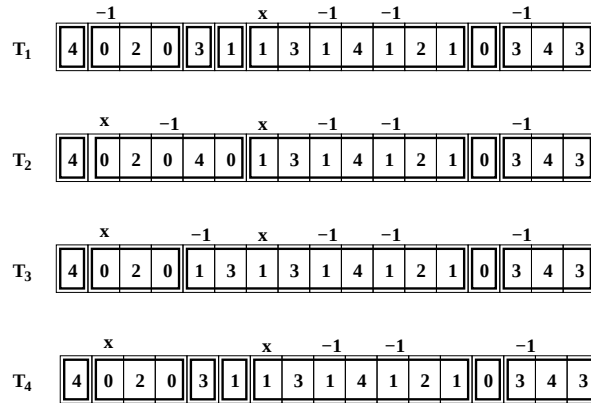


Figure 3: Elements of $\mathcal{O}_{010,17}$.

Next we define a sign-reversing weight-preserving involution $I : \mathcal{O}_{u,n} \rightarrow \mathcal{O}_{u,n}$. If $T = (B, w, L)$, then to define $I(T)$, we scan the cells of $B = (b_1, \dots, b_\ell)$ from left to right looking for the first time where one of the following cases hold.

Case 1. There is a brick b_i of size j whose first cell is labeled with -1 . In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the brick b_i by j bricks of size 1, $w^* = w$, and L^* arises from L by removing the label -1 from the first cell of b_i .

Case 2. There are j consecutive bricks of size 1 in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that the letters in these cells form a u -match. In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size j , $w^* = w$, and L^* arises from L by labeling the first cell of b with a -1 .

Case 3. There is a brick b_i of size $(s+1)(j-1)+1$ where $s \geq 1$ such that all the labels on b_i are x 's except for the cell that is j cells from the right which is labeled with -1 . In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the brick b_i by a brick of size $s(j-1)+1$ followed by $j-1$ bricks of size 1, $w^* = w$, and L^* arises from L by removing the -1 label that was in b_i .

Case 4. There are j consecutive bricks in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i = s(j-1)+1 > 1$ and $b_{i+1}, \dots, b_{i+j-1}$ are of size 1, all the labels on b_i are x 's, and the letters in these bricks form a maximum packing for u of size $(s+1)(j-1)+1$. In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size $(s+1)(j-1)+1$, $w^* = w$, and L^* arises from L by labeling the last cell of b_i with a -1 .

Case 5. There is a brick b_i of size $(s+1)(j-1)+1$ where $s \geq 1$ such that the first cell of b_i is labeled with -1 . In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the brick b_i by $j-1$ bricks of size 1 followed by a brick of size $s(j-1)+1$, $w^* = w$, and L^* arises from L by removing the -1 label that was on the first cell of b_i .

Case 6. There are j consecutive bricks in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i, \dots, b_{i+j-2}$ are bricks of size 1 and $b_{i+j-1} = s(j-1)+1 > 1$ and the letters in these bricks form a maximum packing for u of size $(s+1)(j-1)+1$. In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size $(s+1)(j-1)+1$, $w^* = w$, and L^* arises from L by adding a -1 label on the first cell of b .

Case 7. There is a brick b_i of size $s(j-1)+1$ where $s \geq 3$ such that the first cell is labeled with an x and there is a cell which has a label -1 which is not the j -th cell from the right. Let t be the left-most cell of b_i which is labeled with -1 . Then in this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the brick b_i by j consecutive bricks $c_1, c_2, \dots, c_{j-1}, c_j$ where c_1 contains all the cells of b_i up to and including cell t , $c_2, \dots, c_{j-1}$ are bricks of size 1, and c_j contains the remaining cells of b_i , $w = w^*$, and L^* is the labeling that results from L by removing the -1 label from cell t .

Case 8. There are j consecutive bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i = c(s-1) + 1 > 1$ and there are no -1 labels in $b_i, b_{i+1}, \dots, b_{i+j-2}$ are bricks of size 1, $b_{i+j-1} = d(j-1) + 1 > 1$, and the letters in these three bricks form a maximum packing for u of size $(c+d+1)(j-1) + 1$. In this case, $I(T) = (B^*, w^*, L^*)$ where B^* results from B by replacing the j bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b , $w^* = w$, and L^* results from L by adding a label -1 on the last cell of b_i .

If none of Cases 1-8 apply, then we set $I(T) = T$.

For example, consider images of $T_1, \dots, T_4$ pictured in Figure 3. It is easy to see that T_1 is in Case 1 so that $I(T)$ results by replacing the second brick by three bricks of size 1 and removing the -1 label. This results in $I(T_1)$ pictured in the first row of Figure 4. It is then easy to see that $I(T_1)$ is in Case 2 so that $I^2(T_1) = T_1$. T_2 is in Case 3 where b_i is the second brick. Thus we obtain $I(T_2)$ by replacing b_2 with a brick of size 3 followed by two bricks of size 1 and removing the -1 label from cell 4. $I(T_2)$ is pictured in the second row of Figure 4. It is then easy to see that $I(T_2)$ will be in Case 4 so that $I^2(T_2) = T_2$. T_3 is in Case 5 where b_i is the third brick. Thus we obtain $I(T_3)$ by replacing b_3 by two bricks of size 1 followed by a brick of size 7 and removing the -1 label on cell 5. $I(T_3)$ is pictured in the third row of Figure 4. It is then easy to see that $I(T_3)$ will be in Case 6 so that $I^2(T_3) = T_3$. Finally, T_4 is in Case 7 with $t = 9$ so that we replace the fifth brick by three consecutive bricks of sizes 3, 1, and 3, reading from left to right, and remove the -1 label for cell 9. $I(T_4)$ is pictured in the forth row of Figure 4. It is then easy to see that $I(T_4)$ will be in Case 8 so that $I^2(T_4) = T_4$.

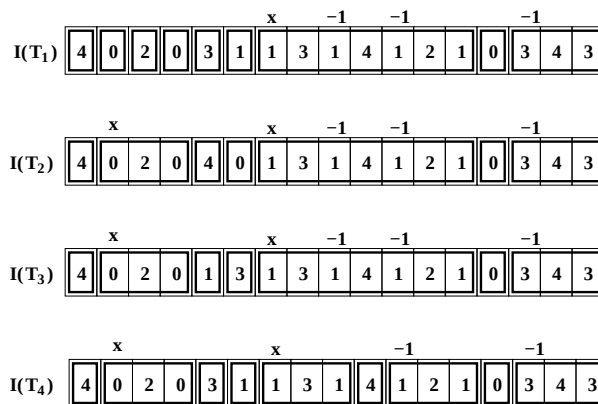


Figure 4: The images under I of the elements in Figure 3.

First we claim that I is an involution. For this we have to do a case by case analysis. Fix some $T = (B, w, L) \in \mathcal{O}_{u,n}$ such that $I(T) \neq T$. Let $B = (b_1, \dots, b_\ell)$. Note that in every case, $I(T)$ is defined by changing the brick structure on some cells $s, s+1, \dots, s+j-1$ where $w_s w_{s+1} \dots w_{s+j-1}$ is a u -match in w .

First suppose that $I(T)$ was defined using Case 1 and that the brick of size j that was used in the definition of $I(T)$ is b_i and b_i covers cells $t, t+1, \dots, t+j-1$. Then

in $I(T)$, we have the possibility to recombine the bricks of size 1 that now cover cells $t, t+1, \dots, t+j-1$. Thus if $I^2(T) \neq T$, then it must be the case that we took some action which involved a u -match $w_s w_{s+1} \dots w_{s+j-1}$ where $s < t$. Now it cannot be that $s+j-1 < t$ since otherwise we could have taken the same action by changing the brick structure on cells $s, s+1, \dots, s+j-1$ in T which would violate the fact that we always take an action on the left-most possible cells that we can when defining $I(T)$. Because u has the minimal overlapping property, the only other possibility is that $t = s+j-1$. Now if $s > 1$, then the minimal overlapping property for u implies that $w_{s-1} w_s \dots w_{s+j-2}$ is not a u -match and hence the cells $s-1, s, \dots, s+j-2$ cannot lie in a single brick since the last j -cells in any brick b of size greater than 1 must correspond to a u -match in w . Thus it must be that in T , cells $s+1, \dots, s+j-2$ must be covered by bricks of size 1. If cell s is also covered by a brick of size 1, then we could apply Case 6 to T using the $j-1$ bricks of size 1 covering cells $s, \dots, s+j-2$ plus b_i which would contradict the fact that for T , we are in Case 1 using brick b_i . If cell s is part of brick b of size > 1 , then we could apply Case 8 to T using b plus the $j-2$ bricks of size 1 covering cells $s+1, \dots, s+j-2$ plus b_i which again would contradict the fact that for T , we are in Case 1 using brick b_i . If $s = 1$, then cells $s, \dots, s+j-2$ must be covered by bricks of size 1 so that we could apply Case 6 to T using the $j-1$ bricks of size 1 covering cells $s, \dots, s+j-2$ plus b_i which would contradict the fact that for T , we are in Case 1 using brick b_i . Thus the left-most u -match that we can use to define the image of I for $I(T)$ is the u -match that lies in the j bricks of size 1 covering cells $t, t+1, \dots, t+j-1$ in which case we know that $I^2(T) = T$. An entirely similar analysis will show that if $I(T)$ is defined using Case 2, then $I^2(T) = T$.

Next suppose that $I(T)$ was defined by Case 3 using a brick b_i of size $a(j-1)+1$ where $a \geq 2$ and b_i covers cells $t, t+1, \dots, t+a(j-1)$. Then in $I(T)$, there is a single brick b covering cells $t, t+1, \dots, t+(a-1)(j-1)$ followed by $j-1$ bricks of size 1 covering cells $t+(a-1)(j-1)+1, \dots, t+a(j-1)$ and all the labels on b are x 's. In this case, if $I^2(T) \neq T$, then it must be the case that we took some action which involved a u -match $w_s w_{s+1} \dots w_{s+j-1}$ where $s < t$. But then we could have taken some action by changing the brick structure on cells $s, s+1, \dots, s+j-1$ in T which would violate the fact that we always take an action on the left-most possible cells that we can when defining $I(T)$. Thus it must be the case that the left-most action that we can take to define I on $I(T)$ is to combine b with the $j-1$ bricks of size 1 that follow b and hence $I^2(T) = T$. A similar analysis will show that if $I(T)$ was defined using Case 4, then $I^2(T) = T$.

Next suppose that $I(T)$ was defined by Case 5 using brick $b_i = a(j-1)+1$ where $a \geq 2$. The analysis in this case is essentially the same as the analysis of Case 1. That is, suppose that b_i covers cells $t, t+1, \dots, t+a(j-1)$. We are assuming that cell t is labeled with -1 . Then in $I(T)$, the first $j-1$ cells of b_i will be covered with bricks of size 1 and the remaining cells of b_i with a single brick b . Thus if $I^2(T) \neq T$, then it must be the case that we took some action which involved a u -match $w_s w_{s+1} \dots w_{s+j-1}$ where $s < t$. Now it cannot be that $s+j-1 < t$ since otherwise we could have taken the same action by changing the brick structure on cells $s, s+1, \dots, s+j-1$ in T which would violate the fact that we always take an action on the left-most possible cells that we can when

defining $I(T)$. Because u has the minimal overlapping property, the only other possibility is that $t = s + j - 1$. Now if $s > 1$, then the minimal overlapping property for u implies that $w_{s-1}w_s \dots w_{s+j-2}$ is not a u -match and hence the cells $s - 1, s, \dots, s + j - 2$ cannot lie in a single brick since the last j -cells in any brick b of size greater than 1 must correspond to a u -match in w . Thus it must be that in T , cells $s + 1, \dots, s + j - 2$ must be covered by bricks of size 1. If cell s is also covered by a brick of size 1, then we could apply Case 6 to T using the $j - 1$ bricks of size 1 covering cells $s, \dots, s + j - 2$ plus b_i which would contradict the fact that for T , we are in Case 5 using brick b_i . If cell s is part of brick b of size > 1 , then we could apply Case 8 to T using b plus the $j - 2$ bricks of size 1 covering cells $s + 1, \dots, s + j - 2$ plus b_i which again would contradict the fact that for T , we are in Case 5 using brick b_i . If $s = 1$, then cells $s, \dots, s + j - 2$ must be covered by bricks of size 1 so that we could apply Case 6 to T using the $j - 1$ bricks of size 1 covering cells $s, \dots, s + j - 2$ plus b_i which would contradict the fact that for T , we are in Case 5 using brick b_i . Thus the left-most u -match that we can use to define the image of I for $I(T)$ is the u -match that lies $j - 1$ bricks of size 1 covering cells $t, t + 1, \dots, t + j - 1$ plus the brick b in which case we know that $I^2(T) = T$. An entirely similar analysis will show that if $I(T)$ is defined using Case 6, then $I^2(T) = T$.

Finally suppose that $I(T)$ was defined using Case 7 using a brick b_i of size $a(j - 1) + 1$ where $a \geq 3$. Suppose that b_i covers cells $t, t + 1, \dots, t + a(j - 1)$. We are assuming that cell t has label x and that the left-most cell of b_i which is labeled with -1 occurs on cell $t + b(j - 1)$ where $1 \leq b < a - 1$. Then in $I(T)$ there is a single brick b^* covering cells $t, t + 1, \dots, t + b(j - 1)$ followed by $j - 2$ bricks of size 1 covering cells $t + b(j - 1) + 1, \dots, t + b(j - 1) + j - 2$ followed by a brick b^{**} covering the remaining cells of b_i . Moreover all the labels on b^* are x 's. In this case, if $I^2(T) \neq T$, then it must be the case that we took some action which involved a u -match $w_s w_{s+1} \dots w_{s+j-1}$ where $s < t$. But then we could have taken some action by changing the brick structure on cells $s, s + 1, \dots, s + j - 1$ in T which would violate the fact that we always take an action on the left-most possible cells that we can when defining $I(T)$. Thus it must be the case that the left-most action that we can take to define is to recombine b^* plus the following $j - 2$ bricks of size 1 plus b^{**} into a single brick so that $I^2(T) = T$. An entirely similar analysis will show that if $I(T)$ is defined using Case 8, then $I^2(T) = T$.

It is easy to see that if $I(T) \neq T$, then $\text{sign}(T)wt(T) = -\text{sign}(I(T))wt(I(T))$. Hence I shows that

$$\Theta(h_n) = \sum_{T \in \mathcal{O}_{u,n}, I(T)=T} \text{sign}(T)wt(T). \quad (26)$$

Thus we must examine the fixed points of I . Suppose that $T = (B, w, L)$ is a fixed point of I where $B = (b_1, \dots, b_\ell)$. There cannot be any -1 labels on any of the bricks in B since otherwise we could apply one of Cases 1, 3, 5, or 7. Thus if $I(T) = T$, $\text{sign}(T) = 1$. It follows that $wt(T) = x^c z(w)$ where c is the number of u -matches in w that lie entirely within some brick b_i in B . We claim that any u -match in w must lie entirely within some brick. That is, suppose that $w = w_1 \dots w_n$ and $w_s w_{s+1} \dots w_{s+j-1}$ a u -match that does not lie in a single brick. Because u has the minimal overlapping property, there are only four possibilities, namely,

(i) cells $s, s+1, \dots, s+j-1$ are covered by bricks of size 1,
(ii) cell s is part of brick b_i of size > 1 and cells $s+1, \dots, s+j-1$ are covered by bricks of size 1,
(iii) cell $s+j-1$ is part of brick b_i of size > 1 and cells $s, \dots, s+j-2$ are covered by bricks of size 1, or
(iv) cell s is part of a brick b_i of size > 1 $b_{i+1}, \dots, b_{i+j-2}$ are bricks of size 1 covering cells $s+1, \dots, s+j-2$, and cell $s+j-1$ is part of brick b_{i+j-1} which is of size > 1 .
In case (i), we could apply Case 2 of the definition of I to cells $s, s+1, \dots, s+j-1$. In case (ii), we could apply Case 4 of the definition of I to cells of b_i plus cells $s+1, \dots, s+j-1$. In case (iii), we could apply Case 6 of the definition of I to cells $s, \dots, s+j-2$ plus the cells of b_i . In case (iv), we can apply Case 8 of the definition of I to the cells of contained in the bricks $b_i, \dots, b_{i+j-1}$. Thus in all the cases (i)-(iv), it would be the case that $I(T) \neq T$ which contradicts our choice of T . Thus we have shown that if $I(T) = T$, then $\text{sign}(T)wt(T) = x^{u\text{-mch}(w)}z(w)$. Finally note that if $w \in \{0, \dots, k-1\}^n$, then we can construct a fixed point of I by placing bricks which cover the maximal length maximum packings in w , covering the remaining cells by bricks of size 1, and labeling the start of each u -match in w by x . It thus follows that

$$\Theta(h_n) = \sum_{w \in \{0, \dots, k-1\}^n} x^{u\text{-mch}(w)} z(w).$$

But then

$$\begin{aligned} \Theta(H(t)) &= \sum_{n \geq 0} t^n \sum_{w \in \{0, \dots, k-1\}^n} x^{u\text{-mch}(w)} z(w) = \Theta(1/E(-t)) = \\ &= \frac{1}{1 + \sum_{n \geq 1} (-t)^n \Theta(e_n)} = \\ &= \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \sum_{n \geq 1} t^{n(j-1)+1} (x-1)^n mp_{u, n(j-1)+1}^k(z_0, \dots, z_{k-1}))} \end{aligned}$$

which proves (20).

If we replace u -matchings by exact u -matching throughout the proof of (20), then we will get a proof of (21). $\square$

Next we prove our main results for k -colored permutations.

Theorem 6. (III) *If $(\tau, u) \in C_k \wr S_j$, $\text{red}(u) = u$, and (τ, u) has the $C_k \wr S_n$ -minimal overlapping property, then*

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, u)\text{-mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \\ \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}} (x-1)^n mp_{(\tau, u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}))}. \end{aligned} \quad (27)$$

(IV) If $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$ -exact match minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, Eu)\text{-mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \quad (28)$$

$$\frac{1}{1 - ((z_0 + \cdots + z_{k-1})t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_q!} (x-1)^n \text{emp}_{(\tau, u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}))}.$$

Proof. We shall see that the proof of Theorem 6 is very similar to the proof of Theorem 5. Suppose that $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$ -minimal overlapping property and $\text{red}(u) = u$. We define a ring homomorphism Γ on Λ by letting

- (1) $\Gamma(e_0) = 1$,
- (2) $\Gamma(e_1) = z_0 + \cdots + z_{k-1}$,
- (3) $\Gamma(e_{s(j-1)+1}) = (-1)^{s(j-1)}(x-1)^s \frac{1}{[s(j-1)+1]_{p,q}!} \text{mp}_{(\tau, u), s(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$ for all $s \geq 1$, and
- (4) $\Gamma(e_n) = 0$ if $n \notin \{1\} \cup \{s(j-1)+1 : s \geq 1\}$.

Note that if $\Gamma(e_n) \neq 0$ and $n \geq 1$, then the sign associated with $\Gamma(e_n)$ is just $(-1)^{n-1}$.

We claim that for all $n \geq 1$,

$$[n]_{p,q}! \Gamma(h_n) = \sum_{(\sigma, w) \in C_k \wr S_n} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} x^{(\tau, u)\text{-mch}(w)} z(w). \quad (29)$$

That is, by (19), we have that

$$[n]_{p,q}! \Gamma(h_n) = [n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu, n} \Gamma(e_\mu). \quad (30)$$

Now if μ is not a partition whose parts come from $\{1\} \cup \{s(j-1)+1 : s \geq 1\}$, then $\Gamma(e_\mu) = 0$. Thus let $P_{j,n}$ denote the set of all partitions of n whose parts come from $\{1\} \cup \{s(j-1)+1 : s \geq 1\}$. It follows that

$$\begin{aligned} [n]_{p,q}! \Gamma(h_n) &= [n]_{p,q}! \sum_{\mu \in P_{j,n}} (-1)^{n-\ell(\mu)} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu, n}} \prod_{i=1}^{\ell(\mu)} \Gamma(e_{b_i}) = \\ &= [n]_{p,q}! \sum_{\mu \in P_{j,n}} (-1)^{n-\ell(\mu)} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu, n}} \prod_{i=1}^{\ell(\mu)} (-1)^{b_i-1} ((z_0 + \cdots + z_{k-1}) \chi(b_i = 1) + \\ &\quad (x-1)^{(b_i-1)/(j-1)} \frac{1}{[b_i]_{p,q}!} \text{mp}_{(\tau, u), b_i}^k(p, q, z_0, \dots, z_{k-1}) \chi(b_i > 1)) = \\ &= \sum_{\mu \in P_{j,n}} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu, n}} p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\mu)} \end{matrix} \right]_{p,q} \prod_{i=1}^{\ell(\mu)} ((z_0 + \cdots + z_{k-1}) \chi(b_i = 1) + \end{aligned}$$

$$(x-1)^{(b_i-1)/(j-1)} p^{-\binom{b_i}{2}} mp_{(\tau,u),b_i}^k(p,q,z_0,\dots,z_{k-1})\chi(b_i > 1)). \quad (31)$$

Next we want to give a combinatorial interpretation to the right hand side of (31). Suppose that we have a brick tabloid $B = (b_1, \dots, b_\ell)$ of size n such that $b_i \in \{1\} \cup \{s(j-1) + 1 : s \geq 1\}$ for all i . By Lemma 4, we can interpret $p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \left[\begin{smallmatrix} n \\ b_1, \dots, b_{\ell(\mu)} \end{smallmatrix} \right]_{p,q}$ as the ways of filling the cells of B with a permutation α such that α is increasing in each brick of B and we weight α by $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)}$. Then if $b_i = 1$, we shall interpret $z_0 + \dots + z_{k-1}$ as allowing us to place any letter from $\{0, 1, \dots, k-1\}$ on top of the integer from α that is in brick b_i . If $b_i = s(j-1) + 1 > 1$, then let $1 \leq a_1^i < \dots < a_{s(j-1)+1}^i \leq n$ be the elements of α which are in the cells of b_i reading from left to right. We shall interpret the term

$$(x-1)^{(b_i-1)/(j-1)} p^{-\binom{b_i}{2}} mp_{(\tau,u),b_i}^k(p,q,z_0,\dots,z_{k-1}) = \\ (x-1)^s p^{-\binom{b_i}{2}} mp_{(\tau,u),s(j-1)+1}^k(p,q,z_0,\dots,z_{k-1})$$

as the number of ways of filling b_i with a pair $(\beta_i, v_i) \in \mathcal{MP}_{(\tau,u),s(j-1)+1}^k$ and then labeling each cell in b_i which is the start of (τ, u) -match in (β_i, v_i) with either x or -1 . In this case, we weight (β_i, v_i) by $p^{-\binom{b_i}{2}} p^{\text{coinv}(\beta_i)} q^{\text{inv}(\beta_i)} z(v_i)$. Finally, we replace the numbers by $1, \dots, s(j-1) + 1$ that occur in β_i by $a_1, \dots, a_{s(j-1)+1}$ respectively. Doing this for each brick will result in a filling of the cells of B with a pair $(\sigma, w) \in C_k \wr S_n$. We claim that $\text{coinv}(\alpha) - \sum_{b_i > 1} \binom{b_i}{2} + \text{coinv}(\beta_i) = \text{coinv}(\sigma)$. That is, $\text{coinv}(\alpha)$ accounts for the coinversions that come from pairs of elements that lie in two different bricks plus $\sum_{b_i > 1} \binom{b_i}{2}$ which accounts for the coinversions that come from pairs that lie in the same brick. Thus $\text{coinv}(\alpha) - \sum_{b_i > 1} \binom{b_i}{2}$ counts only the coinversions that come from pairs of elements in σ that lie in two different bricks. Then $\sum_{b_i > 1} \text{coinv}(\beta_i)$ accounts for the coinversions that come from pairs of σ that lie in the same brick. A similar argument will show that $\text{inv}(\alpha) + \sum_{b_i > 1} \text{inv}(\beta_i) = \text{inv}(\sigma)$. Thus the final weight of (σ, w) will be $p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w)$.

Let $\mathcal{Q}_{(\tau,u),n}$ denote the set of all labeled brick tabloids that can be constructed in this way. Thus a $Q \in \mathcal{Q}_{(\tau,u),n}$ will consist of a triple $T = (B, (\sigma, w), L)$ where $\sigma = \sigma_1 \dots \sigma_n \in S_n$, $w = w_1 \dots w_n \in \{0, 1, \dots, k-1\}^n$, and

1. $B = (b_1, \dots, b_\ell)$ is a brick tabloid of shape (n) such that $b_i \in \{1\} \cup \{s(j-1) + 1 : s \geq 1\}$ for all i ,
2. $(\sigma, w) \in C_k \wr S_n$ is a k -colored permutation,
3. if $b_i = 1$ and covers cell t , then (σ_t, w_t) is allowed to be any pair from $\{1, \dots, n\} \times \{0, 1, \dots, k-1\}$, and
4. if $b_i = s(j-1) + 1 > 1$, then cells of b_i are filled with a pair (γ_i, v_i) such that $(\text{pred}(\gamma_i), v_i)$ is a maximum packing for (τ, u) of size $s(j-1) + 1$ and each cell of b_i which corresponds to the start of (τ, u) -match in (γ_i, v_i) is labeled with either -1 or x .

We then define the weight of T , $wt(T)$, to be $p^{\text{coinv}(\sigma)}q^{\text{inv}(\sigma)}z(w)$ times the product of the x labels in T and the sign of T , $sign(T)$, to be the product of the -1 labels in T . For example, if $k = 5$ and $(\tau, u) = (213, 010)$, then in Figure 5 we have pictured four elements of $\mathcal{Q}_{(213,010),17}$, $T_i = (B^{(i)}, (\sigma^{(i)}, w^{(i)}), L^{(i)})$ for $i = 1, \dots, 4$. Then

$$\begin{aligned} wt(T^{(1)}) &= xz_0^3z_1^5z_2^2z_3^4z_4^3p^{65}q^{71}, \quad sign(T^{(1)}) = 1, \\ wt(T^{(2)}) &= x^2z_0^4z_1^4z_2^2z_3^3z_4^3p^{65}q^{71}, \quad sign(T^{(2)}) = 1, \\ wt(T^{(3)}) &= x^2z_0^3z_1^5z_2^2z_3^4z_4^3p^{66}q^{70}, \quad sign(T^{(3)}) = 1, \\ wt(T^{(4)}) &= x^2z_0^3z_1^5z_2^2z_3^4z_4^3p^{66}q^{70}, \quad \text{and } sign(T^{(4)}) = 1. \end{aligned}$$

It follows that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{T \in \mathcal{Q}_{(\tau,u),n}} sign(T)wt(T). \quad (32)$$

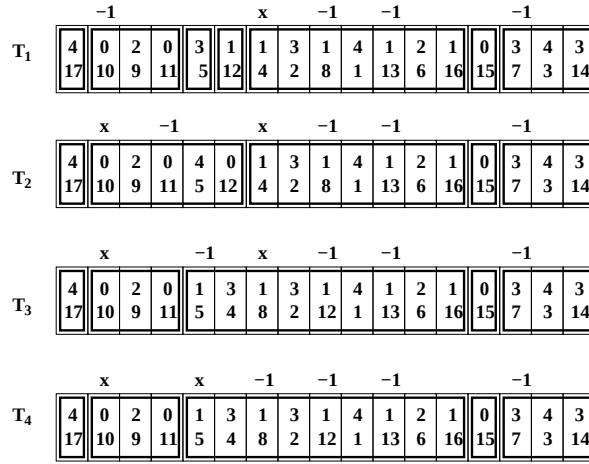


Figure 5: Elements of $\mathcal{Q}_{(213,010),17}$.

Next we define a sign-reversing weight-preserving involution $I : \mathcal{Q}_{(\tau,u),n} \rightarrow \mathcal{Q}_{(\tau,u),n}$ in essentially the same way that we did in Theorem 5. That is, if $T = (B, (\sigma, w), L)$, then to define $I(T)$, we scan the cells of $B = (b_1, \dots, b_\ell)$ from left to right looking for the first time we are in one of the following cases.

Case 1. There is a brick b_i of size j whose first cell is labeled with -1 . In this case, $I(T) = (B^*, (\sigma^*w^*), L^*)$ where B^* results from B by replacing the brick b_i by j bricks of size 1, $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by removing the label -1 from the first cell of b_i .

Case 2. There are j consecutive bricks of size 1 in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that the letters in these cells form a (τ, u) -match. In this case, $I(T) = (B^*, (\sigma^*w^*), L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size j , $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by labeling the first cell of b with a -1 .

Case 3. There is a brick b_i of size $(s+1)(j-1)+1$ where $s \geq 1$ such that all the

labels on b_i are x 's except for the cell that is j cells from the right which is labeled with -1 . In this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the brick b_i by a brick of size $s(j-1) + 1$ followed by $j-1$ bricks of size 1, $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by removing the -1 label that was in b_i .

Case 4. There are j consecutive bricks in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i = s(j-1) + 1 > 1$ and $b_{i+1}, \dots, b_{i+j-1}$ are of size 1, all the labels on b_i are x 's, and the letters in these bricks form a maximum packing for (τ, u) of size $(s+1)(j-1) + 1$. In this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size $(s+1)(j-1) + 1$, $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by labeling the last cell of b_i with a -1 .

Case 5. There is a brick b_i of size $(s+1)(j-1) + 1$ where $s \geq 1$ such that the first cell of b_i is labeled with -1 . In this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the brick b_i by $j-1$ bricks of size 1 followed by a brick of size $s(j-1) + 1$, $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by removing the -1 label that was on the first cell of b_i .

Case 6. There are j consecutive bricks in B , $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i, \dots, b_{i+j-2}$ are of size 1 and $b_{i+j-1} = s(j-1) + 1 > 1$ and the letters in these bricks form a maximum packing for (τ, u) of size $(s+1)(j-1) + 1$. In this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b of size $(s+1)(j-1) + 1$, $(\sigma^*, w^*) = (\sigma, w)$, and L^* arises from L by labeling the first cell of b with a -1 .

Case 7. There is a brick b_i of size $s(j-1) + 1$ where $s \geq 3$ such that the first cell is labeled with an x and there is a cell which has a label -1 which is not the j -th cell from the right. Let t be the left-most cell of b_i which is labeled with -1 . Then in this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the brick b_i by j consecutive bricks $c_1, c_2, \dots, c_{j-1}, c_j$ where c_1 contains all the cells of b_i up to and including cell t , $c_2, \dots, c_{j-1}$ are bricks of size 1, and c_j contains the remaining cells of b_i , $(\sigma^*, w^*) = (\sigma, w)$, and L^* is the labeling that results from L by removing the -1 label from cell t .

Case 8. There are j consecutive bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ such that $b_i = c(s-1) + 1 > 1$ and has no -1 labels, $b_{i+1}, \dots, b_{i+j-2}$ are bricks of size 1, $b_{i+j-1} = d(j-1) + 1 > 1$, and the letters in these three bricks form a maximum packing for (τ, u) of size $(c+d+1)(j-1) + 1$. In this case, $I(T) = (B^*, (\sigma^*, w^*), L^*)$ where B^* results from B by replacing the j bricks $b_i, b_{i+1}, \dots, b_{i+j-1}$ by a single brick b , $(\sigma^*, w^*) = (\sigma, w)$, and L^* results from L by adding a label -1 on the last cell of b_i .

If none of Cases 1-8 apply, then we set $I(T) = T$.

For example, the images of $T_1, \dots, T_4$ as pictured in Figure 5 are pictured in Figure 6.

As in Theorem 5, I is a sign-reversing weight-preserving involution. Hence I shows

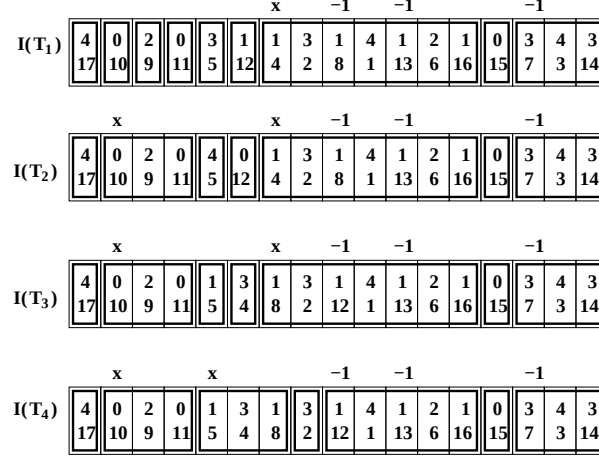


Figure 6: The images under I of the elements in Figure 5.

that

$$[n]_{p,q}! \Gamma(h_n) = \sum_{T \in \mathcal{Q}_{(\tau,u),n}, I(T)=T} \text{sign}(T) \text{wt}(T).$$

Thus we must examine the fixed points of I . We can argue exactly as in Theorem 1 that if $I(T) = T$ where $T = (B, (\sigma, w), L)$, then $\text{sign}(T) = 1$ since there can be no -1 labels and $\text{wt}(T) = x^{(\tau,u)\text{-mch}((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w)$. Moreover if $(\sigma, w) \in C_k \wr S_n$, then we can construct a fixed point of I by placing bricks which cover the maximal length maximum packings for (τ, u) in (σ, w) , covering the remaining cells by bricks of size 1, and labeling the start of each (τ, u) -match in w by x . It thus follows that

$$[n]_{p,q}! \Gamma(h_n) = \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\tau,u)\text{-mch}((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w).$$

But then,

$$\begin{aligned} \Gamma(H(t)) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\tau,u)\text{-mch}((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \Gamma(1/E(-t)) = \\ &= \frac{1}{1 + \sum_{n \geq 1} (-t)^n \Gamma(e_n)} = \\ &= \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n m p_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}))} \end{aligned}$$

which proves (27).

If we replace (τ, u) -matchings by exact (τ, u) -matching throughout the proof of (27), then we will get a proof of (28). $\square$

We note that in the special case of Theorem 6 where $k = 1$, so that the underlying alphabet is just $\{0\}$, a pair $(\tau, 0^j) \in C_1 \wr S_n$ has the 1-minimal overlapping property if

and only if τ has the minimal overlapping property. Thus if $k = 0$ and we set $z_0 = 1$ in Theorem 6, we obtain the following corollary.

Corollary 7. *Suppose that $\tau \in S_j$ has the minimal overlapping property. Then*

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1}{1 - (t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n m p_{\tau, n(j-1)+1}(p, q))}. \quad (33)$$

We end this section by observing that the proofs of Theorems 5 and 6 can be specialized to give another interesting pair of results. That is, we say that a word $u = u_1 \dots u_j \in \{0, \dots, k-1\}^j$ has the k -non-overlapping property (k -exact match non-overlapping property) if the smallest i such that there exists a word $w \in \{0, \dots, k-1\}^i$ with $u\text{-mch}(w) = 2$ ($Eu\text{-mch}(w) = 2$) is $2j$. Thus u has the k -non-overlapping (k -exact match non-overlapping) property if no two u -matches (exact u -matches) can share a letter. For example, if $k = 3$ and $u = 00112$, then it is easy to see that u has both the k -non-overlapping property and the k -exact match non-overlapping property. However, if $k = 3$ and $v = 011$, then v has the k -exact match non-overlapping property but does not have the non-overlapping property since $w = 01122$ has two v -matches. Similarly, we say that $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$ -non-overlapping property ($C_k \wr S_n$ -exact match non-overlapping property) if the smallest i such that there exists a k -colored permutation $(\sigma, w) \in C_k \wr S_i$ with $(\tau, u)\text{-mch}(w) = 2$ ($(\tau, Eu)\text{-mch}(w) = 2$) is $2j$. Clearly, there can be no permutations which have the non-overlapping property so that the only way a $(\tau, u) \in C_k \wr S_j$ can have the $C_k \wr S_n$ -non-overlapping property ($C_k \wr S_n$ -exact match non-overlapping property) is if u has the k -non-overlapping property (k -exact match non-overlapping property).

We can still define the notion of a maximum packing for words $u \in \{0, \dots, k-1\}^*$ which have the k -non-overlapping (k -exact match non-overlapping) property and for k -colored permutations which have the $C_k \wr S_n$ -non-overlapping ($C_k \wr S_n$ -exact match non-overlapping) property, it is just that in such cases the only maximum packings are of size j . It is then easy to check that the proofs of Theorems 5 and 6 go through exactly as before except that only Cases 1 and 2 of the involution apply. Thus we have the following theorems.

Theorem 8. *Let $u \in \{0, 1, \dots, k-1\}^j$ where $j \geq 3$.*

(I) *If u has the k -non-overlapping property and $\text{red}(u) = u$, then*

$$\sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n} x^{u\text{-mch}(w)} z(w) = \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + (x-1) m p_{u,j}^k(z_0, \dots, z_{k-1}) t^j)}. \quad (34)$$

(II) If u has the k -exact match non-overlapping property, then

$$\sum_{n \geq 0} t^n \sum_{w \in \{0,1,\dots,k-1\}^n} x^{Eu-\text{mch}(w)} z(w) = \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + (x-1)\text{emp}_{u,j}^k(z_0, \dots, z_{k-1})t^j)}. \quad (35)$$

Theorem 9. (III) If $(\tau, u) \in C_k \wr S_j$, $\text{red}(u) = u$, and (τ, u) has the $C_k \wr S_n$ -non-overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, u)-\text{mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \frac{t^j}{[j]_{p,q}!} (x-1)mp_{(\tau, u),j}^k(p, q, z_0, \dots, z_{k-1}))}. \quad (36)$$

(IV) If $(\tau, u) \in C_k \wr S_j$ has the $C_k \wr S_n$ -exact match non-overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_k \wr S_n} x^{(\tau, Eu)-\text{mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((z_0 + \dots + z_{k-1})t + \frac{t^j}{[j]_q!} (x-1)\text{emp}_{(\tau, u),j}^k(p, q, z_0, \dots, z_{k-1}))}. \quad (37)$$

4 Computing the number of maximum packings

In this section, we shall give several examples where we can compute the functions $mp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1})$, $\text{emp}_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1})$, $mp_{(\tau, u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$, $\text{emp}_{(\tau, u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$, and $mp_{\tau,n(j-1)+1}(p, q)$.

First suppose that $u \in \{0, 1, \dots, k-1\}^j$ has the k -exact match minimal overlapping property. Then it is trivial to compute $\text{emp}_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1})$. That is, if $u = u_1 \dots u_j$, then we must have $u_1 = u_j$ so that it is simple to see that $\text{emp}_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1}) = z_{u_1}(z_{u_2} \dots z_{u_j})^n$. Thus for example, if $u = 01^a 0$ where $a \geq 1$, then

$$\text{emp}_{01^a 0, n(j-1)+1}^k(z_0, \dots, z_{k-1}) = z_0(z_0 z_1^a)^n = z_0^{n+1} z_1^{an}$$

for all $k \geq 2$.

Next suppose that $u = u_1 \dots u_j \in \{0, 1, \dots, k-1\}^j$ has the k -minimal overlapping property and $\text{red}(u) = u$. In the case where $u_1 = u_j$, it is again relatively straightforward to compute $mp_{u,n(j-1)+1}^k(z_0, \dots, z_{k-1})$. For example if $u = 01^a 0$ where $a \geq 1$, then all maximum packings for u of length $n(a+1)+1$ are of the form

$$i \ b_1^a \ i \ b_2^a \ i \ \dots \ i \ b_n^a \ i$$

where $b_1, \dots, b_n \in \{i+1, \dots, k-1\}$. Thus

$$mp_{01^a 0, 2n+1}^k(z_0, \dots, z_{k-1}) = \sum_{i=0}^{k-2} z_i^{n+1} \left(\sum_{j=i+1}^{k-1} z_j^a \right)^n.$$

If $v = 10^a 1$ where $a \geq 1$, then all maximum packings for v of length $n(a+1) + 1$ are of the form

$$i \ b_1^a \ i \ b_2^a \ i \ \dots \ i \ b_n^a \ i$$

where $b_1, \dots, b_n \in \{0, \dots, i-1\}$. Thus

$$mp_{10^a 1, 2n+1}^k(z_0, \dots, z_{k-1}) = \sum_{i=1}^{k-1} z_i^{n+1} \left(\sum_{j=0}^{i-1} z_j^a \right)^n.$$

However, it is no longer the case that we are forced to have $u_1 = u_j$ if u has the k -minimal overlapping property. For example if $k \geq 3$ and $u = 001$, then $w = 00112$ has 2 u -matches so the 001 has the k -minimal overlapping property for $k \geq 3$. We note that in such cases, there cannot be maximal packings of arbitrary length. That is, suppose that $u_1 < u_j$. Then it is easy to see that in a maximum packing $w = w_1 \dots w_{n(j-1)+1}$ for u , we must have that

$$w_1 < w_j < w_{2(j-1)+1} < w_{3(j-1)+1} < \dots < w_{n(j-1)+1}$$

so that it is impossible to have $n \geq k$. A similar argument applies in the case where $u_1 > u_j$. It follows from Theorem 5 that if $u = u_1 \dots u_j \in \{0, 1, \dots, k-1\}^j$ has the k -minimal overlapping property and $u_1 \neq u_j$, then $\sum_{n \geq 0} t^n \sum_{w \in \{0, 1, \dots, k-1\}^n} x^{u-\text{mch}(w)} z(w)$ is a rational function. In general, it is still rather straightforward to compute $mp_{u, n(j-1)+1}^k(z_0, \dots, z_{k-1})$. For example, if $u = 001$, then all maximum packings for u must be of the form

$$a_1 a_1 a_2 a_2 \dots a_r a_r a_{r+1}$$

where $0 \leq a_1 < \dots < a_{r+1} \leq k-1$. Thus

$$mp_{001, 2n+1}^k(z_0, \dots, z_{k-1}) = \begin{cases} 0 & \text{if } n > k \\ z_{k-1} \prod_{i=0}^{k-2} z_i^2 & \text{if } n = k \\ \sum_{0 \leq a_1 < \dots < a_{n+1} \leq k-1} z_{a_{n+1}} \prod_{i=1}^n z_{a_i}^2 & \text{if } 1 \leq n < k. \end{cases}$$

If $\tau \in S_j$ has the minimal overlapping property, then it seems to be a difficult problem to compute $mp_{\tau, n(j-1)+1}(p, q)$ in general. However, there is at least one special case where it is easy, namely, when τ starts with 1. That is, in that case, we have the following theorem.

Theorem 10. *Suppose that $\tau = \tau_1 \dots \tau_j$ has the minimal overlapping property where $\tau_1 = 1$ and $\tau_j = s$. Then for all $n \geq 1$.*

$$mp_{\tau, (n+1)(j-1)+1}(p, q) =$$

$$p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)} p^{(s-1)n(j-1)} \begin{bmatrix} (n+1)(j-1) + 1 - s \\ j - s \end{bmatrix}_{p,q} mp_{\tau, n(j-1)+1}(p, q)$$

so that

$$mp_{\tau, (n+1)(j-1)+1}(p, q) = (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^{(n+1)} p^{(s-1)(j-1)} \binom{n+1}{2} \prod_{i=1}^{n+1} \begin{bmatrix} i(j-1) + 1 - s \\ j - s \end{bmatrix}_{p,q}. \quad (38)$$

Proof. Suppose that $\sigma = \sigma_1 \dots \sigma_{(n+1)(j-1)+1}$ is a maximum packing for τ . Since $\tau_1 = 1$, it must be the case that σ_1 is less than $\sigma_2, \dots, \sigma_j$, σ_j is less than $\sigma_{j+1}, \dots, \sigma_{j+(j-1)}$, σ_{2j-1} is less than $\sigma_{2j}, \dots, \sigma_{2j-1+(j-1)}$, etc.. It follows that $\sigma_1 = 1$ and that σ_j must be less than $\sigma_{j+1}, \dots, \sigma_{(n+1)(j-1)+1}$. We claim that σ_j must be s . We know that $\sigma_j \geq s$ because the fact that $\sigma_1 \dots \sigma_j$ is a τ -match means that there must be $s-1$ elements of $\sigma_1, \dots, \sigma_{j-1}$ which are less than σ_j . However if $\sigma_j > s$, then $1, \dots, \sigma_j - 1$ must be among $\sigma_1, \dots, \sigma_{j-1}$ which would violate the fact that σ_j is the s -th smallest element among $\sigma_1, \dots, \sigma_j$.

It follows that $1, \dots, s$ must be among $\sigma_1, \dots, \sigma_j$ which means that the positions of $1, \dots, s$ in $\sigma_1 \dots \sigma_j$ must be the same as the positions of $1, \dots, s$ in τ . But then we have $\binom{(n+1)(j-1)+1-s}{j-s}$ ways to choose the remaining $j-s$ elements in $\sigma_1 \dots \sigma_j$ and once we have chosen these $j-s$ elements, their positions are completely determined by τ since $\sigma_1 \dots \sigma_j$ is a τ -match. Moreover $\text{pred}(\sigma_j \dots \sigma_{(n+1)(j-1)+1})$ must be an element of $\mathcal{MP}_{\tau, n(j-1)+1}$. It then follows that

$$mp_{\tau, (n+1)(j-1)+1} = \begin{pmatrix} (n+1)(j-1) + 1 - s \\ j - s \end{pmatrix} mp_{\tau, n(j-1)+1}.$$

Next consider the problem of computing $mp_{\tau, (n+1)(j-1)+1}(p, q)$. For any σ in $\mathcal{MP}_{\tau, (n+1)(j-1)+1}$, we can organize the count of inversions and coinversions of σ by

- (a) counting the inversions and coinversions among $\sigma_1 \dots \sigma_j$ which contributes a factor of $p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)}$ to $p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)}$,
- (b) counting the inversions and coinversions between $1, \dots, s-1$ and $\sigma_{j+1} \dots \sigma_{(n+1)(j-1)+1}$ which contributes a factor of $p^{(s-1)n(j-1)}$ to $p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)}$,
- (c) counting the inversions and coinversions among $\{\sigma_1, \dots, \sigma_j\} - \{1, \dots, s\}$ and $\sigma_{j+1} \dots \sigma_{(n+1)(j-1)+1}$ which contributes a factor of $\begin{bmatrix} (n+1)(j-1)+1-s \\ j-s \end{bmatrix}_{p,q}$ to $mp_{\tau, (n+1)(j-1)+1}(p, q)$ as we vary over all choices $\{\sigma_1, \dots, \sigma_j\} - \{1, \dots, s\}$, and
- (d) counting the inversions and coinversions among $\sigma_j \dots \sigma_{(n+1)(j-1)+1}$.

Thus

$$mp_{\tau, (n+1)(j-1)+1}(p, q) = p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)} p^{(s-1)n(j-1)} \begin{bmatrix} (n+1)(j-1) + 1 - s \\ j - s \end{bmatrix}_{p,q} mp_{\tau, n(j-1)+1}(p, q).$$

Iterating this recursion then yields (38). □

By our remarks in the introduction, if $\tau \in S_j$ has the minimal overlapping property, then τ^r and τ^c have the minimal overlapping property. Thus we can use Theorem 10 to find formulas for $mp_{\tau, (n+1)(j-1)+1}(p, q)$ when either τ starts or ends in j or ends in 1.

It follows from Corollary 7 and Theorem 10 that if $j \geq 3$ and $\tau = \tau_1 \dots \tau_j \in S_j$ has the minimal overlapping property, $\tau_1 = 1$ and $\tau_j = s$, then

$$P_\tau(t, x, p, q) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1}{1 - (t + \sum_{n \geq 1} \frac{(x-1)^n t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^{(n+1)} p^{(s-1)(j-1)} \binom{n+1}{2} \prod_{i=1}^{n+1} \begin{bmatrix} i(j-1)+1-s \\ j-s \end{bmatrix}_{p,q})}.$$
(39)

For example, both 132 and 1342 have the minimal overlapping property. As stated in the introduction, Elizalde and Noy [8] proved that

$$\begin{aligned} P_{132}(t, q) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{132\text{-mch}(\sigma)} \\ &= \frac{1}{1 - \int_0^t e^{(x-1)z^2/2} dz} \\ &= \frac{1}{1 - \sum_{n \geq 0} \frac{(x-1)^n t^{2n+1}}{2^n (n!) (2n+1)}}. \end{aligned}$$

By Theorem 10, we have

$$mp_{132, 2n+1}(p, q) = (p^2 q)^n p^{2 \binom{n}{2}} \prod_{i=1}^n [2i-1]_{p,q}. \quad (40)$$

Plugging equation (40) into Corollary 7, we get the following p, q -analogue of Elizalde and Noy's result for $P_{132}(t, x)$.

$$\begin{aligned} P_{132}(t, x, p, q) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{132\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} \\ &= \frac{1}{1 - (t + \sum_{n \geq 1} \frac{(x-1)^n t^{2n+1}}{[2n+1]_{p,q}!} p^{n^2+n} q^n \prod_{i=1}^n [2i-1]_{p,q})} \\ &= \frac{1}{1 - \sum_{n \geq 0} \frac{p^{n^2+n} q^n (x-1)^n t^{2n+1}}{[2n+1]_{p,q} \prod_{i=1}^n [2i]_{p,q}}}. \end{aligned}$$

Similarly, Elizalde and Noy [8] proved that

$$P_{1342}(t, x) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{1342\text{-mch}(\sigma)}$$

$$\begin{aligned}
&= \frac{1}{1 - \int_0^t e^{(x-1)z^3/6} dz} \\
&= \frac{1}{1 - \sum_{n \geq 0} \frac{(x-1)^n t^{3n+1}}{6^n (n!) (3n+1)}}.
\end{aligned}$$

By Theorem 10, we have

$$\begin{aligned}
mp_{1342, 2n+1} &= (p^3 q^2)^n p^{3 \binom{n}{2}} \prod_{i=1}^n \left[\begin{matrix} 3n+1-2 \\ 2 \end{matrix} \right]_{p,q} \\
&= p^{(3n^2+3n)/2} q^{2n} \prod_{i=1}^n \frac{[3n-1]_{p,q} [3n-2]_{p,q}}{[2]_{p,q}}. \tag{41}
\end{aligned}$$

Plugging equation (41) into Corollary 7, we get the following p, q -analogue of Elizalde and Noy's result for $P_{1342}(t, x)$.

$$\begin{aligned}
P_{1342}(t, x, p, q) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{1342\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} \\
&= \frac{1}{1 - (t + \sum_{n \geq 1} \frac{(x-1)^n t^{3n+1}}{[3n+1]_{p,q}!} p^{(3n^2+3n)/2} q^{2n} \frac{1}{[2]_{p,q}^n} \prod_{i=1}^n [3i-1]_{p,q} [3i-2]_{p,q})} \\
&= \frac{1}{1 - \sum_{n \geq 0} \frac{p^{(3n^2+3n)/2} q^{2n} (x-1)^n t^{3n+1}}{[3n+1]_{p,q} [2]_{p,q}^n \prod_{i=1}^n [3i]_{p,q}}}.
\end{aligned}$$

Now if $\tau = 12 \dots a \sigma a + 1$ where σ is a permutation of $\{a+2, \dots, k+1\}$, then τ has the minimal overlapping property. Note that

$$\begin{aligned}
\text{coinv}(\tau) &= \binom{a+1}{2} + a(k-a) + \text{coinv}(\sigma) \text{ and} \\
\text{inv}(\tau) &= (k-a) + \text{inv}(\sigma).
\end{aligned}$$

Thus using Corollary 7 and Theorem 10, we can derive the following generalization of Kitaev's Theorem 2.

Theorem 11. *Let $\tau = 12 \dots a \sigma a + 1$ where σ is a permutation of $\{a+2, \dots, k+1\}$. Then*

$$\begin{aligned}
&\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \\
&\frac{1}{1 - (t + \sum_{i \geq 0} \frac{(x-1)^{i+1} t^{ik+1}}{[ik+1]_{p,q}!} (p^{a(k-a)+\text{coinv}(\sigma)+\binom{a+1}{2}} q^{k-a+\text{inv}(\sigma)})^{(i+1)} p^{ak\binom{i+1}{2}} \prod_{j=2}^i \left[\begin{matrix} jk-a \\ k-a \end{matrix} \right]_{p,q})}. \tag{42}
\end{aligned}$$

Next following a suggestion of Sergey Kitaev, we show that if α and β are two permutations in S_j which have the minimal overlapping property and have the same first and last elements, then for any n , $mp_{\alpha, n(j-1)+1} = mp_{\beta, n(j-1)+1}$. That is, we shall prove the following.

Theorem 12. Suppose $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then for all $n \geq 1$,

$$mp_{\alpha, n(j-1)+1} = mp_{\beta, n(j-1)+1}. \quad (43)$$

If in addition, $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}$, then

$$mp_{\alpha, n(j-1)+1}(p, q) = mp_{\beta, n(j-1)+1}(p, q). \quad (44)$$

Proof. Suppose that we wanted to construct all maximum packings $\sigma = \sigma_1 \dots \sigma_{n(j-1)+1}$ of size $n(j-1)+1$ for α or β . One way to do this is to partition $\{1, \dots, n(j-1)+1\}$ into sets $T_1, \dots, T_n$ where $|T_1| = j$ and $|T_i| = j-1$ for $i \geq 2$ and use the elements of T_1 for $\sigma_1 \dots \sigma_j$, use the elements of T_2 for $\sigma_{j+1} \dots \sigma_{2j-1}$, use the elements of T_3 for $\sigma_{2j} \dots \sigma_{3j-2}$, etc.. Of course, this will not work for all choices of $T_1, \dots, T_n$. That is, if $\alpha = 132$ and we pick $T_1 = \{4, 5, 6\}$ and $T_2 = \{1, 2\}$, then there will be no way to use T_1 for the elements $\sigma_1 \sigma_2 \sigma_3$ and use T_2 for the elements $\sigma_4 \sigma_5$ to produce a maximum packing for 132. That is, in such a situation, we must let $\sigma_1 = 3$, $\sigma_2 = 5$, and $\sigma_3 = 4$. But then σ_3 will be greater than σ_4 and σ_5 so that this choice will not allow us to construct a maximum packing for 132. Our claim is that for any choice of $T_1, \dots, T_n$, either we cannot construct a maximum packing for either α or β in this way or we can construct a maximum packing for both α and β in this way. For example, if $\alpha = 24153$, then we have pictured in Figure 7 choices for T_1, T_2, T_3 , and T_4 and the steps needed to construct a maximum packing for α . Clearly, given T_1 , there is only one way to place the elements of T_1 so that $\sigma_1 \dots \sigma_5$ is a α -match. This is pictured in the second row of Figure 7. In this process, $\sigma_5 = 6$ must be the third largest element of T_1 since $\alpha_5 = 3$. But then to continue, it must be the case that 6 is the second largest element of $\{6\} \cup T_2$ if $\sigma_5 \dots \sigma_9$ is an α -match. Since 6 is the second largest element in $\{6\} \cup T_2$, the positions of the elements of T_2 are then forced by the requirement that $\sigma_5 \dots \sigma_9$ be an α -match which is pictured in row three of Figure 7. In particular, σ_9 must be the third largest element of $\{6\} \cup T_2$ so that $\sigma_9 = 8$. To continue, it must be the case that 8 is the second largest element of $\{8\} \cup T_3$ if $\sigma_9 \dots \sigma_{13}$ is an α -match and σ_{13} must be the third largest element of $\{8\} \cup T_3$. In this case, 8 is the second largest element of $\{8\} \cup T_3$ and we are forced to have $\sigma_{13} = 11$ since 11 is the third largest element of $\{8\} \cup T_3$. Finally, it must be the case that 11 is the second largest element of $\{11\} \cup T_4$ which is true in this case so that we can complete the construction of a maximum packing for α using $T_1, \dots, T_4$.

In general, suppose that $\alpha_1 = \beta_1 = s$ and $\alpha_j = \beta_j = t$. Then to be able to use T_2 to continue the construction of a maximum packing for either α or β , we must have that the t -th largest element a_1 of T_1 is the s -th largest element of $\{a_1\} \cup T_2$. If not, we cannot use $T_1, \dots, T_n$ to construct a maximum packing for either α or β . If so, then the t -th largest element a_2 of $\{a_1\} \cup T_2$ must be the s -largest element of $\{a_2\} \cup T_3$. If not, we cannot use $T_1, \dots, T_n$ to construct a maximum packing for either α or β . If so, then the t -th largest element a_3 of $\{a_2\} \cup T_3$ must be the s -largest element of $\{a_3\} \cup T_4$, etc. Thus it is easy to see that we can use $T_1, \dots, T_n$ to construct a maximum packing σ for α if and only if we can use $T_1, \dots, T_n$ to construct a maximum packing σ^* for β .

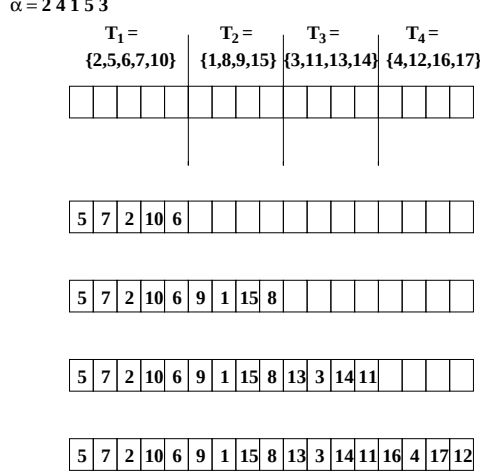


Figure 7: Constructing maximum packings for $\alpha = 24153$.

Moreover, it is easy to see that if $p^{\text{coinv}(\alpha)}q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)}q^{\text{inv}(\beta)}$, then it will be the case that $p^{\text{coinv}(\sigma)}q^{\text{inv}(\sigma)} = p^{\text{coinv}(\sigma^*)}q^{\text{inv}(\sigma^*)}$. Thus (43) and (44) hold. $\square$

We can now use Theorem 12 and Corollary 7 to prove Elizalde's conjecture stated in the introduction. That is, suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$. Elizalde [7] conjectured that it must be the case that α and β are strongly c-Wilf equivalent. By Theorem 12, we know that in such a situation, $mp_{\alpha,n(j-1)+1} = mp_{\beta,n(j-1)+1}$ for all $n \geq 1$ and, hence, by Corollary 7 with $p = q = 1$, we know that

$$\begin{aligned}
 \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\alpha\text{-mch}(\sigma)} &= \\
 \frac{1}{1 - (t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{(n(j-1)+1)!} (x-1)^n mp_{\alpha,n(j-1)+1})} &= \\
 \frac{1}{1 - (t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{(n(j-1)+1)!} (x-1)^n mp_{\beta,n(j-1)+1})} &= \\
 \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\beta\text{-mch}(\sigma)}. &
 \end{aligned}$$

Thus Elizalde's conjecture follows. If in addition, $p^{\text{coinv}(\alpha)}q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)}q^{\text{inv}(\beta)}$, then by the same reasoning, we get the even stronger conclusion that

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\alpha\text{-mch}(\sigma)} p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\beta\text{-mch}(\sigma)} p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}.$$

We note that Elizalde's conjecture has been proved independently by Vladimir Dot-senko and Anton Khoroshkin [4] using cluster algebras and ideas similar to Theorem 12.

Next we consider the problem of computing $mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$ ($emp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$) for $(\tau, u) \in C_k \wr S_n$ which have the $C_k \wr S_n$ -minimal overlapping property ($C_k \wr S_n$ -exact match minimal overlapping property). There are several reasons why a $(\tau, u) \in C_k \wr S_j$ can have the $C_k \wr S_n$ -minimal overlapping property. For example, it could be that u has the k -minimal overlapping property such as $(\tau, u) = (12 \dots a + 2, 01^a 0)$. In this case it is easy to see that

$$mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) = p^{\binom{n(j-1)+1}{2}} mp_{u,n(j-1)+1}^k(z_0, \dots, z_k).$$

It could also be that τ has the minimal overlapping property such as $(\tau, u) = (132, 000)$. In this case, it is easy to see that

$$mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) = mp_{\tau,n(j-1)+1}^k(p, q) \sum_{i=0}^{k-1} z_i^{n(j-1)+1}.$$

If it is the case that u has the k -minimal overlapping property and τ has the minimal overlapping property, then it is easy to see that

$$mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) = mp_{\tau,n(j-1)+1}^k(p, q) mp_{u,n(j-1)+1}^k(z_0, \dots, z_k). \quad (45)$$

However, there are cases like $(\tau, u) = (152364, 001100)$ which has the $C_k \wr S_n$ -minimal overlapping property where neither u has the k -minimal overlapping property nor τ has the minimal overlapping property. Now if τ starts with 1, then we can mimic the reasoning in Theorem 10 to find an explicit formula for $mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1})$. That is, suppose $\tau = \tau_1 \dots \tau_j \in S_j$ is a permutation such that $\tau_1 = 1$ and $\tau_j = s$. Then let $MP_{\tau,n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}$ denote the set of permutations in $S_{n(j-1)+1}$ which have τ -matches starting at positions $1, j, 2j-1, 3j-2, \dots, (n-1)j-(n-2)$ and let

$$mp_{\tau,n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) = \sum_{\sigma \in MP_{\tau,n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)}.$$

If $u = u_1 \dots u_j \in \{0, 1, \dots, k-1\}^j$ and $\text{red}(u) = u$, we let $MP_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}$ denote the set of all words $w \in \{0, 1, \dots, k-1\}^{n(j-1)+1}$ which have u -matches starting at positions $1, j, 2j-1, 3j-2, \dots, (n-1)j-(n-2)$ and let

$$mp_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(z_0, \dots, z_{k-1}) = \sum_{w \in MP_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}} z(w).$$

Similarly, if $u = u_1 \dots u_j \in \{0, 1, \dots, k-1\}^j$, we let $EMP_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}$ denote the set of all words $w \in \{0, 1, \dots, k-1\}^{n(j-1)+1}$ which have exact u -matches starting at positions $1, j, 2j-1, 3j-2, \dots, (n-1)j-(n-2)$ and let

$$emp_{\tau,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(z_0, \dots, z_{k-1}) = \sum_{w \in EMP_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}} z(w).$$

Then using the same reasoning as in Theorem 10, we can show that

$$mp_{\tau, (n+1)(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2),nj-(n-1)\}}(p, q) = p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)} p^{(s-1)n(j-1)} \left[\begin{matrix} (n+1)(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q} mp_{\tau, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) \quad (46)$$

so that

$$mp_{\tau, (n+1)(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q, z_0, \dots, z_{k-1}) = (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^{(n+1)} p^{(s-1)(j-1)} \binom{n+1}{2} \prod_{i=1}^{n+1} \left[\begin{matrix} i(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q}. \quad (47)$$

We then have the following theorem.

Theorem 13. Suppose that $(\tau, u) \in C_k \wr S_n$ has the $C_k \wr S_n$ -minimal overlapping property, $\text{red}(u) = u$, and $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = s$. Then

$$\begin{aligned} mp_{(\tau,u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) &= \\ mp_{\tau, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) &mp_{u, n(j-1)+1}^k(p, q, z_0, \dots, z_k) = \\ mp_{u, n(j-1)+1}^k(p, q, z_0, \dots, z_k) &\times \\ (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^n p^{(s-1)(j-1)} \binom{n}{2} &\prod_{i=1}^n \left[\begin{matrix} i(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q}. \end{aligned}$$

Similarly if $(\tau, u) \in C_k \wr S_n$ has the $C_k \wr S_n$ -exact match minimal overlapping property and $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = s$, then

$$\begin{aligned} emp_{(\tau,u), n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) &= \\ mp_{\tau, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) &emp_{u, n(j-1)+1}^k(p, q, z_0, \dots, z_k) = \\ emp_{u, n(j-1)+1}^k(p, q, z_0, \dots, z_k) &\times \\ (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^n p^{(s-1)(j-1)} \binom{n}{2} &\prod_{i=1}^n \left[\begin{matrix} i(j-1) + 1 - s \\ j - s \end{matrix} \right]_{p,q}. \end{aligned}$$

Now suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are two permutations in S_j such that $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$ and $u \in \{0, \dots, k-1\}^j$ is such that (α, u) and (β, u) have the $C_k \wr S_n$ -minimal overlapping property ($C_k \wr S_n$ -exact match minimal overlapping property). Then we can use the same reasoning at Theorem 12 to show that for all $n \geq 1$,

$$mp_{\alpha, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}} = mp_{\beta, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}.$$

If, in addition, $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}$, then

$$mp_{\alpha, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) = mp_{\beta, n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q).$$

Since for $\tau \in \{\alpha, \beta\}$,

$$mp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) = mp_{\tau,n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) mp_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(z_0, \dots, z_k)$$

and

$$emp_{(\tau,u),n(j-1)+1}^k(p, q, z_0, \dots, z_{k-1}) = mp_{\tau,n(j-1)+1}^{\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(p, q) emp_{u,n(j-1)+1}^{k,\{1,j,2j-1,\dots,(n-1)j-(n-2)\}}(z_0, \dots, z_k),$$

we have the following extension of Theorem 12.

Theorem 14. *Suppose $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$. If $u \in \{0, 1, \dots, k-1\}^j$ is such that $\text{red}(u) = u$ and (α, u) and (β, u) have the $C_k \wr S_n$ -minimal overlapping property, then*

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\alpha,u)-mch((\sigma,w))} z(w) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\beta,u)-mch((\sigma,w))} z(w). \quad (48)$$

If, in addition, $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}$, then

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\alpha,u)-mch((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \\ \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{(\beta,u)-mch((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w). \end{aligned} \quad (49)$$

If $u \in \{0, 1, \dots, k-1\}^j$ is such that (α, u) and (β, u) have the $C_k \wr S_n$ -exact match minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{E(\alpha,u)-mch((\sigma,w))} z(w) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{E(\beta,u)-mch((\sigma,w))} z(w). \quad (50)$$

If, in addition, $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}$, then

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{E(\alpha,u)-mch((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \\ \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} x^{E(\beta,u)-mch((\sigma,w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w). \end{aligned} \quad (51)$$

5 Extensions

In this section, we shall briefly remark about two simple ways to extend the results of the previous sections. First, we can replace single occurrences of words, permutations, and k -colored permutations by sets of words, permutations, and k -colored permutations. That is, suppose Υ is a set of words in $\{0, 1, \dots, k-1\}^j$ such that $\text{red}(u) = u$ for all $u \in \Upsilon$, then we say $w = w_1 \dots w_n \in \{0, 1, \dots, k-1\}^n$ has an Υ -match starting at position i if $\text{red}(w_i \dots w_{i+j-1}) \in \Upsilon$. Let $\Upsilon\text{-mch}(w)$ denote the number of Υ -matches in w . We say that Υ has the *k -minimal overlapping property* if the smallest i such that there exists a $w \in \{0, 1, \dots, k-1\}^i$ with $\Upsilon\text{-mch}(w) = 2$ is $2j-1$. Similarly, if Υ is a set of words in $\{0, 1, \dots, k-1\}^j$, then we say $w = w_1 \dots w_n \in \{0, 1, \dots, k-1\}^n$ has an exact Υ -match starting at position i if $w_i \dots w_{i+j-1} \in \Upsilon$. Let $E\Upsilon\text{-mch}(w)$ denote the number of exact Υ -matches in w . We say that u has the *k -exact match minimal overlapping property* if the smallest i such that there exists a $w \in \{0, 1, \dots, k-1\}^i$ with $E\Upsilon\text{-mch}(w) = 2$ is $2j-1$. If Υ is a set of permutations in S_j , then we say that a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ has an Υ -match starting at position i if $\text{pred}(\sigma_i \dots \sigma_{i+j-1}) \in \Upsilon$. Let $\Upsilon\text{-mch}(\sigma)$ denote the number of Υ -matches in σ . We say that Υ has the *minimal overlapping property* if the smallest i such that there exists a $\sigma \in S_i$ with $\Upsilon\text{-mch}(\sigma) = 2$ is $2j-1$. If Υ is a set of colored permutations in $C_k \wr S_n$ such that $\text{red}(u) = u$ for all $(\tau, u) \in \Upsilon$, then we say $(\sigma, w) \in C_k \wr S_n$ where $\sigma = \sigma_1 \dots \sigma_n$ and $w = w_1 \dots w_n$ has an Υ -match starting at position i if $(\text{pred}(\sigma_i \dots \sigma_{i+j-1}), \text{red}(w_i \dots w_{i+j-1})) \in \Upsilon$. Let $\Upsilon\text{-mch}((\sigma, w))$ denote the number of Υ -matches in (σ, w) . We say that Υ has the *$C_k \wr S_n$ -minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with $\Upsilon\text{-mch}((\sigma, w)) = 2$ is $2j-1$. If Υ is a set of colored permutations in $C_k \wr S_n$, then we say that $(\sigma, w) \in C_k \wr S_n$ where $\sigma = \sigma_1 \dots \sigma_n$ and $w = w_1 \dots w_n$ has an exact Υ -match starting at position i if $(\text{pred}(\sigma_i \dots \sigma_{i+j-1}), w_i \dots w_{i+j-1}) \in \Upsilon$. Let $E\Upsilon\text{-mch}((\sigma, w))$ denote the number of exact Υ -matches in (σ, w) . We say that Υ has the *$C_k \wr S_n$ -exact match minimal overlapping property* if the smallest i such that there exists a $(\sigma, w) \in C_k \wr S_i$ with $E\Upsilon\text{-mch}((\sigma, w)) = 2$ is $2j-1$.

It should be clear that the obvious analogues of Theorem 5 in Section 3 for words $u \in \{0, 1, \dots, k-1\}^j$ which have the k -minimal overlapping or the k -exact match minimal overlapping property also holds for sets of words $\Upsilon \subseteq \{0, 1, \dots, k-1\}^j$ which have the k -minimal overlapping or the k -exact match minimal overlapping property with the same proofs. Similarly, the obvious analogues of Theorem 6 in Section 3 for k -colored permutations $(\tau, u) \in C_k \wr S_j$ which have the k -minimal overlapping or the k -exact match minimal overlapping property also hold for sets of k -colored permutations $\Upsilon \subseteq C_k \wr S_j$ which have the k -minimal overlapping or the k -exact match minimal overlapping property with the same proofs.

We also note that the proofs of Theorems 5 and 6 did not depend on the fact that we used a finite alphabet $\{0, 1, \dots, k-1\}$. That is, if $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers, then we can extend our definitions to words or sets of words in $\mathbb{N}^*$ and $\mathbb{N}$ -colored permutations or sets of $\mathbb{N}$ -colored permutations in the obvious way. That is, if $\Upsilon \subseteq \mathbb{N}^j$ has the $\mathbb{N}$ -minimal overlapping property (the $\mathbb{N}$ -exact match minimal overlapping property),

then we let $\mathcal{MP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}$ ($\mathcal{EMP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}$) denote the set of all $w \in \mathbb{N}^{n(j-1)+1}$ which are maximum packings (exact match maximum packings) for Υ . We let $C_{\mathbb{N}} \wr S_n$ denote the set of all pairs (σ, w) such $\sigma \in S_n$ and $w \in \mathbb{N}^n$ and we call the elements of $C_{\mathbb{N}} \wr S_n$ $\mathbb{N}$ -colored permutations. Then if $\Upsilon \subseteq C_{\mathbb{N}} \wr S_n$ has the $C_{\mathbb{N}} \wr S_n$ -minimal overlapping property (the $C_{\mathbb{N}} \wr S_n$ -exact match minimal overlapping property), then we let $\mathcal{MP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}$ ($\mathcal{EMP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}$) denote the set of all $(\sigma, w) \in C_{\mathbb{N}} \wr S_n$ which are maximum packings (exact match maximum packings) for Υ . Then we define

$$\begin{aligned} mp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(z_0, z_1, \dots) &= \sum_{w \in \mathcal{MP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}} z(w), \\ emp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(z_0, z_1, \dots) &= \sum_{w \in \mathcal{EMP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}} z(w), \\ mp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(p, q, z_0, z_1, \dots) &= \sum_{(\sigma, w) \in \mathcal{MP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) \text{ and} \\ emp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(p, q, z_0, z_1, \dots) &= \sum_{(\sigma, w) \in \mathcal{EMP}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w). \end{aligned}$$

Then if $j \geq 3$, the following hold.

(I) ^{$\mathbb{N}$} If $\Upsilon \subseteq \mathbb{N}^j$ has the $\mathbb{N}$ -minimal overlapping property and is such that for all $u \in \Upsilon$, $\text{red}(u) = u$, then

$$\sum_{n \geq 0} t^n \sum_{w \in \mathbb{N}^n} x^{\Upsilon\text{-mch}(w)} z(w) = \frac{1}{1 - ((\sum_{i \geq 0} z_i)t + \sum_{n \geq 1} t^{n(j-1)+1} (x-1)^n mp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(z_0, z_1, \dots))}. \quad (52)$$

(II) ^{$\mathbb{N}$} If $\Upsilon \subseteq \mathbb{N}^j$ has the k -exact match minimal overlapping property, then

$$\sum_{n \geq 0} t^n \sum_{w \in \mathbb{N}^n} x^{E\Upsilon\text{-mch}(w)} z(w) = \frac{1}{1 - ((\sum_{i \geq 0} z_i)t + \sum_{n \geq 1} t^{n(j-1)+1} (x-1)^n emp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(z_0, z_1, \dots))}. \quad (53)$$

(III) ^{$\mathbb{N}$} If $\Upsilon \subseteq C_{\mathbb{N}} \wr S_j$ has the $C_{\mathbb{N}} \wr S_n$ -minimal overlapping property and is such that for all $(\tau, u) \in \Upsilon$, $\text{red}(u) = u$, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_{\mathbb{N}} \wr S_n} x^{\Upsilon\text{-mch}((\sigma, w))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((\sum_{i \geq 0} z_i)t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n mp_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(p, q, z_0, z_1, \dots))}. \quad (54)$$

(IV)^N If $\Upsilon \subseteq C_{\mathbb{N}} \wr S_j$ has the $C_{\mathbb{N}} \wr S_n$ -exact match minimal overlapping property, then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{(\sigma, w) \in C_{\mathbb{N}} \wr S_n} x^{E\Upsilon\text{-mch}((\sigma, u))} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} z(w) = \frac{1}{1 - ((\sum_{i \geq 0} z_i)t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_q!} (x-1)^n \text{emp}_{\Upsilon, n(j-1)+1}^{\mathbb{N}}(p, q, z_0, z_1, \dots))}. \quad (55)$$

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