# The Double Riordan Group

Dennis E. Davenport*
Mathematics Department
Howard University, Washington, DC
dennis.davenport@howard.edu

Louis W. Shapiro
Mathematics Department
Howard University, Washington, DC
lshapiro@howard.edu

Leon C. Woodson
Department of Mathematics
Morgan State University, Baltimore, MD
leon.woodson@morgan.edu

Submitted: Sep 28, 2011; Accepted: Jan 31, 2012; Published: Feb 7, 2012
Mathematics Subject Classifications: 05A05, 05A10

**Abstract**

The Riordan group is a group of infinite lower triangular matrices that are defined by two generating functions, $g$ and $f$. The kth column of the matrix has the generating function $gf^k$. In the Double Riordan group there are two generating function $f_1$ and $f_2$ such that the columns, starting at the left, have generating functions using $f_1$ and $f_2$ alternately. Examples include Dyck paths with level steps of length 2 allowed at even height and also ordered trees with differing degree possibilities at even and odd height(perhaps representing summer and winter). The Double Riordan group is a generalization not of the Riordan group itself but of the checkerboard subgroup. In this context both familiar and far less familiar sequences occur such as the Motzkin numbers and the number of spoiled child trees. The latter is a slightly enhanced cousin of ordered trees which are counted by the Catalan numbers.

## 1 Introduction

This article is simply dedicated to Doron on the occasion of his $|A_5|$th birthday.

In 1991, Shapiro, Getu, Woan, and Woodson introduced a group of infinite lower triangular matrices called the Riordan group, see [5]. Since then about a hundred papers

*The first author acknowledges support received from the National Science Foundation via Independent Research Development while serving as a Visiting Scientist at NSF.

have appeared that involve the Riordan group, see [7], and typing Riordan group or Riordan arrays into the Google search engine gives about 11,200 hits, albeit with considerable duplications. The elements of the group are defined by two power series $g$ and $f$, where the coefficients of $g$ gives the left most column and the $i^{th}$ column is given by the coefficients of $g \cdot f^i$, for $i = 0, 1, 2, 3, \ldots$. We call $f$ the multiplier function. A natural question to ask is, "what happens when more than one multiplier function is given?" We investigate this question for two multiplier functions, $f_1$ and $f_2$. In Section 2, we define the Double Riordan Group, in Section 3 the group structure is investigated, and in Section 4 combinatorial applications are given.

In this paper we consider ordinary generating functions. However, the techniques used apply to exponential generating functions as well.

Before defining the double Riordan group we define the Riordan group, state the Fundamental Theorem of Riordan Arrays and give some examples of elements in the Riordan group.

Let $g(z) = 1 + \sum_{k=1}^{\infty} g_k z^k$ and $f(z) = \sum_{k=1}^{\infty} f_k z^k$, where $f_1 \neq 0$. Let $d_{n,k}$ be the coefficient of $z^n$ in $g(z)(f(z))^k$. Then $D = (d_{n,k})_{n,k \geq 0}$ is a Riordan array and an element of the Riordan group. We write $D = (g(z), f(z))$.

**Example 1.1:** The identity matrix is

$$(1, z) = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \\ & & \cdots & & \end{bmatrix}$$

**Example 1.2:** Pascal's matrix is

$$(\frac{1}{1 - \mathrm{z}}, \frac{\mathrm{z}}{1 - \mathrm{z}}) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \cdots & & \end{bmatrix}$$

**Example 1.3:** The Fibonacci matrix with Pascal like columns and Fibonacci row sums is

$$(1, z(1+z)) = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 2 & 1 & \\ 0 & 0 & 1 & 3 & 1 \\ & & \cdots & & \end{bmatrix}$$

**Theorem 1.** ***(Fundamental Theorem of Riordan Arrays):*** *Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ and $B(z) = \sum_{k=0}^{\infty} b_k z^k$ and let $A$ and $B$ be the column vectors $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$ . Then*
*$(g, f)A = B$, if and only if $B(z) = g(z)A(f(z))$.*

**Theorem 2.** *Let $(g, f)$ and $(G, F)$ be two Riordan arrays. Then the operation *, given by $(g, f) * (G, F) = (g(z)G(f(z)), F(f(z)))$ is matrix multiplication which is an associative binary operation, $(1, z)$ is the identity element and the inverse of $(g, f)$ is$(\frac{1}{g(\overline{f})}, \overline{f})$, where $\overline{f}$ is the compositional inverse of $f$.*

With the Fundamental Theorem of Riordan Arrays in hand one can easily prove many combinatorial identities and the group structure gives a systematic way to invert identities. Also, other topics such as the Stieltjes transform, Hankel matrix decomposition, and determinant sequences can be developed using the Riordan Group.

The Riordan Group has many interesting and important subgroups. The set of all elements $(g, f)$, such that $g$ is even (with leading one) and $f$ odd, is called the Checkerboard Subgroup. The terminology comes from the fact that $(g, f)$ has the appearance of a checkerboard. We also say that a generating function or an array is aerated if it has alternating zeros. Other subgroups are defined in Section 3.

# 2 Double Riordan Arrays

In a Riordan array we use one multiplier function. Hence, to move from one column to the next we multiply by $f$ to make the change. Suppose alternating rules are used to generate an infinite matrix similar to a Riordan array. To consider this case we use two multiplier functions. So, if $g$ gives column zero and $f_1$ and $f_2$ are the multiplier functions, then the first column is $gf_1$, the second is $gf_1f_2$, the third is $gf_1f_2f_1$, and so on. In general the set of double Riordan arrays is not closed under multiplication. However if we require that $g$ be an even function and $f_1$ and $f_2$ be odd functions we can develop an analog of the fundamental theorem and thus obtain a group structure.

**Definition 1.** Let

$$g(z) = \sum_{k=0}^{\infty} g_{2k} z^{2k}, \quad f_1(z) = \sum_{k=0}^{\infty} f_{1,2k+1} z^{2k+1}, \text{ and } f_2(z) = \sum_{k=0}^{\infty} f_{2,2k+1} z^{2k+1}.$$

Then the double Riordan matrix (or array) of $g$, $f_1$ and $f_2$, denoted by $(g; f_1, f_2)$, has column vectors

$$(g,\ gf_1,\ gf_1f_2, gf_1^2f_2, gf_1^2f_2^2,\ \cdots),$$

The set of all aerated double Riordan matrices is denoted as $\mathcal{DR}$.

**Theorem 3.** ***(The Fundamental Theorem of Double Riordan Arrays):*** *Let* $g(z) = \sum_{k=0}^{\infty} g_{2k} z^{2k}$*,* $f_1(z) = \sum_{k=0}^{\infty} f_{1,2k+1} z^{2k+1}$*, and* $f_2(z) = \sum_{k=0}^{\infty} f_{2,2k+1} z^{2k+1}$*.*

*Case 1: If* $A(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}$ *and* $B(z) = \sum_{k=0}^{\infty} b_{2k} z^{2k}$*, and* $A = (a_0, 0, a_2, 0, \cdots)^T$ *and* $B = (b_0, 0, b_2, 0, \cdots)^T$ *are column vectors. Then* $(g, f_1, f_2)A = B$ *if and only if* $B(z) = g(z)A(\sqrt{f_1(z) f_2(z)})$*.*

*Case 2: If* $A(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}$ *and* $B(z) = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}$ *with* $(g, f_1, f_2)A = B$ *then* $B(z) = g(z)\sqrt{f_1/f_2}A(\sqrt{f_1(z) f_2(z)})$*.*

*Proof.* Case 1: $A(z)$ is an even function. Then

$$(g, f_1, f_2)A = \begin{bmatrix} \uparrow & 0 & 0 & 0 & 0 & 0 & \\ & \uparrow & 0 & 0 & 0 & 0 & \\ & & \uparrow & 0 & 0 & 0 & \\ & & & \uparrow & 0 & 0 & \cdots \\ g & gf_1 & gf_1 f_2 & gf_1^2 f_2 & \uparrow & 0 & \\ \downarrow & \downarrow & \downarrow & \downarrow & & \ddots & \\ & & & \vdots & & & \end{bmatrix} \begin{bmatrix} a_0 \\ 0 \\ a_2 \\ 0 \\ a_4 \\ 0 \\ a_6 \\ \cdots \end{bmatrix} \Leftrightarrow$$

$$(a_0 g + a_2 g f_1 f_2 + a_4 g f_1^2 f_2^2 + ...) = g(a_0 + a_2 f_1 f_2 + a_4 f_1^2 f_2^2 + ...) = g(z)A(\sqrt{f_1(z) f_2(z)})$$

Case 2: $A(z)$ is an odd function. Similarly

$$\begin{aligned} &(g, f_1, f_2)A \iff (a_1 g f_1 + a_3 g f_1^2 f_2 + a_5 g f_1^3 f_2^2 + ...) \\ &= g(a_1 f_1 + a_3 f_1^2 f_2 + a_5 f_1^3 f_2^2 + ...) = (g/f_2)(a_1 f_1 f_2 + a_3 f_1^2 f_2^2 + a_5 f_1^3 f_2^3 + ...) \\ &= (g/f_2)(a_1 \sqrt{f_1 f_2}^2 + a_3 \sqrt{f_1 f_2}^4 + a_5 \sqrt{f_1 f_2}^6 + ...) \\ &= (g/f_2)\sqrt{f_1 f_2}(a_1 \sqrt{f_1 f_2} + a_3 \sqrt{f_1 f_2}^3 + a_5 \sqrt{f_1 f_2}^5 + ...) \\ &= g(z)\sqrt{f_1(z)/f_2(z)}A(\sqrt{f_1(z) f_2(z)}) \end{aligned}$$

□

Using the Fundamental Theorem of Double Riordan Arrays, we can define a binary operation on $\mathcal{DR}$. The development of this algebraic structure is similar to what was done with single Riordan arrays.

**Definition 2.** Let $(g, f_1, f_2)$ and $(G, F_1, F_2)$ be elements of $\mathcal{DR}$. Then

$$(g; f_1, f_2)(G; F_1, F_2) = (gG(\sqrt{f_1 f_2}); \sqrt{f_1/f_2}F_1(\sqrt{f_1 f_2}), \sqrt{f_2/f_1}F_2(\sqrt{f_1 f_2})).$$

The following theorem is analogous to Theorem 2.

**Theorem 4.** $(\mathcal{DR}, *)$ *is a group.*

*Proof.* The matrix $(1; z, z)$ is the identity. Matrix multiplication is associative.

Let $(g; f_1, f_2)$ be in $\mathcal{DR}$ and let $h = \sqrt{f_1 f_2}$ and also denote by $\bar{h}$ the compositional inverse of $h$. Then $((1/g(\bar{h}); z\bar{h}/f_1(\bar{h}), z\bar{h}/f_2(\bar{h}))$ is the inverse of $(g; f_1, f_2)$. □

**Example 2.2:** If $(g, f)$ is an element of the Riordan group with $g$ even and $f$ odd then $(g; f, f) = (g, f)$ is an element of $\mathcal{DR}$. In particular $(1, z) = (1; z, z)$ is the identity matrix. In fact, the mapping $(g; f) \to (g; f, f)$ is an isomorphism from the Checkerboard subgroup of the Riordan Group to $\mathcal{DR}$.

The following theorem shows how to find the generating function, $\Sigma(z)$, for the row sums of a double Riordan array.

**Theorem 5.** *Let $D = (g(z); f_1(z), f_2(z))$ be a double Riordan array. Then*

$$\Sigma(z) = \frac{g(1+f_1)}{1-f_1f_2} = \frac{g}{1-f_1f_2} + \frac{gf_1}{1-f_1f_2}$$

*Proof.* According to the Fundamental Theorem of Double Riordan Arrays (Theorem 3), we should distinguish the sums of even and odd rows, since they are associated with different formulas. Thus $B(z) = B_E(z) + B_O(z)$.
For even rows, $A_E(z) = (1-z^2)^{-1} = 1 + z^2 + z^4 + z^6 + ...$, so we get

$$B_E(z) = \frac{g(z)}{1 - f_1(z)f_2(z)}.$$

For odd rows, $A_O(z) = z\,(1-z^2)^{-1} = z + z^3 + z^5 + z^7 + ...$ and we get

$$B_O(z) = \frac{g(z)f_1(z)}{1 - f_1(z)f_2(z)}.$$

Hence,

$$\Sigma(z) = B_E(z) + B_O(z) = \frac{g(z)}{1 - f_1(z)f_2(z)} + \frac{g(z)f_1(z)}{1 - f_1(z)f_2(z)}.$$

□

# 3 Special Subgroups

For any functions $f$ and $g$ such that $(g, f)$ is a Riordan array, we can map $(g(z), f(z))$ to $(g(z^2); \frac{f(z^2)}{z}, \frac{f(z^2)}{z})$. Hence, the Riordan group, $\mathcal{R}$, can be mapped one-to-one onto a subset of $\mathcal{DR}$. This will not give us an isomorphism, consider the Pascal matrix squared. Question: Is there a subgroup of the double Riordan group that is isomorphic to the Riordan group? Since this obvious approach does not work it remains an open question.

Many subgroups of $\mathcal{R}$ have been studied, both for their combinatorial and algebraic properties, for example the Bell subgroup which is given by $\{(g, f) \in \mathcal{R} : f = zg\} = \{(g, zg)\}$, the Associated subgroup, given by $\{(g, f) \in \mathcal{R} : g = 1\} = \{(1, f)\}$, and the Appel subgroup, given by $\{(g, f) \in \mathcal{R} : f = z\} = \{(g, z)\}$. Of these subgroups of $\mathcal{R}$, the Appel subgroup is normal. Note that for all functions $f$ and $g$, $(g, f) = (g, z)(1, f)$. Thus $\mathcal{R}$ is the semidirect product of the Associated and Appel subgroups. A natural question is, can we find a similar subgroup structure in $\mathcal{DR}$?

**Theorem 6.** *Let $\mathcal{A}$=$\{(g; f_1, f_2) \in \mathcal{DR} : g = 1\}$ and*
*$\mathcal{B}_1$=$\{(g; f_1, f_2) \in \mathcal{DR} : f_1 = zg\}$ and $\mathcal{B}_2$=$\{(g; f_1, f_2) \in \mathcal{DR} : f_2 = zg\}$. Then $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{A}$ are subgroups of $\mathcal{DR}$.*

**Theorem 7.** *$\{(g;z,z)\in\mathcal{R}\}$ is a normal subgroup of $\mathcal{DR}$ and $\mathcal{DR}$ is the semidirect product of $\{(g;z,z)\in\mathcal{R}: f=z\}$ and $\mathcal{A}$.*

We see that $\mathcal{DR}$ has some of the same subgroup properties as $\mathcal{R}$.

# 4 Combinatorial Examples

In this section we look at three places where double Riordan arrays occur. Sloane's EIS [6] is an excellent source for some of these examples and the A-numbers mentioned refer to this source.

**Example 1:** Consider ordered trees with no branch points at odd heights. This is discussed in [3]. There is another view that is more applied. Consider a plant species that reproduces in the summer but during the winter at most survives. Thus a vertex at odd (or winter) height has outdegree 0 or 1. Let the generating function be $m(z)$ or simply $m$. Then by looking at the left most branch we get, $m=1+zm+z^2m^2$. Here 1 counts the trivial tree consisting of just the root, $zm$ counts trees whose left most principal subtree is just one edge, and $z^2m^2$ counts trees whose left hand branch has at least two edges. This is the generating function equation that defines the Motzkin numbers. See [3] where there is a bijective proof involving complete binary trees, and [2] and [8] for more information about the Motzkin numbers.

If we convert these ordered trees to Dyck paths we have Dyck paths with no valley at odd height and $z$ becomes $z^2$, since down and up edges are both counted. We define $M(z)=m(z^2)$ and the key generating function identity becomes $M=1+z^2M+z^4M^2$ and $M=1+z^2+2z^4+4z^6+9z^8+21z^{10}+...,A001006$. Thus

$$M=\frac{1-z^2-\sqrt{1-2z^2-3z^4}}{2z^4}$$

If we look at paths with no valleys at odd heights, but ending at height $k$, we get the following matrix where $d(n,k)$ is the number of paths with $n$ edges that end at height $k$.

$$D=(d(n,k))_{n,k\geq 0}=\begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 3 & 0 & 1 \\ 4 & 0 & 5 & 0 & 3 & 0 & 1 \\ 0 & 9 & 0 & 8 & 0 & 4 & 0 & 1 \\ 9 & 0 & 12 & 0 & 9 & 0 & 4 & 0 & 1 \\ 0 & 21 & 0 & 21 & 0 & 13 & 0 & 5 & 0 & 1 \\ 21 & 0 & 30 & 0 & 25 & 0 & 14 & 0 & 5 & 0 & 1 \\ & & & & \cdots \\ \end{bmatrix}$$

We want to show that $D$ is a double Riordan array. First we look at the $k^{th}$ column where $k = 2m$ is even. In this column we have the recurrence $d(n,k) = d(n-2,k-2) + d(n-1,k+1)$ since any path ending at height $k$ after $n$ steps must have finished either with a down step or, to avoid a valley at odd height, two up steps. With a finish at odd height with $k = 2m+1$ we obtain the simpler $d(n,k) = d(n-1,k-1) + d(n-1,k+1)$. Thus, there exist generating functions $g$, $f_1$, and, $f_2$ such that $D = (g; f_1, f_2)$. Note that if $l = 2n$ where $n$ is any positive integer, then a recurrence relation for the $l^{th}$ column of $D$ is $g(f_1f_2)^n = z^2g(f_1f_2)^{n-1} + zg(f_1f_2)^n f_1$. Hence $f_1f_2 = z^2 + z{f_1}^2 f_2$.

Likewise, if $l = 2n+1$ where $n$ is any non-negative integer, then a recurrence relation of the $l^{th}$ column of $D$ is given by $g(f_1f_2)^n f_1 = zg(f_1f_2)^n + zg(f_1f_2)^{n+1}$. Thus $f_1 = z + zf_1f_2$. If we set $H = f_1f_2/z^2$ these equations become

$$f_1 = z\left(1 + z^2H\right) \text{ and } H = 1 + zf_1H$$

Thus $H = 1 + zH\left(z + z^3H\right)$ so that

$$H = 1 + z^2H + z^4H^2$$

and since this is the defining equation for $M$ we have

$$H = \frac{f_1f_2}{z^2} = M.$$

Then

$$f_1 = z\left(1 + z^2H\right) = z\left(1 + z^2M\right).$$

Next we have that $z\left(1 + z^2M\right) f_2 = z^2M$, so that

$$\begin{aligned} f_2 &= \frac{zM}{1+z^2M} \cdot \frac{1+z^2M}{1+z^2M} \\ &= \frac{zM\left(1+z^2M\right)}{1+z^2M+z^4M^2+z^2M} \\ &= \frac{zM\left(1+z^2M\right)}{M+z^2M} \\ &= \frac{z\left(1+z^2M\right)}{1+z^2} \\ &= z + z^5 + z^7 + 3z^9 + 6z^{11} + 15z^{13}..., \text{(Riordan numbers) } A005043 \end{aligned}$$

To find the row sums for $D$, we apply Theorem 5 and get

$$\Sigma(z) = \frac{M}{1-z^2M} + \frac{f_1M}{1-z^2M}.$$

If $E$ is the even part of $\Sigma(z)$ and $O$ the odd part, then $E = \frac{M}{1-z^2M}$ and $O = \frac{f_1M}{1-z^2M}$.

Further, it can be shown that

$$\begin{aligned} E &= \frac{f_1}{z\sqrt{1-2z^2-3z^4}} \\ &= 1+2z^2+5z^4+13z^6+35z^8+..., \text{(directed animals) } A005773 \end{aligned}$$

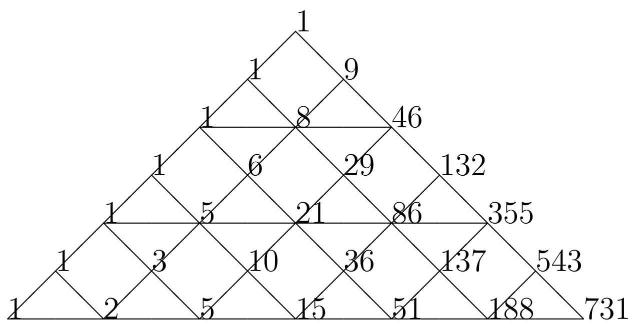
and

$$\begin{aligned} O &= \frac{1-z^2}{2z^3}\sqrt{\frac{1+z^2}{1-3z^2}} - \frac{1+z^2}{2z^3} \\ &= z+3z^3+8z^5+22z^7+61z^9+..., A025566 \end{aligned}$$

Thus

$$\begin{aligned} \Sigma(z) &= E(z)+O(z) = \frac{1+z-z^2}{2z^3}\sqrt{\frac{1+z^2}{1-3z^2}} - \frac{1+z+z^2}{2z^3} \\ &= 1+z+2z^2+3z^3+5z^4+8z^5+13z^6+22z^7+35z^8+61z^9+.... \end{aligned}$$

**Example 2:** Consider Schröder paths with no level steps at odd height.




Arranging these numbers as a lower triangular array we get the following $DR$ matrix.

$$S = \begin{bmatrix} 1 \\ 0 & 1 \\ 2 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 5 & 0 & 5 & 0 & 1 \\ 0 & 10 & 0 & 6 & 0 & 1 \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 \\ 0 & 36 & 0 & 29 & 0 & 9 & 0 & 1 \\ 51 & 0 & 86 & 0 & 46 & 0 & 11 & 0 & 1 \\ 0 & 137 & 0 & 132 & 0 & 57 & 0 & 12 & 0 & 1 \\ 188 & 0 & 355 & 0 & 235 & 0 & 80 & 0 & 14 & 0 & 1 \\ & & & & \cdots \end{bmatrix}$$

We have the following relations.

$$\begin{aligned} g &= 1 + z^2 g + zgf_1 \\ gf_1 &= zg + zgf_1f_2 \implies f_1 = z + zf_1f_2 \\ gf_1f_2 &= zgf_1 + z^2gf_1f_2 + zg{f_1}^2f_2 \implies f_2 = z + z^2f_2 + zf_1f_2 \end{aligned}$$

Solving this system of equations we get the following for $f_1$, $f_2$, and $g$.

$$\begin{aligned} g &= \frac{1 - z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z^2(1 - z^2)} \\ &= 1 + 2z^2 + 5z^4 + 15z^6 + 51z^8 + ... \end{aligned}$$

$$\begin{aligned} f_1 &= \frac{1 - z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z} \\ &= z + z^3 + 3z^5 + 10z^7 + 36z^9 + 137z^{11}..., \text{(hex numbers) } A002212 \end{aligned}$$

$$\begin{aligned} f_2 &= \frac{1 - z^2 - \sqrt{1 - 6z^2 + 5z^4}}{2z(1 - z^2)} \\ &= z + 2z^3 + 5z^5 + 15z^7 + 51z^9 + ..., A0007317 \end{aligned}$$

Using Theorem 5, we can find the generating function of the row sums for $S$. We get

$$\begin{aligned} E &= \frac{1}{\sqrt{1 - 6z^2 + 5z^4}} \\ &= 1 + 3z^2 + 11z^4 + 45z^6 + ..., A026375 \end{aligned}$$

and

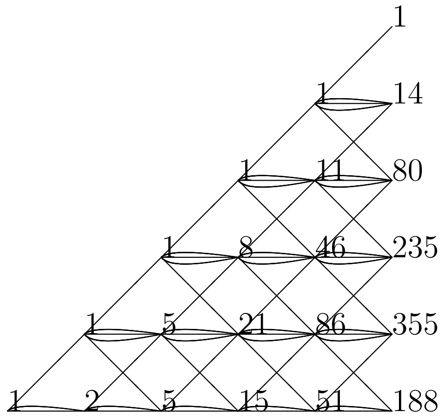
$$\begin{aligned} O &= \frac{1}{2z}\left(\frac{1 - z^2}{\sqrt{1 - 6z^2 + 5z^4}} - 1\right) \\ &= z + 4z^3 + 17z^5 + 75z^7 + 339z^9 + ..., A026378 \end{aligned}$$

Thus

$$\begin{aligned} \Sigma(z) &= \frac{1 + 2z - z^2}{2z\sqrt{1 - 6z^2 + 5z^4}} - \frac{1}{2z} \\ &= 1 + z + 3z^2 + 4z^3 + 11z^4 + 17z^5 + 45z^6 + 75z^7 + ... \end{aligned}$$

The sequence 1, 2, 5, 15, 51, 188 ..., appears in a variety of combinatorial settings. We will discuss three of these settings briefly and put the details in another paper. Also see [6].

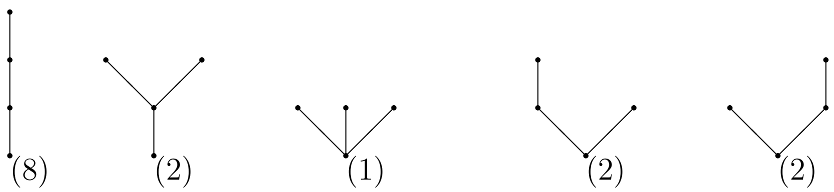
**a.** $3_2$-Motzkin path is one where level steps come in three colors except at level 0, where only 2 colors are permitted. We get the following lattice diagram.




**b.** The Spoiled Child Tree: Consider rooted trees with the following condition. If an edge (child) has no siblings (comes from a vertex of outdegree 1) we allow two possibilities, the child can be spoiled or not. Thus if $R$ (for spoiled rotten) is the generating function counting Spoiled Child Trees, then

$$\begin{aligned} R(z) &= 1 + 2zR(z) + z^2R^2(z) + z^3R^3(z) + ... \\ R(z) &= zR(z) + \frac{1}{1 - zR(z)} \end{aligned}$$

Spoiled Child trees when $n = 3$ are illustrated below.

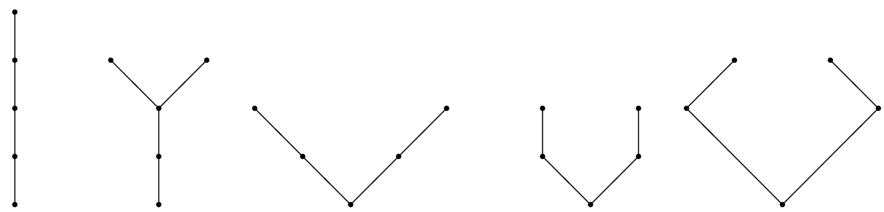



Note that there are 15 Spoiled Child Trees when $n = 3$, since there are two choices for each edge which comes from a vertex with outdegree 1. **c.** Consider Hex trees. The term

Hex refers to ways of joining hexagons or benzene rings together, see [4] for details. Hex trees are rooted trees satisfying the following conditions. The outdegree of any vertex is 0, 1, or 2 and if 1, then the edge is either left, center, or right. If the outdegree is 2, then the children are left and right (this keeps 3 hexagons from meeting in a common point). Hence we get $H(z) = 1 + 3zH(z) + z^2H^2(z)$. Solving for $H$ yields

$$\begin{aligned} H(z) &= \frac{1 - 3z - \sqrt{1 - 6z + 5z^2}}{2z^2} \\ &= 1 + 3z + 10z^2 + 36z^3 + 137z^4 + ..., A002212 \end{aligned}$$

A Symmetric Hex tree is a Hex tree which is symmetric with respect to the vertical line passing through the root. We define Symmetric Hex trees to have an even number of edges. The Symmetric Hex trees when $n = 4$ are illustrated below.



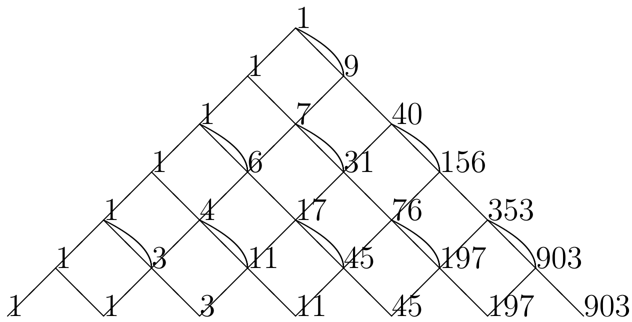
If $\widehat{H}$ denotes the generating function counting Symmetric Hex trees then,

$$\begin{aligned}\widehat{H}(z) &= 1 + z^2\widehat{H}(z) + z^2H(z^2)\\ \widehat{H}(z) &= \frac{1}{(1-z^2)} + \frac{z^2}{(1-z^2)}H(z^2)\end{aligned}$$

Here $\frac{1}{1-z^2}$ is the generating function for the vertical stem below the split. The next theorem can be proved by using generating functions.

**Theorem 8.** $\widehat{H}(z) = R(z)$

**Example 3:** Consider Dyck paths with two choices for down steps that start at even heights. This example is a small modification of an array dealing with a Schröder version of the tennis ball problem, [1]




From here we get the following matrix.

$$S = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 4 & 0 & 1 \\ 0 & 11 & 0 & 6 & 0 & 1 \\ 11 & 0 & 17 & 0 & 7 & 0 & 1 \\ 0 & 45 & 0 & 31 & 0 & 9 & 0 & 1 \\ 45 & 0 & 76 & 0 & 40 & 0 & 10 & 0 & 1 \\ 0 & 197 & 0 & 156 & 0 & 60 & 0 & 12 & 0 & 1 \\ 197 & 0 & 353 & 0 & 216 & 0 & 72 & 0 & 13 & 0 & 1 \\ & & & & \cdots \\ \end{bmatrix}$$

It can be shown that the numbers in the zeroth column are the aerated little Schröder numbers. Hence $g = s$, where

$$s = \frac{1 + z^2 - \sqrt{1 - 6z^2 + z^4}}{4z^2}.$$

From the first column we get

$$gf_1z + 1 = g$$

Recall that

$$s = \frac{1}{1 - z^2 r} = 1 + z^2 + 3z^4 + 11z^6 + ...$$

Where $r$ is the generating function for the aerated big Schröder numbers, so that

$$r = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2} = 1 + 2z^2 + 6z^4 + 22z^6 + ..., A006318$$

Also note that

$$s = \frac{r+1}{2}.$$

Hence,

$$\begin{aligned} sf_1z + 1 &= s \\ s &= \frac{1}{1 - zf_1} \\ \therefore \frac{1}{1 - zf_1} &= \frac{1}{1 - z^2 r}. \end{aligned}$$

Thus

$$f_1 = zr.$$

In terms of generating functions, the path requirements translate to

$$gf_1 = z(g + 2gf_1f_2) \implies f_1 = z(1 + 2f_1f_2)$$

and

$$gf_1f_2 = z(gf_1 + g{f_1}^2 f_2) \implies f_2 = z(1 + f_1f_2).$$

Hence,

$$\begin{aligned} f_2 &= z + \frac{f_1 - z}{2} = z + \frac{zr - z}{2} = \frac{z}{2}(1 + r) = zs \\ &= 1 + z^3 + 3z^5 + 11z^7 + 45z^9 + ..., A001003. \end{aligned}$$

For the row sums $\Sigma(z)$, of $S$, we again use Theorem 5 to get

$\Sigma(z) = \frac{(1+2z)\sqrt{1-6^2+z^4}-1+5z^2-2z^3}{2z(1-6z^2)} = 1 + z + 2z^2 + 4z^3 + 8z^4 + 18z^5 + 36z^6 + 86z^7 + 172z^8...$

Is there a triple Riordan Group? Yes, with $g, \frac{f_1}{z}, \frac{f_2}{z}, \frac{f_3}{z}$ all invertible elements of $\mathbb{C}[z^3]$ with nonzero leading term. One example is the Riordan group element $(C(z^3); zC(z^3))$ which has the property that $(C(z^3); zC(z^3))^{-1} = (C(-z^3); zC(-z^3))$. The equivalent of $h = \sqrt{f_1 f_2}$ is $h = (f_1 f_2 f_3)^{\frac{1}{3}}$. For each positive integer $k$, the extension to the $k$-tuple Riordan group is similar.

We would like to thank the referee for a careful reading and several very helpful suggestions.

# References

[1] Cameron, N. and Matthews, L., A Motzkin variation on the tennis ball problem (to appear in Ars Combinatoria)

[2] Donaghey, R. and Shapiro, L. W., A Survey of the Motzkin Numbers. *J. Combinatorial Theroy (A)*, (1977), 291-301

[3] Kuznetkov, A., Pak, I., and Postnikov, A., Trees Associated with the Motzkin Numbers. *Journal of Combinatorial Theory, Series A*, 76, (1996), 145-147

[4] Harary, F. and Read, R. C., The Enumeration of Tree-Like Polyhexes. *Proc. Edinburgh Math. Soc.* 17, (1970), 1-13.

[5] Shapiro, L. W., Getu, S., Woan, W. and, Woodson, L. C., The Riordan Group. *Discrete Applied Mathematics* 34 (1991), 229-239.

[6] *Sloane's Online Encyclopedia of Integer Sequences*, www.research.att.com/njas/sequences/

[7] Sprugnoli, R. *A bibliography on Riordan arrays*,www.dsi.unifi.it/~resp/BibRioMio.pdf

[8] Stanley, R.P., *Enumerative Combinatorics*, Vol. 2, Cambridge University Press (1999)