Bivariate identities for values of the Hurwitz zeta function and supercongruences

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Submitted: Apr 17, 2011; Accepted: Feb 14, 2012; Published: Feb 23, 2012
Mathematics Subject Classifications: 33F10, 05A30, 11B65, 05A19, 33D15, 11M06

Dedicated to Doron Zeilberger on the occasion of his 60th birthday

Abstract

In this paper, we prove a new identity for values of the Hurwitz zeta function which contains as particular cases Koecher's identity for odd zeta values, the Bailey-Borwein-Bradley identity for even zeta values and many other interesting formulas related to values of the Hurwitz zeta function. We also get an extension of the bivariate identity of Cohen to values of the Hurwitz zeta function. The main tool we use here is a construction of new Markov-WZ pairs. As application of our results, we prove several conjectures on supercongruences proposed by J. Guillera, W. Zudilin, and Z. W. Sun.

1 Introduction

The Riemann zeta function for $\mathrm{Re}(s) > 1$ is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = {}_{s+1}F_s\left(\left.\begin{matrix} 1, \dots, 1 \\ 2, \dots, 2 \end{matrix}\right| 1\right),$$

where

$${}_pF_q\left(\left.\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

is the generalized hypergeometric function and $(a)_n$ is the shifted factorial defined by $(a)_n = a(a+1)\cdots(a+n-1)$, $n \ge 1$, and $(a)_0 = 1$.

*[1] This research was in part supported by a grant from IPM (No. 90110029)
†[2] This research was in part supported by a grant from IPM (No. 90110030)

In 1978, R. Apéry used the faster convergent series for $\zeta(3)$,

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \tag{1}$$

to derive the irrationality of this number [21]. The series (1) first obtained by A. A. Markov [17] in 1890 converges exponentially faster than the original series for $\zeta(3)$, since by Stirling's formula,

$$\frac{1}{k^3\binom{2k}{k}}\sim\frac{\sqrt{\pi}}{k^{5/2}}4^{-k} \qquad (k\to+\infty).$$

A general formula giving analogous Apéry-like series for all $\zeta(2n+3)$, $n\ge 0$, was proved by Koecher [15] (and independently in an expanded form by Leshchiner [16]). For $|a|<1$, it reads

$$\sum_{n=0}^{\infty}\zeta(2n+3)a^{2n}=\sum_{k=1}^{\infty}\frac{1}{k(k^2-a^2)}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\frac{5k^2-a^2}{k^2-a^2}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right). \tag{2}$$

Expanding the right-hand side of (2) by powers of a^2 and comparing coefficients of a^{2n} on both sides leads to the Apéry-like series for $\zeta(2n+3)$ [16]:

$$\zeta(2n+3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}(-1)^n e_n^{(2)}(k)+2\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2j+3}\binom{2k}{k}}(-1)^{n-j}e_{n-j}^{(2)}(k), \tag{3}$$

where for positive integers r, s,

$$e_r^{(s)}(k):=[\,t^r]\prod_{j=1}^{k-1}(1+j^{-s}t)=\sum_{1\le j_1<j_2<\ldots<j_r\le k-1}(j_1j_2\cdots j_r)^{-s},$$

and $[\,t^r]$ means the coefficient of t^r. In particular, substituting $n=0$ in (3) recovers Markov's formula (1) and setting $n=1,2$ gives the following two formulas:

$$\zeta(5)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}-\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2}, \tag{4}$$

$$\zeta(7)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}-2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2}+\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{m=1}^{k-1}\frac{1}{m^2}\sum_{j=1}^{m-1}\frac{1}{j^2}, \tag{5}$$

respectively. In 1996, inspired by this result, J. Borwein and D. Bradley [4] applied extensive computer searches on the base of integer relations algorithms looking for additional zeta identities of this sort. This led to the discovery of the new identity

$$\zeta(7)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}+\frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}, \tag{6}$$

which is simpler than Koecher's formula for $\zeta(7)$, and similar identities for $\zeta(9)$, $\zeta(11)$, $\zeta(13)$, etc. This allowed them to conjecture that certain of these identities, namely those for $\zeta(4n+3)$ are given by the following generating function formula [3]:

$$\sum_{n=0}^{\infty}\zeta(4n+3)a^{4n}=\sum_{k=1}^{\infty}\frac{k}{k^4-a^4}=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{\binom{2k}{k}(k^4-a^4)}\prod_{m=1}^{k-1}\left(\frac{m^4+4a^4}{m^4-a^4}\right),\quad |a|<1. \tag{7}$$

The validity of (7) was proved later by G. Almkvist and A. Granville [1] in 1999. Expanding the right-hand side of (7) in powers of a^4 gives the following Apéry-like series for $\zeta(4n+3)$ [3]:

$$\zeta(4n+3)=\frac{5}{2}\sum_{j=0}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{4j+3}\binom{2k}{k}}\sum_{r=0}^{n-j}4^r h_{n-j-r}^{(4)}(k)e_r^{(4)}(k), \tag{8}$$

where

$$h_r^{(s)}(k):=[\,t^r]\prod_{j=1}^{k-1}(1-j^{-s}t)^{-1}.$$

In particular, substituting $n=0$ in (8) gives (1) and putting $n=1$ yields (6). It is easily seen that for $n\ge 1$, formula (8) contains fewer summations than the corresponding formula for $\zeta(4n+3)$ given by (3).

There exists a bivariate unifying formula for identities (2) and (7)

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\binom{2k}{k}}\frac{5k^2-x^2}{k^4-x^2k^2-y^4}\prod_{m=1}^{k-1}\frac{(m^2-x^2)^2+4y^4}{m^4-x^2m^2-y^4}. \tag{9}$$

It was originally conjectured by H. Cohen and then proved by D. Bradley [7] and, independently, by T. Rivoal [22]. Their proof consists of reduction of (9) to a finite non-trivial combinatorial identity which can be proved on the basis of Almkvist and Granville's work [1]. Another proof of (9) based on application of WZ pairs was given by the authors in [12]. Since

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{n}\zeta(2n+4m+3)x^{2n}y^{4m},\quad |x|^2+|y|^4<1, \tag{10}$$

the formula (9) generates Apéry-like series for all $\zeta(2n+4m+3)$, $n,m\ge 0$, convergent at the geometric rate with ratio $1/4$ and contains, as particular cases, both identities (2) and (7). Indeed, setting $x=a$ and $y=0$ yields Koecher's identity (2), and setting $x=0$, $y=a$ yields the Borwein-Bradley identity (7). Putting

$$a^2:=\frac{x^2+\sqrt{x^4+4y^4}}{2},\qquad b^2:=\frac{x^2-\sqrt{x^4+4y^4}}{2}, \tag{11}$$

we can rewrite (9) in a more symmetrical way

$$\sum_{k=1}^{\infty}\frac{k}{(k^2-a^2)(k^2-b^2)}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(5n^2-a^2-b^2)(1\pm a\pm b)_{n-1}}{n\binom{2n}{n}(1\pm a)_n(1\pm b)_n}. \tag{12}$$

Here and below $(u\pm v\pm w)$ means that the product contains the factors $u+v+w$, $u+v-w$, $u-v+w$, $u-v-w$.

In [12], the authors showed that the generating function (10) also has a much more rapidly convergent representation, namely

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}r(n)}{n\binom{2n}{n}}\frac{\prod_{m=1}^{n-1}((m^2-x^2)^2+4y^4)}{\prod_{m=n}^{2n}(m^4-x^2m^2-y^4)}, \tag{13}$$

where

$$r(n)=205n^6-160n^5+(32-62x^2)n^4+40x^2n^3+(x^4-8x^2-25y^4)n^2+10y^4n+y^4(x^2-2).$$

The identity (13) produces accelerated series for all $\zeta(2n+4m+3)$, $n,m\ge$, convergent at the geometric rate with ratio 2^{-10}. In particular, if $x=y=0$ we get Amdeberhan-Zeilberger's series [2] for $\zeta(3)$,

$$\zeta(3)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(205n^2-160n+32)}{n^5\binom{2n}{n}^5}. \tag{14}$$

It is worth pointing out that both identities (9) and (13) were proved in [12] by using the same Markov-WZ pair (see also [14, p. 702] for the explicit expression), but with the help of different summation formulas.

A more general form of the bivariate identity (9) for the generating function

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n}(A_0\zeta(2n+4m+4)+B_0\zeta(2n+4m+3)+C_0\zeta(2n+4m+2))x^{2n}y^{4m}$$
$$=\sum_{k=1}^{\infty}\frac{A_0+B_0k+C_0k^2}{k^4-x^2k^2-y^4},\qquad |x|^2+|y|^4<1,$$

where A_0,B_0,C_0 are arbitrary complex numbers, was proved in [12] by means of the Markov-Wilf-Zelberger theory. More precisely, we have

$$\sum_{k=1}^{\infty}\frac{A_0+B_0k+C_0k^2}{k^4-x^2k^2-y^4}=\sum_{n=1}^{\infty}\frac{d_n}{\prod_{m=1}^{n}(m^4-x^2m^2-y^4)}, \tag{15}$$

where

$$d_n=\frac{(-1)^{n-1}B_0(5n^2-x^2)}{2n\binom{2n}{n}}\prod_{m=1}^{n-1}((m^2-x^2)^2+4y^4)$$
$$+\frac{(40n+10)L_n+(35n^5-35n^3x^2+4n(3x^4+10y^4))L_{n-1}}{4(5n^2-2x^2)}$$

and L_n is a solution of a certain second order linear difference equation with polynomial coefficients in n and x, y with the initial values $L_0 = C_0$, $L_1 = (5-2x^2)A_0/15 + (5x^2 - 1 - 4(x^4+6y^4))C_0/30$. If we take $A_0 = C_0 = 0$, $B_0 = 1$ in (15), then $L_n = 0$ for all $n \ge 0$ and we get the bivariate identity (9).

First results related to generating function identities for even zeta values belong to Leshchiner [16] who proved (in an expanded form) that for $|a| < 1$,

$$\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{2n+1}}\right)\zeta(2n+2)a^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\,\frac{3k^2+a^2}{k^2-a^2}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right). \tag{16}$$

Comparing constant terms on both sides of (16) yields

$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.$$

In 2006, D. Bailey, J. Borwein and D. Bradley [5] proved another identity

$$\sum_{n=0}^{\infty}\zeta(2n+2)a^{2n} = \sum_{k=1}^{\infty}\frac{1}{k^2-a^2} = 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-a^2)}\prod_{m=1}^{k-1}\left(\frac{m^2-4a^2}{m^2-a^2}\right). \tag{17}$$

It generates similar Apéry-like series for the numbers $\zeta(2n+2)$, which are not covered by Leshchiner's result (16). In the same paper [5], a generating function producing fast convergent series for the sequence $\zeta(2n+4)$, $n = 0, 1, 2, \dots$, was found, which for $|a| < 1$, has the form

$$\frac{1}{2}\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{2n+3}}-\frac{3}{6^{2n+4}B_{2n+4}}\right)\zeta(2n+4)a^{2n} = \sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}(k^2-a^2)}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right), \tag{18}$$

where $B_{2n} \in \mathbb{Q}$ are the even indexed Bernoulli numbers generated by

$$x\coth(x) = \sum_{n=0}^{\infty}B_{2n}\frac{(2x)^{2n}}{(2n)!}.$$

It was shown that the left-hand side of (18) represents a Maclaurin expansion of the function

$$\frac{\pi a\csc(\pi a) + 3\cos(\pi a/3) - 4}{4a^4}.$$

Comparing constant terms in (18) implies that

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

The identity (18) gives a formula for $\zeta(2n+4)$ which for $n \ge 0$ involves fewer summations then the corresponding formula generated by (16). Note that a unifying formula for identities generating even zeta values similar to the bivariate formula (9) for the odd cases is not known.

The Hurwitz zeta function defined by

$$\zeta(s,v)=\sum_{k=0}^{\infty}\frac{1}{(k+v)^s}$$

for $s\in\mathbb{C}$, $\operatorname{Re}s>1$ and $v\neq 0,-1,-2,\ldots$ is a generalization of the Riemann zeta function $\zeta(s)=\zeta(s,1)$. In this paper, we prove a new identity for values $\zeta(2n,v)$ which contains as particular cases Koecher's identity (2), the Bailey-Borwein-Bradley identity (17), some special case of identity (9) and many other interesting formulas related to values of the Hurwitz zeta function. We also get extensions of identities (9) and (13) to values of the Hurwitz zeta function. The main tool we use here is a construction of new Markov-WZ pairs. As application of our results, we prove several conjectures on supercongruences proposed by J. Guillera and W. Zudilin [10], and Z. W. Sun [25, 26].

2 Background

We start by recalling several definitions and known facts related to the Markov-Wilf-Zeilberger theory (see [17]–[19]). A function $H(n,k)$, in the integer variables n and k, is called *hypergeometric* or *closed form (CF)* if the quotients

$$\frac{H(n+1,k)}{H(n,k)}\quad\text{and}\quad\frac{H(n,k+1)}{H(n,k)}$$

are both rational functions of n and k. A hypergeometric function that can be written as a ratio of products of factorials is called *pure-hypergeometric.* A pair of CF functions $F(n,k)$ and $G(n,k)$ is called a *WZ pair* if

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{19}$$

A *P-recursive* function is a function that satisfies a linear recurrence relation with polynomial coefficients. If for a given hypergeometric function $H(n,k)$, there exists a polynomial $P(n,k)$ in k of the form

$$P(n,k)=a_0(n)+a_1(n)k+\cdots+a_L(n)k^L,$$

for some non-negative integer L, and P-recursive functions $a_0(n),\ldots,a_L(n)$ such that

$$F(n,k):=H(n,k)P(n,k)$$

satisfies (19) with some function G, then a pair (F,G) is called a *Markov-WZ pair* associated with the kernel $H(n,k)$ (MWZ pair for short). We call $G(n,k)$ an *MWZ mate* of $F(n,k)$. If $L=0$, then (F,G) is simply a WZ pair.

In 2005, M. Mohammed [18] showed that for any pure-hypergeometric kernel $H(n,k)$, there exists a non-negative integer L and a polynomial $P(n,k)$ as above such that the function $F(n,k) = H(n,k)P(n,k)$ has an MWZ mate $G(n,k) = F(n,k)Q(n,k)$, where $Q(n,k)$ is a ratio of two P-recursive functions. Paper [19] is accompanied by the Maple package MarkovWZ which, for a given $H(n,k)$ outputs the polynomial $P(n,k)$ and the $G(n,k)$ as above.

From relation (19) we get the following summation formulas.

Proposition A. [18, Theorem 2(b)] *Let (F,G) be an MWZ pair. If $\lim_{n\to\infty} F(n,k) = 0$ for every $k \geq 0$, then*

$$\sum_{k=0}^{\infty} F(0,k) - \lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,0), \tag{20}$$

whenever both sides converge.

Proposition B. [18, Cor. 2] *Let (F,G) be an MWZ pair. If $\lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) = 0$, then*

$$\sum_{k=0}^{\infty} F(0,k) = \sum_{n=0}^{\infty} (F(n,n) + G(n,n+1)), \tag{21}$$

whenever both sides converge.

Formulas (20), (21) with an appropriate choice of MWZ pairs can be used to convert a given hypergeometric series into a different rapidly converging one.

To ensure wider applications of WZ pairs for proving hypergeometric identities we use an approach due to I. Gessel [8] (see also [20, §7.3, 7.4]). It is based on the fact that if we have a WZ pair (F,G), then we can easily find other WZ pairs by the following rules.

Proposition C. [8, Th. 3.1] *Let (F,G) be a WZ pair.*

(i) *For any complex numbers α and β, $(F(n+\alpha, k+\beta), G(n+\alpha, k+\beta))$ is a WZ pair.*

(ii) *For any complex number γ, $(\gamma F(n,k), \gamma G(n,k))$ is a WZ pair.*

(iii) *If $p(n,k)$ is a gamma product such that $p(n+1,k) = p(n,k+1) = p(n,k)$ for all n and k for which $p(n,k)$ is defined, then $(p(n,k)F(n,k), p(n,k)G(n,k))$ is a WZ pair.*

(iv) *$(F(-n,k), -G(-n-1,k))$ is a WZ pair.*

(v) *$(F(n,-k), -G(n,-k+1))$ is a WZ pair.*

(vi) *$(G(k,n), F(k,n))$ is a WZ pair.*

The WZ pairs obtained from (F,G) by any combination of (i)–(v) are called the *associates* of (F,G). The WZ pair of the form (vi) and all its associates are called the *duals* of (F,G).

3 The identities

Theorem 1 *Let $a, \alpha, \beta \in \mathbb{C}$, $|a| < 1$, $\alpha \neq \beta$, and $\alpha \pm a$, $\beta \pm a$ be distinct from $0, -1, -2, \ldots$. Then we have*

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\sum_{n=0}^{\infty} a^{2n}(\zeta(2n+2,\alpha)-\zeta(2n+2,\beta)) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+\alpha-\beta)_{n-1}(1+\beta-\alpha)_{n-1}(1+2a)_{n-1}(1-2a)_{n-1}}{(\alpha+a)_n(\alpha-a)_n(\beta+a)_n(\beta-a)_n} \\
&\quad \times \frac{(5n^2+3n(\alpha+\beta-2)+2(\alpha-1)(\beta-1)-2a^2)}{n\binom{2n}{n}}.
\end{aligned} \tag{22}$$

Proof. By the definition of the Hurwitz zeta function, we have

$$\frac{1}{\beta-\alpha}\sum_{n=0}^{\infty} a^{2n}(\zeta(2n+2,\alpha)-\zeta(2n+2,\beta)) = \sum_{k=0}^{\infty} \frac{2k+\alpha+\beta}{((k+\alpha)^2-a^2)((k+\beta)^2-a^2)}. \tag{23}$$

Now define a Markov kernel $H(n,k)$ by the formula

$$H(n,k) = \frac{(\alpha+a)_k(\alpha-a)_k(\beta+a)_k(\beta-a)_k(n+2k+\alpha+\beta)}{(\alpha+a)_{n+k+1}(\alpha-a)_{n+k+1}(\beta+a)_{n+k+1}(\beta-a)_{n+k+1}}.$$

Applying the Maple package Markov-WZ we get the associated WZ pair

$$F(n,k) = H(n,k)\frac{(-1)^n(1+\alpha-\beta)_n(1+\beta-\alpha)_n(1+2a)_n(1-2a)_n}{\binom{2n}{n}},$$

$$G(n,k) = F(n,k)\frac{5n^2+n(3\alpha+3\beta+4)+2\alpha\beta-2a^2+(2k+1)(1+\alpha+\beta)+2k(k+3n)}{2(2n+1)(n+2k+\alpha+\beta)}.$$

Now by Proposition A, we obtain

$$\sum_{k=0}^{\infty} F(0,k) = \sum_{n=0}^{\infty} G(n,0),$$

which implies (22). □

Multiplying both sides of (22) by $\beta-\alpha$ and letting β tend to infinity we get an extension of the Bailey-Borwein-Bradley identity to values of the Hurwitz zeta function:

$$\sum_{n=0}^{\infty} a^{2n}\zeta(2n+2,\alpha) = \sum_{n=0}^{\infty} \frac{(3n+2\alpha-2)(1+2a)_{n-1}(1-2a)_{n-1}}{n\binom{2n}{n}(\alpha+a)_n(\alpha-a)_n}. \tag{24}$$

Setting $\alpha = 1$ in (24) yields the Bailey-Borwein-Bradley identity (17).

Replacing a by $a/2$ and α, β by $1+a/2$, $1-a/2$, respectively, in (22) and taking into account (23), we get Koecher's identity (2).

Letting β tend to α in (22) and using the equality

$$\frac{d}{dv}\zeta(s,v) = -s\zeta(s+1,v),$$

we get the following.

Corollary 1 *Let $a, \alpha \in \mathbb{C}$, $|a| < 1$, and $\alpha \pm a \ne 0, -1, -2, \ldots$. Then*

$$\sum_{n=0}^{\infty}(n+1)\zeta(2n+3,\alpha)a^{2n} = \sum_{k=0}^{\infty}\frac{k+\alpha}{((k+\alpha)^2-a^2)^2}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(5n^2+6n(\alpha-1)+2(\alpha-1)^2-2a^2)}{n\binom{2n}{n}}\frac{(n-1)!^2(1+2a)_{n-1}(1-2a)_{n-1}}{(\alpha+a)_n^2(\alpha-a)_n^2}.$$

Taking $\alpha = 1$ in Corollary 1, we get the following identity for odd zeta values.

Corollary 2 *Let $a \in \mathbb{C}$, $|a| < 1$. Then*

$$\sum_{n=1}^{\infty}n\zeta(2n+1)a^{2n-2} = \sum_{k=1}^{\infty}\frac{k}{(k^2-a^2)^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(5n^2-a^2)}{n\binom{2n}{n}(n^2-a^2)^2}\prod_{m=1}^{n-1}\frac{1-4a^2/m^2}{(1-a^2/m^2)^2}. \quad (25)$$

Note that the right-hand side equality of (25) also follows from the bivariate identity (9) or (12) as was shown by D. Bradley (see [7, Cor. 1]).

It is clear that identity (25) gives formulas for odd zeta values which are linear combinations of series generated by the bivariate identity (9). Thus comparing constant terms on both sides of (25) gives Apéry's series (1) for $\zeta(3)$. Similarly, comparing coefficients of a^2 gives formula (4) for $\zeta(5)$. It produces the following complicated expression for $\zeta(7)$:

$$\zeta(7) = \frac{11}{6}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^7\binom{2k}{k}} - \frac{8}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} - \frac{25}{6}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}$$
$$+ \frac{10}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2}\sum_{m=1}^{j-1}\frac{1}{m^2},$$

which can be written as

$$\zeta(7) = \frac{1}{3}(4K - B),$$

where K and B are right-hand sides of formulas (5) and (6), respectively. More generally, if we denote

$$g_r^{(s)}(k) := [\,t^r]\prod_{j=1}^{k-1}(1-j^{-s}t)^{-2},$$

then taking into account that

$$\frac{5k^2-2a^2}{(k^2-a^2)^2} = \frac{2}{1-a^2/k^2} + \frac{3}{(1-a^2/k^2)^2} = \sum_{j=0}^{\infty}(3j+5)\frac{a^{2j}}{k^{2j}}$$

and comparing the coefficients of a^{2n} on both sides of (25), we get

Corollary 3 *Let n be a non-negative integer. Then*

$$\zeta(2n+3) = \frac{1}{2n+2}\sum_{j=0}^{n}(3j+5)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2j+3}\binom{2k}{k}}\sum_{r=0}^{n-j}(-4)^r e_r^{(2)}(k) g_{n-j-r}^{(2)}(k). \qquad (26)$$

Consider several other particular cases of Theorem 1. Replacing a by $a/2$, α by $1/2$, and β by 1 in (22) and noting that

$$\sum_{k=0}^{\infty}\frac{2k+3/2}{((k+1/2)^2-a^2/4)((k+1)^2-a^2/4)} = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2},$$

we get the following identity.

Corollary 4 *Let a be a complex number, distinct from a non-zero integer. Then*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(10n^2-3n-a^2)}{n(2n-1)(n^2-a^2)\binom{2n}{n}\prod_{j=1}^{n}(1-a^2/(n+j)^2)}.$$

In particular,

$$\zeta(2) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(10n-3)}{n^2(2n-1)\binom{2n}{n}}.$$

Substituting $\alpha = 1/3$, $\beta = 2/3$, $a = 0$ in Theorem 1, we get

$$\zeta(2,1/3) - \zeta(2,2/3) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n!^2(15n-4)}{n^3\binom{2n}{n}(1/3)_n(2/3)_n}.$$

Now observing that

$$\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n = \frac{(3n)!}{27^n n!}$$

and

$$\zeta(2,1/3) - \zeta(2,2/3) = 9\sum_{n=1}^{\infty}\frac{\left(\frac{n}{3}\right)}{n^2} =: 9K$$

(where $\left(\frac{n}{p}\right)$ is the Legendre symbol), we get the following formula

$$K = \sum_{n=1}^{\infty}\frac{(15n-4)(-27)^{n-1}}{n^3\binom{2n}{n}^2\binom{3n}{n}},$$

which was conjectured by Z. W. Sun in [25].

Substituting $\alpha = 1/4$, $\beta = 3/4$ recovers Theorem 3 from [13] and in particular (when $a = 0$), it gives the following formula for Catalan's constant $G := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$:

$$G = \frac{1}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}256^n(40n^2-24n+3)}{\binom{4n}{2n}^2\binom{2n}{n}n^3(2n-1)}.$$

Applying Proposition B to the Markov-WZ pair found in the proof of Theorem 1, we get the following identity which generates Apéry-like series for the differences $\zeta(2n+2,\alpha)-\zeta(2n+2,\beta)$ converging exponentially fast as 2^{-10}.

Theorem 2 *Let $a,\alpha,\beta\in\mathbb{C}$, $|a|<1$, $\alpha\neq\beta$, and $\alpha\pm a$, $\beta\pm a$ be distinct from $0,-1,-2,\dots$. Then*

$$\begin{aligned}&\frac{1}{\beta-\alpha}\sum_{n=0}^{\infty}(\zeta(2n+2,\alpha)-\zeta(2n+2,\beta))a^{2n}\\&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}p_{\alpha,\beta}(n)}{n\binom{2n}{n}}\frac{\prod_{j=1}^{n-1}(j^2-(\alpha-\beta)^2)(j^2-4a^2)}{\prod_{j=n-1}^{2n-1}((j+\alpha)^2-a^2)((j+\beta)^2-a^2)},\end{aligned}\tag{27}$$

where

$$\begin{aligned}p_{\alpha,\beta}(n)&=2(2n-1)(3n+\alpha+\beta-3)((2n-1+\alpha)^2-a^2)((2n-1+\beta)^2-a^2)\\&+((n+\alpha-1)^2-a^2)((n+\beta-1)^2-a^2)(13n^2+5n(\alpha+\beta-2)+2((1-\alpha)(1-\beta)-a^2)).\end{aligned}$$

Setting $\alpha=1/3$, $\beta=2/3$, $a=0$ in Theorem 2, we get the following fast converging series for the constant K :

$$K=\sum_{n=1}^{\infty}\frac{(-27)^{n-1}(5535n^3-4689n^2+1110n-80)}{n^3(3n-1)(3n-2)\binom{6n}{3n}^2\binom{3n}{n}}.$$

Setting $\alpha=1/4$, $\beta=3/4$, we recover Theorem 4 from [13].

Multiplying both sides of (27) by $\beta-\alpha$ and letting β tend to infinity we obtain an extension of Theorem 2 from [11] to values of the Hurwitz zeta function

$$\sum_{n=0}^{\infty}\zeta(2n+2,\alpha)a^{2n}=\sum_{n=1}^{\infty}\frac{p(n)}{n\binom{2n}{n}}\frac{\prod_{m=1}^{n-1}(m^2-4a^2)}{\prod_{m=n-1}^{2n-1}((m+\alpha)^2-a^2)},\tag{28}$$

where $p(n)=2(2n-1)((2n-1+\alpha)^2-a^2)+(5n+2\alpha-2)((n+\alpha-1)^2-a^2)$. In particular, setting $\alpha=1$, $a=0$ in (28) we get Zeilberger's series [30, §12] for $\zeta(2)$,

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3}.$$

Replacing a by $a/2$ and α,β by $1+a/2$, $1-a/2$, respectively, we recover Theorem 4 from [11]. Letting β tend to α in (27), we get the following

Corollary 5 *Let $a,\alpha\in\mathbb{C}$, $|a|<1$, and $\alpha\pm a$ be distinct from $0,-1,-2,\dots$. Then*

$$\begin{aligned}&\sum_{k=1}^{\infty}k\zeta(2k+1,\alpha)a^{2k-2}\\&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}p_{\alpha}(n)}{n\binom{2n}{n}^5((n+\alpha-1)^2-a^2)^2((n+\alpha-1)^2-a^2/4)^2}\prod_{m=1}^{n-1}\frac{1-4a^2/m^2}{((1+\frac{\alpha-1}{m+n})^2-\frac{a^2}{(m+n)^2})^2},\end{aligned}$$

where

$$p_\alpha(n) := p_{\alpha,\alpha}(n) = 2(2n-1)(3n+2\alpha-3)((2n+\alpha-1)^2-a^2)^2 \\ + ((n+\alpha-1)^2-a^2)^2(13n^2+10n(\alpha-1)+2((1-\alpha)^2-a^2)).$$

Setting $\alpha = 1$ in Corollary 5 we get the following identity.

Corollary 6 *Let $a \in \mathbb{C}$, $|a| < 1$. Then*

$$\sum_{k=1}^{\infty} k\zeta(2k+1)a^{2k-2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}p(n)}{n\binom{2n}{n}^5(n^2-a^2)^2(n^2-a^2/4)^2} \prod_{m=1}^{n-1} \frac{1-4a^2/m^2}{(1-a^2/(m+n)^2)^2},$$

where $p(n) = 2(2n-1)(3n-1)(4n^2-a^2)^2 + (n^2-a^2)^2(13n^2-2a^2)$.

Setting $a = 0$ in Corollary 6 we get Amdeberhan-Zeilberger's series (14) for $\zeta(3)$.

The next theorem gives a generalization of identity (12).

Theorem 3 *Let $\alpha, a, b \in \mathbb{C}$ and $\alpha \pm a$, $\alpha \pm b$ be distinct from $0, -1, -2, \ldots$. Then the following identity holds:*

$$\sum_{k=0}^{\infty} \frac{k+\alpha}{((k+\alpha)^2-a^2)((k+\alpha)^2-b^2)} \\ = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1\pm a\pm b)_{n-1}(5n^2-6n(1-\alpha)+2(1-\alpha)^2-a^2-b^2)}{n\binom{2n}{n}(\alpha\pm a)_n(\alpha\pm b)_n}. \tag{29}$$

Proof. Taking the kernel

$$H(n,k) = \frac{(\alpha+a)_k(\alpha-a)_k(\alpha+b)_k(\alpha-b)_k(n+2k+2\alpha)}{(\alpha+a)_{n+k+1}(\alpha-a)_{n+k+1}(\alpha+b)_{n+k+1}(\alpha-b)_{n+k+1}}$$

and applying the Maple package MarkovWZ we get that

$$F(n,k) = \frac{(-1)^n}{\binom{2n}{n}}(1\pm a\pm b)_n H(n,k)$$

and

$$G(n,k) = F(n,k)\frac{5n^2+6\alpha n+4n+2\alpha+2\alpha^2+1-a^2-b^2+k(2k+6n+4\alpha+2)}{2(2n+1)(n+2k+2\alpha)}$$

give a WZ pair, i.e.,

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Now by Proposition A, we get

$$\sum_{k=0}^{\infty} F(0,k) = \sum_{n=0}^{\infty} G(n,0),$$

which implies (29). □.

Making the substitution (11) in (29) we get a generalization of Cohen's identity to values of the Hurwitz zeta function.

Corollary 7 *Let $x, y, \alpha \in \mathbb{C}$, $|x|^2 + |y|^4 < 1$ and $\alpha \neq 0, -1, -2, \dots$. Then*

$$\sum_{k=0}^{\infty} \frac{k+\alpha}{(k+\alpha)^4 - x^2(k+\alpha)^2 - y^4} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{n+m}{n} \zeta(2n+4m+3, \alpha) x^{2n} y^{4m}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(5n^2 - 6n(1-\alpha) + 2(1-\alpha)^2 - x^2)}{n\binom{2n}{n}((n+\alpha-1)^4 - x^2(n+\alpha-1)^2 - y^4)} \prod_{j=1}^{n-1} \frac{(j^2-x^2)^2 + 4y^4}{(j+\alpha-1)^4 - x^2(j+\alpha-1)^2 - y^4}.$$

Setting $\alpha = 1/2$, $x = y = 0$ in Corollary 7 we get the following formula:

$$\zeta(3) = \frac{1}{28}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(10n^2 - 6n + 1)256^n}{n^5\binom{2n}{n}^5}.$$

Applying Proposition B to the Markov-WZ pair used in the proof of Theorem 3 we get the following identity.

Theorem 4 *Let $\alpha, a, b \in \mathbb{C}$ and $\alpha \pm a$, $\alpha \pm b \neq 0, -1, -2, \dots$. Then*

$$\sum_{k=0}^{\infty} \frac{k+\alpha}{((k+\alpha)^2 - a^2)((k+\alpha)^2 - b^2)}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1 \pm a \pm b)_{n-1}(\alpha \pm a)_{n-1}(\alpha \pm b)_{n-1}}{n\binom{2n}{n}(\alpha \pm a)_{2n}(\alpha \pm b)_{2n}} q(n),$$

where

$$q(n) = 2(2n-1)(3n+2\alpha-3)((2n+\alpha-1)^2 - a^2)((2n+\alpha-1)^2 - b^2)$$
$$+ ((n+\alpha-1)^2 - a^2)((n+\alpha-1)^2 - b^2)(13n^2 - 10n(1-\alpha) + 2(1-\alpha)^2 - a^2 - b^2).$$

Making the change of variables (11) in Theorem 4 we get a generalization of the identity (13) to values of the Hurwitz zeta function.

Corollary 8 *Let $x, y, \alpha \in \mathbb{C}$, $|x|^2 + |y|^4 < 1$ and $\alpha \neq 0, -1, -2, \dots$. Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \zeta(2n+4m+3, \alpha) x^{2n} y^{4m}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}Q(n)}{n\binom{2n}{n}} \frac{\prod_{j=1}^{n-1}((j^2-x^2)^2 + 4y^4)}{\prod_{j=0}^{n}((n+j+\alpha-1)^4 - x^2(n+j+\alpha-1)^2 - y^4)}, \tag{30}$$

where

$$Q(n) = 2(2n-1)(3n+2\alpha-3)((2n+\alpha-1)^4 - x^2(2n+\alpha-1)^2 - y^4)$$
$$+ ((n+\alpha-1)^4 - x^2(n+\alpha-1)^2 - y^4)(13n^2 - 10n(1-\alpha) + 2(1-\alpha)^2 - x^2).$$

Formula (30) produces accelerated series for the values $\zeta(2n+4m+3, \alpha)$, $n, m \geq 0$, convergent at the geometric rate with ratio 2^{-10}.

Remark. Note that Theorem 4 can also be obtained from Proposition A applied to the Markov-WZ pair associated with the kernel

$$H_1(n,k) = \frac{(\alpha+a)_{k+n}(\alpha-a)_{k+n}(\alpha+b)_{k+n}(\alpha-b)_{k+n}}{(\alpha+a)_{2n+k+1}(\alpha-a)_{2n+k+1}(\alpha+b)_{2n+k+1}(\alpha-b)_{2n+k+1}} (3n+2k+2\alpha). \tag{31}$$

4 Hypergeometric summation formulas

In this section we derive several interesting summation formulas for special classes of certain hypergeometric series.

Corollary 9 *Let n be a non-negative integer. Suppose that α, β are complex numbers such that 2β and $\alpha+\beta$ are distinct from $-n, -n-1, -n-2, \ldots$, and $\alpha-\beta \neq n, n-1, n-2, \ldots$. Then we have*

$$
{}_7F_6\left(\begin{matrix} n+1, n+1\pm(\alpha-\beta), n+1\pm(2n+2\beta), n+\frac{7}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D_1}}{10} \\ \frac{3}{2}+n, n+\alpha+1\pm(\beta+n), 2n+2\beta+1, n+\frac{2}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D_1}}{10} \end{matrix}\middle| -\frac{1}{4}\right)
= \frac{(n+\alpha+\beta)_{n+1}(n+2)_{n+1}}{(1+\alpha-\beta)_{n+1}(1+2n+2\beta)_{n+1}}, \tag{32}
$$

where $D_1 = 40((n+\beta)^2-(\alpha-1)(\beta-1))+9(\alpha+\beta-2)^2$.

Proof. To prove (32), we rewrite identity (22) in terms of the generalized hypergeometric series as

$$
\sum_{k=0}^{\infty}\frac{2k+\alpha+\beta}{((k+\beta)^2-a^2)((k+\alpha)^2-a^2)} = \frac{(1+\alpha)(1+\beta)+\alpha\beta-2a^2}{2(\alpha^2-a^2)(\beta^2-a^2)}
$$
$$
\times\, {}_8F_7\left(\begin{matrix} 1,1,1+\alpha-\beta, 1+\beta-\alpha, 1\pm 2a, \frac{7}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D}}{10} \\ \frac{3}{2}, 1+\alpha\pm a, 1+\beta\pm a, \frac{2}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D}}{10} \end{matrix}\middle| -\frac{1}{4}\right),
$$

where $D = 40a^2+9(\alpha-\beta)^2-4(\alpha-1)(\beta-1)$. Now considering both sides of the above equality as meromorphic functions of variable a and comparing corresponding residues at the simple pole $a = \beta+n$, where n is a non-negative integer, on both sides, we get

$$
\begin{aligned}
1 &= \lim_{a\to n+\beta}(n+\beta-a)\frac{(n+\beta)(\alpha-\beta)((1+\alpha)(1+\beta)+\alpha\beta-2a^2)}{(\alpha^2-a^2)(\beta^2-a^2)} \\
&\times\sum_{k=n}^{\infty}\frac{k!(1\pm(\alpha-\beta))_k(1\pm 2a)_k(7/5+3(\alpha+\beta)/10\pm\sqrt{D}/10)_k}{(-4)^k(3/2)_k(1+\alpha\pm a)_k(1+\beta\pm a)_k(2/5+3(\alpha+\beta)/10\pm\sqrt{D}/10)_k} \\
&= \frac{(n+\beta)(\alpha-\beta)((1+\alpha)(1+\beta)+\alpha\beta-2(n+\beta)^2)}{(\alpha^2-(n+\beta)^2)(n+2\beta)(-1)^n} \\
&\times\sum_{k=n}^{\infty}\binom{k}{n}\frac{(1\pm(\alpha-\beta))_k(1\pm(2n+2\beta))_k((14+3(\alpha+\beta)\pm\sqrt{D_1})/10)_k}{(-4)^k(3/2)_k(1+\alpha\pm(\beta+n))_k(1+2\beta+n)_k((4+3(\alpha+\beta)\pm\sqrt{D_1})/10)_k} \\
&= \frac{(n+\beta)(\alpha-\beta)(1\pm(\alpha-\beta))_n(1\pm(2n+2\beta))_n(10n+4+3(\alpha+\beta)\pm\sqrt{D_1})}{20(\alpha^2-(n+\beta)^2)(n+2\beta)4^n(3/2)_n(1+\alpha\pm(\beta+n))_n(1+2\beta+n)_n} \\
&\times\, {}_7F_6\left(\begin{matrix} n+1, n+1\pm(\alpha-\beta), n+1\pm(2n+2\beta), n+\frac{7}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D_1}}{10} \\ \frac{3}{2}+n, n+\alpha+1\pm(\beta+n), 2n+2\beta+1, n+\frac{2}{5}+\frac{3(\alpha+\beta)}{10}\pm\frac{\sqrt{D_1}}{10} \end{matrix}\middle| -\frac{1}{4}\right).
\end{aligned}
$$

Now taking into account that

$$\frac{\alpha-\beta}{\alpha-\beta-n}\cdot\frac{(1+\beta-\alpha)_n}{(1+\alpha-\beta-n)_n}=(-1)^n, \qquad \frac{n+\beta}{n+2\beta}\cdot\frac{(1-2n-2\beta)_n}{(1+2\beta+n)_n}=\frac{(-1)^n}{2},$$

$$\frac{(10n+4+3(\alpha+\beta))^2-D_1}{20}=(\alpha-\beta+n+1)(3n+2\beta+1),$$

and $4^n(3/2)_n=(2n+1)!/n!$, we get the required identity. □

Corollary 10 *Let n be a non-negative integer and a,α be complex numbers such that $2\alpha\ne -n,-n-1,\dots,$ and $\alpha\pm a\ne -n-1,-n-2,\dots,-2n-1$. Then we have*

$$\begin{aligned}{}_7F_6&\left(\left.\begin{matrix} n+1, 2n+\alpha+1\pm a, 1-\alpha\pm a, n+\frac{7+3\alpha}{5}\pm\sqrt{\frac{a^2+(n+\alpha)^2}{5}-(\frac{1-\alpha}{5})^2}\\ \frac{3}{2}+n, 2n+2\alpha+1, n+\alpha+1\pm a, n+\frac{2+3\alpha}{5}\pm\sqrt{\frac{a^2+(n+\alpha)^2}{5}-(\frac{1-\alpha}{5})^2}\end{matrix}\right| -\frac{1}{4}\right)\\ &=\frac{(n+2)_{n+1}(n+2\alpha)_{n+1}}{(n+\alpha+1\pm a)_{n+1}}.\end{aligned} \tag{33}$$

Proof. To prove (33), we rewrite identity (29) in the form

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k+\alpha}{((k+\alpha)^2-a^2)((k+\alpha)^2-b^2)}&=\frac{((1+\alpha)^2+\alpha^2-a^2-b^2)}{4(\alpha^2-a^2)(\alpha^2-b^2)}\\ \times\,{}_8F_7&\left(\left.\begin{matrix}1,1,1\pm a\pm b, \frac{7+3\alpha}{5}\pm\sqrt{D(b^2)}\\ \frac{3}{2}, 1+\alpha\pm a, 1+\alpha\pm b, \frac{2+3\alpha}{5}\pm\sqrt{D(b^2)}\end{matrix}\right| -\frac{1}{4}\right),\end{aligned} \tag{34}$$

where

$$D(b^2):=\frac{a^2+b^2}{5}-\left(\frac{1-\alpha}{5}\right)^2,$$

replace b^2 by z and consider both sides of (34) as meromorphic functions of variable z. Suppose that 2α is distinct from $-n,-n-1,\dots$. This restriction ensures that $(n+\alpha)^2\ne(j+\alpha)^2$ for any non-negative integer $j\ne n$. Now equating residues on both sides of (34) at the simple pole $z=(n+\alpha)^2$, where n is a non-negative integer, we have

$$\frac{n+\alpha}{(n+\alpha)^2-a^2} = \lim_{z\to(n+\alpha)^2}((n+\alpha)^2-z)\cdot\frac{(1+\alpha)^2+\alpha^2-a^2-z}{4(\alpha^2-a^2)(\alpha^2-z)}$$

$$\times\sum_{k=n}^{\infty}\frac{k!\left(\frac{7+3\alpha}{5}\pm\sqrt{D(z)}\right)_k}{(-4)^k(3/2)_k\left(\frac{2+3\alpha}{5}\pm\sqrt{D(z)}\right)_k}\prod_{j=1}^{k}\frac{((j\pm a)^2-z)}{((j+\alpha)^2-z)((j+\alpha)^2-a^2)}$$

$$=\frac{(1+\alpha)^2-a^2-n^2-2n\alpha}{(-1)^{n-1}(2\alpha+n)_n}\sum_{k=n}^{\infty}\binom{k}{n}\frac{(1\pm a\pm(n+\alpha))_k}{(-4)^{k+1}(3/2)_k(\alpha\pm a)_{k+1}(2\alpha+2n+1)_{k-n}}$$

$$\times\frac{\left(\frac{7+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\right)_k}{\left(\frac{2+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\right)_k}=\frac{(1\pm a\pm(n+\alpha))_n((2n+\alpha+1)^2-a^2)}{4^{n+1}(3/2)_n(2\alpha+n)_n(\alpha\pm a)_{n+1}}$$

$$\times{}_7F_6\left(\left.\begin{matrix}n+1,2n+\alpha+1\pm a,1-\alpha\pm a,n+\frac{7+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\\ \frac{3}{2}+n,2n+2\alpha+1,n+\alpha+1\pm a,n+\frac{2+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\end{matrix}\right|-\frac{1}{4}\right).$$

Here in the last equality we used the fact that

$$\frac{\left(\frac{7+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\right)_n}{\left(\frac{2+3\alpha}{5}\pm\sqrt{D((n+\alpha)^2)}\right)_n}=\frac{(2n+\alpha+1)^2-a^2}{(1+\alpha)^2-a^2-n^2-2n\alpha}.$$

Finally, after simplifying

$$\frac{(1-a-n-\alpha)_n}{(\alpha+a)_{n+1}}=\frac{(-1)^n}{\alpha+a+n},\qquad\frac{(1+a-n-\alpha)_n}{(\alpha-a)_{n+1}}=\frac{(-1)^n}{\alpha-a+n},$$

we get the required identity. □

Setting $\alpha=1$ and replacing n by $n-1$ in (33) we get

Corollary 11 *Let n be a positive integer and a be a complex number such that $\pm a\neq -n-1,-n-2,\ldots,-2n$. Then we have*

$${}_7F_6\left(\left.\begin{matrix}n,2n\pm a,\pm a,n+1\pm\sqrt{\frac{a^2+n^2}{5}}\\ n+\frac{1}{2},2n+1,n+1\pm a,n\pm\sqrt{\frac{a^2+n^2}{5}}\end{matrix}\right|-\frac{1}{4}\right)=\prod_{j=n+1}^{2n}\frac{j^2}{j^2-a^2}.\qquad(35)$$

Formula (35) generalizes similar identities from [7, §2]. In particular, substituting $a=\sqrt{c}/n$ gives [7, Corollary 2] and substituting $a=i\sqrt{b+n^2}$ gives [7, Corollary 3].

5 Supercongruences arising from the Amdeberhan-Zeilberger series for $\zeta(3)$

In this section, we consider supercongruences arising from the Amdeberhan-Zeilberger series (14) for $\zeta(3)$. In [10], Guillera and Zudilin proposed conjecturally a p-adic analogue:

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^5}{(1)_k^5}(205k^2+160k+32)(-1)^k 2^{10k} \equiv 32p^2 \pmod{p^5} \quad \text{for prime } p>3.$$

In [26], Z. W. Sun formulated more general conjectures: let p be an odd prime, then

$$\sum_{k=0}^{p-1}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2+64p^3H_{p-1} \pmod{p^7} \quad \text{for } p \neq 5,$$

where $H_{p-1} = \sum_{k=1}^{p-1} 1/k$, and

$$\sum_{k=0}^{(p-1)/2}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2+\frac{896}{3}p^5B_{p-3} \pmod{p^6} \quad \text{for } p>3, \quad (36)$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

Moreover, Sun [26] introduced the related sequence

$$a_n = \frac{1}{8n^2\binom{2n}{n}^2}\sum_{k=0}^{n-1}(205k^2+160k+32)(-1)^{n-1-k}\binom{2k}{k}^5$$

and conjectured that for any positive integer n, a_n should be a positive integer.

In this section, we confirm these conjectures (with the only exception that we prove (36) modulo p^5) and prove the following theorems.

Theorem 5 *Let n be a positive integer and let*

$$A_n := \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2k}{k}^5(205k^2+160k+32).$$

Then the following two alternative representations are valid:

$$A_n = 16n\binom{2n}{n}\sum_{k=0}^{n-1}\binom{n+k-1}{k}^4(2k+n)$$

and

$$A_n = 8n^2\binom{2n}{n}^2\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{n+k}\binom{2n-k-2}{n-k-1}^2.$$

Theorem 5 implies immediately the following

Corollary 12 *For any positive integer* n, $a_n := \frac{A_n}{8n^2\binom{2n}{n}^2}$ *is a positive integer.*

Theorem 6 *Let* p *be an odd prime. Then the following supercongruences take place:*

$$\sum_{k=0}^{p-1}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2+64p^3H_{p-1} \pmod{p^7} \quad \text{for } p\ne 5,$$

$$\sum_{k=0}^{(p-1)/2}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2 \pmod{p^5} \quad \text{for } p>3.$$

The proof of Theorem 5 is contained in the following two lemmas.

Lemma 1 *For any positive integer* N, *the following identity holds:*

$$A_N = 16N\binom{2N}{N}\sum_{k=0}^{N-1}\binom{N+k-1}{k}^4(2k+N).$$

Proof. Consider the Markov kernel $H_1(n,k)$ defined in (31) for $\alpha=1$, $a=b=0$:

$$H_1(n,k)=\frac{(k+n)!^4}{(k+2n+1)!^4}(3n+2k+2).$$

Then the corresponding to it WZ pair has the form

$$F(n,k)=\frac{(-1)^n n!^6}{(2n)!}H_1(n,k), \qquad G(n,k)=\frac{(-1)^n n!^6(k+n)!^4}{2(2n+1)!(k+2n+2)!^4}q(n,k),$$

where

$$\begin{aligned} q(n,k) &= (n+1)^4(205n^2+250n+77)+k(254+344k+1526n+3628n^2+888k^2n \\ &+2928kn^2+4268n^3+248k^2+101k^3+1648kn+574n^5+2486n^4+22k^4+2k^5 \\ &+664kn^4+2288kn^3+408k^2n^3+1048k^2n^2+141k^3n^2+240k^3n+26k^4n). \end{aligned}$$

It is easy to show (see Proposition C or [20, Ch. 7]) that the pair $(\bar{F},\bar{G})$ given by

$$\bar{G}(n,k)=(-1)^{k-1}k\binom{2k}{k}\binom{n+2k-1}{n+k}^4(3k+2n),$$

$$\bar{F}(n,k)=(-1)^k\binom{2k}{k}\left(\frac{(n+2k-1)!}{k!(n+k)!}\right)^4 q_1(n,k)$$

with

$$\begin{aligned} q_1(n,k) &= 205k^6+160k^5+32k^4+2n^6+n^5(4+26k)+n^4(2+42k+141k^2) \\ &+n^3k(16+176k+408k^2)+n^2k^2(48+368k+664k^2)+nk^3(64+384k+574k^2) \end{aligned}$$

is its dual WZ pair, for which

$$\bar{F}(n+1,k) - \bar{F}(n,k) = \bar{G}(n,k+1) - \bar{G}(n,k). \tag{37}$$

It turns out for our further proof that it is useful to consider the usual binomial coefficient $\binom{r}{k}$ in a more general setting, i.e., to allow an arbitrary real number to appear in the upper index of $\binom{r}{k}$, and to allow an arbitrary integer in the lower. We give the following formal definition (see [9, §5.1]):

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1}, & \text{if integer} \quad k \geq 0; \\ 0, & \text{if integer} \quad k < 0. \end{cases}$$

After this elaboration we see that $\bar{G}(n,k)$ is defined for all integers n, k. Rewriting $\bar{F}(n,k)$ in the form

$$\bar{F}(n,k) = \begin{cases} \frac{(-1)^k}{k^4}\binom{2k}{k}\binom{n+2k-1}{n+k}^4 q_1(n,k), & \text{if} \quad k \neq 0; \\ 2(n+1)^2, & \text{if} \quad k = 0, n \geq 0; \\ 0, & \text{if} \quad k = 0, n < 0; \end{cases}$$

we can conclude that $\bar{F}(n,k)$ is well defined for all $n, k \in \mathbb{Z}$. Now we can show that relation (37) takes place for all integers n and k. Indeed, if $k < 0$ or if $k = 0$ and $n < -1$, then all parts in (37) are zero. If $k = 0$ and $n \geq -1$, then $\bar{G}(n,k) = 0$, $\bar{G}(n,k+1) = 2(2n+3)$, $\bar{F}(n+1,k) = 2(n+2)^2$, $F(n,k) = (n+1)^2$, and relation (37) is equivalent to the obvious equality

$$2(n+2)^2 - (n+1)^2 = 2(2n+3).$$

If $k > 0$ and $n + k < 0$, then $\bar{F}(n,k) = \bar{G}(n,k) = 0$,

$$\bar{F}(n+1,k) = \frac{(-1)^k}{k^4}\binom{2k}{k}\binom{n+2k}{n+k+1}^4 q_1(n+1,k),$$

and

$$\bar{G}(n,k+1) = (-1)^k(k+1)\binom{2k+2}{k+1}\binom{n+2k+1}{n+k+1}^4 (3k+2n+3).$$

If moreover, $n + k + 1 < 0$, then $\bar{F}(n+1,k) = \bar{G}(n,k+1) = 0$ and (37) holds. If $n + k + 1 = 0$, then $\bar{G}(n,k+1) = (-1)^k(k+1)^2\binom{2k+2}{k+1}$ and

$$\bar{F}(n+1,k) = \frac{(-1)^k}{k^4}\binom{2k}{k} q_1(-k,k) = (-1)^k\binom{2k}{k}(2k+2)(2k+1) = \bar{G}(n,k+1),$$

and therefore (37) holds. If $k > 0$ and $n + k \geq 0$, then canceling common factorials on both sides of (37) we get the equality

$$\begin{aligned} &(n+2k)^4 q_1(n+1,k) - (n+k+1)^4 q_1(n,k) \\ &= 2(2k+1)(n+2k)^4(n+2k+1)^4(3k+2n+3) + k^5(n+k+1)^4(3k+2n), \end{aligned}$$

which can be easily checked by straightforward verification.

Now let $N \in \mathbb{N}$. Considering relation (37) at the point $(n-N,k)$ we have

$$\bar{F}(n+1-N,k)-\bar{F}(n-N,k)=\bar{G}(n-N,k+1)-\bar{G}(n-N,k). \tag{38}$$

Summing both sides of (38) over k from 0 to $N-1$ we have

$$\sum_{k=0}^{N-1}(\bar{F}(n+1-N,k)-\bar{F}(n-N,k))=\bar{G}(n-N,N)-\bar{G}(n-N,0)=\bar{G}(n-N,N). \tag{39}$$

Now summing (39) over n from 0 to $N-1$ we get

$$\sum_{k=0}^{N-1}(\bar{F}(0,k)-\bar{F}(-N,k))=\sum_{n=0}^{N-1}\bar{G}(n-N,N).$$

Since $\bar{F}(-N,k)=0$ for $k=0,1,\ldots,N-1$, we obtain

$$\sum_{k=0}^{N-1}\bar{F}(0,k)=\sum_{n=0}^{N-1}\bar{G}(n-N,N)$$

or

$$\frac{1}{16}\sum_{k=0}^{N-1}(-1)^k\binom{2k}{k}^5(205k^2+160k+32)=(-1)^{N-1}N\binom{2N}{N}\sum_{n=0}^{N-1}\binom{N+n-1}{n}^4(2n+N),$$

and the lemma is proved. □

Lemma 2 *For any positive integer N, the following identity holds:*

$$A_N=8N^2\binom{2N}{N}^2\cdot\sum_{k=0}^{N-1}(-1)^k\binom{2N-1}{N+k}\binom{2N-k-2}{N-1-k}^2.$$

Proof. Let $N \in \mathbb{Z}$, $N \geq 0$. Put

$$S_N:=2\sum_{k=0}^{N}\binom{k+N}{k}^4(N+2k+1). \tag{40}$$

Now rewriting S_N in the form of a terminating hypergeometric series, we get

$$S_N=2(N+1)\sum_{k=0}^{N}\frac{(N+1)_k^4\,(\frac{N+3}{2})_k}{(1)_k^4\,(\frac{N+1}{2})_k}.$$

Changing the order of summation and noticing that

$$(\alpha)_{N-k}=\frac{(-1)^k(\alpha)_N}{(1-\alpha-N)_k}$$

we get

$$S_N = 2\binom{2N}{N}^4 (3N+1) \cdot {}_6F_5\left(\begin{matrix} 1, \frac{1}{2} - \frac{3}{2}N, -N, -N, -N, -N \\ -\frac{1}{2} - \frac{3}{2}N, -2N, -2N, -2N, -2N \end{matrix} \middle| 1 \right). \tag{41}$$

To evaluate the hypergeometric series on the right-hand side of (41), we apply Whipple's transformation [29, (7.7)] which transforms a Saalschutian ${}_4F_3(1)$ series into a well-poised ${}_7F_6(1)$ series (see [23, p. 61, (2.4.1.1)]):

$$\begin{aligned} {}_4F_3\left(\begin{matrix} f-a_1-a_2, d_1, d_2, -N \\ f-a_1, f-a_2, g \end{matrix} \middle| 1 \right) &= \frac{(g-d_1)_N (g-d_2)_N}{(g)_N (g-d_1-d_2)_N} \\ \times {}_7F_6 &\left(\begin{matrix} f-1, \frac{1}{2}f+\frac{1}{2}, a_1, a_2, d_1, d_2, -N \\ \frac{1}{2}f-\frac{1}{2}, f-a_1, f-a_2, f-d_1, f-d_2, f+N \end{matrix} \middle| 1 \right). \end{aligned} \tag{42}$$

Setting $a_1 = a_2 = -N$, $d_1 = -N$, $d_2 = 1$, $f = -3N$ in (42), we get

$$\begin{aligned} &{}_6F_5\left(\begin{matrix} 1, \frac{1}{2} - \frac{3}{2}N, -N, -N, -N, -N \\ -\frac{1}{2} - \frac{3}{2}N, -2N, -2N, -2N, -2N \end{matrix} \middle| 1 \right) \\ &\quad = \frac{(2N+1)^2}{(3N+1)(N+1)} {}_4F_3\left(\begin{matrix} 1, -N, -N, -N \\ -2N, -2N, N+2 \end{matrix} \middle| 1 \right). \end{aligned} \tag{43}$$

Therefore from (41) and (43) we get

$$S_N = 2\binom{2N}{N}^4 \frac{(2N+1)^2}{N+1} \cdot {}_4F_3\left(\begin{matrix} 1, -N, -N, -N \\ -2N, -2N, N+2 \end{matrix} \middle| 1 \right)$$

or

$$S_N = \frac{2(2N+1)!^2 (2N)!^2}{(N+1)! N!^7} \sum_{k=0}^{N} \frac{(-N)_k^3}{(-2N)_k^2 (N+2)_k}. \tag{44}$$

Replacing Pocchammer's symbols by factorials in (44) we arrive at

$$S_N = (N+1)\binom{2N+2}{N+1} \sum_{k=0}^{N} (-1)^k \binom{2N+1}{N-k} \binom{2N-k}{N-k}^2. \tag{45}$$

Now replacing N by $N-1$ in (45), and using (40) and Lemma 1 we get the required identity. □

To prove Theorem 6, we need several results concerning harmonic sums modulo a power of prime p. The multiple harmonic sum is defined by

$$H(a_1, a_2, \dots, a_r; n) = \sum_{1 \le k_1 < k_2 < \cdots < k_r \le n} \frac{1}{k_1^{a_1} k_2^{a_2} \cdots k_r^{a_r}},$$

where $n \ge r \ge 1$ and $(a_1, a_2, \dots, a_r) \in \mathbb{N}^r$. For $r = 1$ we will also use the notation

$$H_n^{(a)} := H(a; n) = \sum_{k=1}^{n} \frac{1}{k^a} \quad \text{and} \quad H_n := H_n^{(1)}.$$

The values of many harmonic sums modulo a power of prime p are well known. We need the following results.

Lemma 3 [24, Theorem 5.1] *Let p be a prime greater than 5. Then*

$$H_{p-1} \equiv H_{p-1}^{(3)} \equiv 0 \pmod{p^2}, \qquad H_{p-1}^{(2)} \equiv H_{p-1}^{(4)} \equiv 0 \pmod{p}.$$

Lemma 4 *Let $p > 5$ be a prime. Then*

$$H(\{1\}^2; p-1) \equiv -\frac{1}{2} H_{p-1}^{(2)} \pmod{p^4}, \qquad H(\{1\}^3; p-1) \equiv 0 \pmod{p^2},$$

$$H(\{1\}^4; p-1) \equiv 0 \pmod{p}.$$

Proof. Since for $n \geq 1$ (see, for example, [28])

$$H(\{1\}^2; n) = \frac{1}{2}(H_n^2 - H_n^{(2)}),$$

$$H(\{1\}^3; n) = \frac{1}{6}(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}),$$

$$H(\{1\}^4; n) = \frac{1}{24}(H_n^4 - 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 - 6H_n^{(4)}),$$

then by Lemma 3, we get the required congruences. □

Lemma 5 *Let $p > 5$ be a prime. Then*

$$\frac{1}{2}\binom{2p}{p} \equiv 1 + pH_{p-1} - \frac{p^2}{2} H_{p-1}^{(2)} \pmod{p^5}.$$

Proof. It is readily seen that

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} = \prod_{j=1}^{p-1}\left(1 + \frac{p}{j}\right) = \sum_{j=0}^{p-1} p^j H(\{1\}^j; p-1)$$

and therefore

$$\frac{1}{2}\binom{2p}{p} \equiv 1 + pH_{p-1} + p^2 H(\{1\}^2; p-1) + p^3 H(\{1\}^3; p-1) + p^4 H(\{1\}^4; p-1) \pmod{p^5}.$$

Now by Lemma 4, we get the required congruence. □

For similar congruences related to the central binomial coefficients, see [27].

Proof of Theorem 6.

From Theorem 5 with $n = p$, where $p > 5$ is a prime, we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^5(205k^2+160k+32) = 8p^2\binom{2p}{p}^2\sum_{k=0}^{p-1}(-1)^k\binom{2p-1}{p+k}\binom{2p-k-2}{p-1-k}^2. \quad (46)$$

For the sum on the right-hand side of (46), changing the order of summation we have

$$\begin{aligned}\sum_{k=0}^{p-1}(-1)^k\binom{2p-1}{p+k}\binom{2p-k-2}{p-1-k}^2 &= \sum_{k=0}^{p-1}(-1)^k\binom{2p-1}{k}\binom{p-1+k}{k}^2 \\ &= 1+\sum_{k=1}^{p-1}(-1)^k\frac{(2p-1)(2p-2)\cdots(2p-k)}{k!}\frac{p^2(p+1)^2\cdots(p+k-1)^2}{k!^2} \\ &= 1+p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\prod_{j=1}^{k}\left(1-\frac{2p}{j}\right)\prod_{j=1}^{k-1}\left(1+\frac{p}{j}\right)^2.\end{aligned} \quad (47)$$

Since

$$\prod_{j=1}^{k}\left(1-\frac{2p}{j}\right) \equiv 1-2pH_k+4p^2H(\{1\}^2;k) = 1-2pH_k+2p^2(H_k^2-H_k^{(2)}) \pmod{p^3}$$

and

$$\begin{aligned}\prod_{j=1}^{k-1}\left(1+\frac{p}{j}\right)^2 &= \prod_{j=1}^{k-1}\left(1+\frac{2p}{j}+\frac{p^2}{j^2}\right) \equiv 1+2pH_{k-1}+p^2H_{k-1}^{(2)}+4p^2H(\{1\}^2;k-1) \\ &= 1+2pH_{k-1}+2p^2H_{k-1}^2-p^2H_{k-1}^{(2)} \pmod{p^3},\end{aligned}$$

substituting these congruences in (47) and simplifying we obtain

$$\sum_{k=0}^{p-1}(-1)^k\binom{2p-1}{p+k}\binom{2p-k-2}{p-1-k}^2 \equiv 1+p^2H_{p-1}^{(2)}-2p^3H_{p-1}^{(3)}-3p^4\sum_{k=1}^{p-1}\frac{H_{k-1}^{(2)}}{k^2} \pmod{p^5}. \quad (48)$$

Note that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k-1}^{(2)}}{k^2} &= \sum_{k=1}^{p-1}\frac{1}{k^2}\sum_{j=1}^{k-1}\frac{1}{j^2} = \sum_{k=1}^{p-1}\frac{1}{k^2}\sum_{j=1}^{k}\frac{1}{j^2}-H_{p-1}^{(4)} \\ &= \sum_{j=1}^{p-1}\frac{1}{j^2}\left(H_{p-1}^{(2)}-H_{j-1}^{(2)}\right)-H_{p-1}^{(4)} = \left(H_{p-1}^{(2)}\right)^2-\sum_{j=1}^{p-1}\frac{H_{j-1}^{(2)}}{j^2}-H_{p-1}^{(4)}\end{aligned}$$

and therefore,

$$2\sum_{k=1}^{p-1}\frac{H_{k-1}^{(2)}}{k^2} = \left(H_{p-1}^{(2)}\right)^2-H_{p-1}^{(4)}. \quad (49)$$

Now by Lemma 3, from (48) and (49) we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{2p-1}{p+k}\binom{2p-k-2}{p-1-k}^2 \equiv 1+p^2H_{p-1}^{(2)} \pmod{p^5}. \tag{50}$$

From Lemma 5 we find easily

$$\binom{2p}{p}^2 \equiv 4\Big(1+2pH_{p-1}-p^2H_{p-1}^{(2)}\Big) \pmod{p^5}. \tag{51}$$

Now from (46), (50) and (51), by Lemma 3, for any prime $p>5$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^5(205k^2+160k+32) &\equiv 32p^2\Big(1+p^2H_{p-1}^{(2)}\Big)\Big(1+2pH_{p-1}-p^2H_{p-1}^{(2)}\Big)\\ &\equiv 32p^2+64p^3H_{p-1} \pmod{p^7}.\end{aligned}$$

The validity of this congruence for $p=3$ can be easily checked by straightforward verification. Taking into account that for an odd prime p,

$$\binom{2k}{k}^5 \equiv 0 \pmod{p^5}$$

for $k=\frac{p+1}{2},\dots,p-1$, and applying Lemma 3, we get the second congruence of Theorem 6.

6 A supercongruence arising from a series for the constant K.

In Section 3 we proved the accelerated convergence formula

$$K=\sum_{n=1}^{\infty}\frac{(-27)^{n-1}(15n-4)}{n^3\binom{2n}{n}^2\binom{3n}{n}},$$

which was earlier conjectured by Z. W. Sun. Motivated by this series, Z. W. Sun [25, Conj. 5.6] formulated the following conjecture on supercongruences: for any prime $p>3$ and a positive integer a,

$$\sum_{k=0}^{p^a-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 4\left(\frac{p^a}{3}\right)p^a \pmod{p^{2+a}}.$$

In this connection we prove here the following theorem.

Theorem 7 *Let p be a prime greater than 3. Then*

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 4\left(\frac{p}{3}\right)p \pmod{p^2}.$$

Proof Consider the WZ pair (F, G) defined in the proof of Theorem 1 with $\alpha = 1/3$, $\beta = 2/3$, $a = 0$:

$$F(n,k) = \frac{(-1)^n(3n+1)!n!^3}{(2n)!27^n} \cdot \frac{\left(\frac{1}{2}\right)_k^2 \left(\frac{2}{3}\right)_k^2 (n+2k+1)}{\left(\frac{1}{3}\right)_{n+k+1}^2 \left(\frac{2}{3}\right)_{n+k+1}^2},$$

$$G(n,k) = F(n,k) \cdot \frac{45n^2+63n+22+18k(3n+k+2)}{18(2n+1)(n+1+2k)}.$$

Then it is readily seen that the pair $(\bar{F}, \bar{G})$ given by

$$\bar{F}(n,k) = \frac{(-27)^{-k}(2k)!(3k+3n)!^2 n!^2}{(3k+1)!k!^3(k+n)!^2(3n)!^2}\left((15k+4)(3k+1)+18n(3k+n+1)\right),$$

$$\bar{G}(n,k) = \frac{3k^3(3k-1)(-27)^{1-k}(2k)!(3k+3n)!^2 n!^2(2n+k+1)}{(3k)!k!^3(k+n)!^2(3n+2)!^2}$$

is its dual WZ pair (see Proposition C or [20, Ch. 7]), for which we have

$$\bar{F}(n+1,k) - \bar{F}(n,k) = \bar{G}(n,k+1) - \bar{G}(n,k), \qquad n,k \geq 0. \tag{52}$$

Summing (52) over $k = 0, 1, \ldots, p-1$ and observing that $\bar{G}(n,0) = 0$, we obtain

$$\sum_{k=0}^{p-1} \bar{F}(n+1,k) - \sum_{k=0}^{p-1} \bar{F}(n,k) = \bar{G}(n,p). \tag{53}$$

Further, for every integer n satisfying $0 \leq n < \frac{p-2}{3}$ we have

$$\bar{G}(n,p) = \frac{(-27)^{1-p}(2p)!(3p+3n)!^2 n!^2(2n+p+1)}{(3p-2)!(p-1)!^2 p!(p+n)!^2(3n+2)!^2} \equiv 0 \pmod{p^3}, \tag{54}$$

since the numerator is divisible by p^8 and the denominator is divisible by p^5 and not divisible by p^6. Now from (53) and (54) for any non-negative integer $n < \frac{p-2}{3}$, we have

$$\sum_{k=0}^{p-1} \bar{F}(0,k) \equiv \sum_{k=0}^{p-1} \bar{F}(1,k) \equiv \cdots \equiv \sum_{k=0}^{p-1} \bar{F}(n+1,k) \pmod{p^3}. \tag{55}$$

Moreover, from (55) we obtain

$$\bar{F}(0,k) \equiv \begin{cases} \sum_{k=0}^{p-1} \bar{F}\left(\frac{p-1}{3}, k\right) \pmod{p^3}, & \text{if } p \equiv 1 \pmod 3; \\ \sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \pmod{p^3}, & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{56}$$

Since

$$\sum_{k=0}^{p-1} \bar{F}(0,k) = \sum_{k=0}^{p-1} \frac{(15k+4)}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k},$$

it is sufficient to prove the required congruence for the right-hand side of (56). We consider separately two cases depending on the sign of $(\frac{p}{3})$.

First, let $p \equiv 2 \pmod 3$, then we have

$$\bar{F}\left(\frac{p-2}{3}, k\right) = \frac{(-27)^{-k}(2k)!(3k+p-2)!^2\left(\frac{p-2}{3}\right)!^2}{(3k+1)!k!^3\left(k+\frac{p-2}{3}\right)!^2(p-2)!^2}((15k+4)(3k+1)+2(p-2)(9k+p+1)).$$

Note that if $1 \le k \le \frac{p-2}{3}$, then the denominator of $\bar{F}(\frac{p-2}{3}, k)$ is not divisible by p and the numerator is divisible by p^2. Therefore, we have

$$\bar{F}\left(\frac{p-2}{3}, k\right) \equiv 0 \pmod{p^2}, \qquad k = 1, 2, \ldots, \frac{p-2}{3}. \tag{57}$$

Let $\operatorname{ord}_p n$ be the p-adic order of n that is the exponent of the highest power of p dividing n. It is clear that

$$\operatorname{ord}_p \bar{F}\left(\frac{p-2}{3}, \frac{p+1}{3}\right) = \operatorname{ord}_p \frac{3^{-p-1}\left(\frac{2p+2}{3}\right)!(2p-1)!^2\left(\frac{p-2}{3}\right)!^2}{(p+2)!\left(\frac{p+1}{3}\right)!^3\left(\frac{2p-1}{3}\right)!^2(p-2)!^2} = 1. \tag{58}$$

Similarly, considering the disjointed intervals $\frac{p+4}{3} \le k \le \frac{p-1}{2}$, $\frac{p+1}{2} \le k \le \frac{2p-1}{3}$, and $\frac{2p+2}{3} \le k \le p-1$, we obtain

$$\operatorname{ord}_p \bar{F}\left(\frac{p-2}{3}, k\right) \ge 3, \qquad k = \frac{p+4}{3}, \frac{p+4}{3}+1, \ldots, p-1. \tag{59}$$

Thus from (57)–(59) we have

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \equiv \bar{F}\left(\frac{p-2}{3}, 0\right) + \bar{F}\left(\frac{p-2}{3}, \frac{p+1}{3}\right) \pmod{p^2}$$

or

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \equiv 4 + 2(p-2)(p+1) + \frac{3^{-p-1}\left(\frac{2p+2}{3}\right)!(2p-1)!^2\left(\frac{p-2}{3}\right)!^2}{(p+2)!\left(\frac{p+1}{3}\right)!^3\left(\frac{2p-1}{3}\right)!^2(p-2)!^2} \times ((5p+9)(p+2)+8(p-2)(p+1)) \pmod{p^2}.$$

Taking into account that

$$\frac{(2p-1)!^2}{(p-2)!^2(p+2)!} = \frac{(p-1)^2 p}{(p+1)(p+2)} \frac{(p+1)^2(p+2)^2\cdots(2p-1)^2}{(p-1)!}$$
$$\equiv \frac{(p-1)^2 p}{(p+1)(p+2)} \cdot (p-1)! \pmod{p^2} \equiv \frac{p!}{2} \pmod{p^2},$$

$$(5p+9)(p+2)+8(p-2)(p+1) \equiv 2 \pmod p,$$

and simplifying, we get

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \equiv -2p + \frac{2\cdot 3^{-p}\cdot p!}{\left(\frac{2p-1}{3}\right)!\left(\frac{p+1}{3}\right)!} \pmod{p^2}.$$

For primes $p > 3$, by Fermat's theorem, we have $3^p \equiv 3 \pmod{p}$ and therefore,

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \equiv -2p + \frac{2}{3}\frac{p!}{\left(\frac{2p-1}{3}\right)!\left(\frac{p+1}{3}\right)!} \pmod{p^2}. \tag{60}$$

Now put $n = \frac{p+1}{3}$ and note that

$$\begin{aligned}\frac{p\,!}{\left(\frac{2p-1}{3}\right)!\left(\frac{p+1}{3}\right)!} &= \binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!} = p\frac{(-1)^{n-1}}{n}\prod_{j=1}^{n-1}\left(1-\frac{p}{j}\right)\\ &\equiv \frac{(-1)^{n-1}}{n}p \pmod{p^2} \equiv -3p \pmod{p^2},\end{aligned} \tag{61}$$

where in the last equality we used the fact that $n-1 = \frac{p-2}{3}$ is odd (otherwise, a prime $p > 3$ must be even). Finally, substituting (61) in (60) we obtain

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-2}{3}, k\right) \equiv -2p - 2p = -4p = 4\left(\frac{p}{3}\right)p \pmod{p^2},$$

which proves the theorem for primes $p \equiv 2 \pmod{3}$.

Now suppose that $p \equiv 1 \pmod{3}$. Then we have

$$\sum_{k=0}^{p-1} \bar{F}(0,k) \equiv \sum_{k=0}^{p-1} \bar{F}\left(\frac{p-1}{3}, k\right) \pmod{p^3},$$

where

$$\bar{F}\left(\frac{p-1}{3}, k\right) = \frac{(-27)^{-k}(2k)!(3k+p-1)!^2\left(\frac{p-1}{3}\right)!^2}{(3k+1)!k!^3\left(k+\frac{p-1}{3}\right)!^2(p-1)!^2}((15k+4)(3k+1)+2(p-1)(9k+p+2)).$$

Note that if $1 \le k < \frac{p-1}{3}$, then the denominator of $\bar{F}(\frac{p-1}{3}, k)$ is not divisible by p. On the other hand, $\mathrm{ord}_p\,(3k+p-1)!^2 = 2$ and therefore, $\bar{F}(\frac{p-1}{3}, k) \equiv 0 \pmod{p^2}$. It is clear that

$$\mathrm{ord}_p\,\bar{F}\left(\frac{p-1}{3}, \frac{p-1}{3}\right) = 1.$$

Similarly, if $\frac{p+2}{3} \le k \le \frac{2p-2}{3}$, then $\mathrm{ord}_p\,\bar{F}(\frac{p-1}{3}, k) \ge 3$, and if $\frac{2p+1}{3} \le k \le p-1$, then $\mathrm{ord}_p\,\bar{F}(\frac{p-1}{3}, k) = 3$. Therefore, we have

$$\sum_{k=0}^{p-1} \bar{F}\left(\frac{p-1}{3}, k\right) \equiv \bar{F}\left(\frac{p-1}{3}, 0\right) + \bar{F}\left(\frac{p-1}{3}, \frac{p-1}{3}\right) \pmod{p^2}, \tag{62}$$

where

$$\bar{F}\left(\frac{p-1}{3},0\right)=4+2(p-1)(p+2)\equiv 2p \pmod{p^2} \tag{63}$$

and

$$\bar{F}\left(\frac{p-1}{3},\frac{p-1}{3}\right)=\frac{3^{1-p}(2p-2)!^2(p(5p-1)+2(p-1)(4p-1))}{\left(\frac{2p-2}{3}\right)!\left(\frac{p-1}{3}\right)!p!(p-1)!^2}.$$

Noting that

$$\frac{(2p-2)!^2}{p!(p-1)!^2}=\frac{p}{p-1}\frac{(p+1)^2(p+2)^2\cdots(2p-2)^2}{(p-2)!}\equiv\frac{p}{p-1}(p-2)!\equiv p! \pmod{p^2},$$

and applying Fermat's theorem, we have

$$\bar{F}\left(\frac{p-1}{3},\frac{p-1}{3}\right)\equiv\frac{2\cdot p!}{\left(\frac{2p-2}{3}\right)!\left(\frac{p-1}{3}\right)!} \pmod{p^2}.$$

Now setting $n=\frac{p-1}{3}$ and taking into account that

$$\frac{(p-1)!}{\left(\frac{2p-2}{3}\right)!\left(\frac{p-1}{3}\right)!}=\binom{p-1}{n}=(-1)^n\prod_{j=1}^{n}\left(1-\frac{p}{j}\right)\equiv 1 \pmod{p}$$

we have

$$\bar{F}\left(\frac{p-1}{3},\frac{p-1}{3}\right)\equiv 2p \pmod{p^2}. \tag{64}$$

Now by (62)–(64), we obtain

$$\sum_{k=0}^{p-1}\bar{F}\left(\frac{p-1}{3},k\right)\equiv 4p=4\left(\frac{p}{3}\right)p \pmod{p^2},$$

which proves the theorem for primes $p\equiv 1 \pmod 3$. □

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