Andrews' conjecture on a ${}_4\phi_3$ summation and its extensions

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Dedicated to Doron Zeilberger on the occasion of his 60th birthday.

Abstract

We give a computer proof of Andrews' conjecture on a ${}_4\phi_3$ summation and extend the result to a family of ${}_4\phi_3$ summations.

Keywords: q-Zeilberger algorithm, Andrews' ${}_4\phi_3$ summation, creative symmetrizing

Let

$$\Phi_n(x,u,v,y,z) = {}_4\phi_3\left[\begin{matrix} q^{-2n}, a, b, q^{x-2n}/ab \\ q^{u-2n}/a, q^{v-2n}/b, abq^y \end{matrix} ; q^2, q^z\right]$$

be a basic hypergeometric series with parameters x, u, v, y, z. (See [4, Section 1.2] for the notation.) Andrews [1] proved that

$$\Phi_n(1,2,2,1,2) = \frac{(a,b,-q;q)_n(ab;q^2)_n}{q^n(ab;q)_n(a,b;q^2)_n}, \tag{1}$$

and conjectured that

$$\Phi_n(3,2,4,1,2) = -\frac{(a,-q;q)_n(b;q)_{n-1}(ab;q^2)_{n-1}}{q^{n+1}(ab;q)_{n-1}(a,b/q^2;q^2)_n(1-abq^{2n-1})} \times \left(abq^{2n-2}(q^2-b) + abq^{n-1}(1-q) - q + b\right). \tag{2}$$

Guo [5] proved the conjecture by utilizing contiguous relations. Here we provide a computer proof based on the q-Zeilberger algorithm [2, 7, 9].

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Proof of Equation (2). Let $t_{n,k}$ be the kth summand of $\Phi_n(3,2,4,1,2)$. Notice that the ratio

$$r(n,k) = \frac{t_{n,n-k}}{t_{n,k}} = \frac{(-bq^{2n}+q^{2k+2})(-q+abq^{2k})q^{2k-n}}{(-q^2+bq^{2k})(q^{2k+1}-abq^{2n})}$$

is a rational function of q^n and q^k. We can apply Paule's method of creative symmetrizing [6, 8], i.e.,

$$\sum_{k=0}^{n} t_{n,k} = \sum_{k=0}^{n} \frac{t_{n,k}+t_{n,n-k}}{2} = \frac{1}{2}\sum_{k=0}^{n} \big(1+r(n,k)\big)t_{n,k}.$$

Now applying the q-Zeilberger algorithm to $\big(1+r(n,k)\big)t_{n,k}$, we find that the sum $S_n = \Phi_n(3,2,4,1,2)$ satisfies

$$\begin{aligned}(1-bq^{n-1})&(1-abq^{2n-1})(1-abq^{2n-2})(1-aq^n)(1+q^{n+1})P(n)S_n\\ &-(1-abq^{n-1})(1-aq^{2n})(1-abq^{2n+1})(1-bq^{2n-2})qP(n-1)S_{n+1}=0,\end{aligned}$$

where

$$P(n) = (ab^2q^{2n} - abq^{2n+2} - abq^n + abq^{n+1} - b + q).$$

Since $S_0 = 1$, we immediately derive Equation (2). □

Remark. Applying the q-Zeilberger algorithm directly to $t_{n,k}$, we will obtain a recursion on n of order 2.

Let

$$H_n = \Phi_n(1,2,2,1,2) = \frac{(a,b,-q;q)_n(ab;q^2)_n}{q^n(ab;q)_n(a,b;q^2)_n}.$$

We have the following theorem on $\Phi_n(x,u,v,y,z)$ with integer parameters.

Theorem 1. *Let i,j,k,ℓ,m be integers such that*

$$i \geq j+k+\ell+m \quad \text{and} \quad k,\ell,m \geq 0.$$

Then the quotient

$$\Phi_n(2j+1,-2\ell+2,2i,-2m+1,2k+2)/H_n$$

is a rational function in q^n.

Proof. With the same method as in the proof of Equation (2), we can derive that the assertion holds for

$$\begin{array}{lll}\Phi_n(-1,2,-2,1,2), & \Phi_n(1,2,0,1,2), & \Phi_n(3,2,2,1,2),\\ \Phi_n(-1,2,0,1,2), & \Phi_n(1,2,2,1,2), & \Phi_n(3,2,4,1,2),\\ \Phi_n(-1,2,2,1,2), & \Phi_n(1,2,4,1,2). & \end{array}$$

By the q-Zeilberger algorithm, $S(v) = \Phi_n(-1,2,2v,1,2)$ satisfies a recursion of the form

$$P_3(n)S(v+3) + P_2(n)S(v+2) + P_1(n)S(v+1) + P_0(n)S(v) = 0, \tag{3}$$

where each $P_i(n)$ is a polynomial in q^n. In particular, we have

$$P_3(n) = q^{2n}(q^{2v+4} - abq^{2n})(q^{2v+4} - b^2q^{2n})(q^{2v+4} - b)(1 - aq^{2v+5}),$$
$$P_0(n) = -abq^6(1 - q^{2v+4})(q^{2v} - bq^{2n})(q^{2v+4} - bq^{2n})(q^{2v+2} - bq^{2n}).$$

By iterating the recursion (3), we derive that for each integer $i \geq -1$, the quotient $\Phi_n(-1, 2, 2i, 1, 2)/H_n$ is a rational function in q^n.

Now by the extended q-Zeilberger algorithm [3], we find that $S(x, v) = \Phi_n(x, 2, v, 1, 2)$ satisfies

$$S(x+2, v) = \frac{b(-q^{x+2} + aq^v)q^{-v+2n}}{-q^x + abq^{2n}} S(x, v) + \frac{q^{x-v}(-q^v + bq^{2n+2})}{-q^x + abq^{2n}} S(x, v-2).$$

More precisely, the relation is built by the command `qExt_Zeil` of the package `EZA` (available from `http://www.combinatorics.net.cn/homepage/hou`):

```
> qExt_Zeil([subs(x=x+2,Sk), Sk, subs(v=v-2,Sk)], q, k);
```

where `Sk` is the kth summand of $S(x, v)$. An alternative way is to use the command `qTelescope` of Riese's package `qZeil` [7] with a suitable setting of the parameter `qParameterized`.

By iterating the above recursion, we derive that for any integers $i \geq j \geq -1$, the quotient $\Phi_n(2j+1, 2, 2i, 1, 2)/H_n$ is a rational function in q^n. We also find by the extended q-Zeilberger algorithm that

$$S(x-2, v-2) = R_0(n)S(x, v) + R_1(n)S(x, v+2) + R_2(n)S(x, v+4),$$

where $R_i(n)$ are rational functions in q^n. Thus the restriction $j \geq -1$ can be removed so that the assertion holds for $\Phi_n(2j+1, 2, 2i, 1, 2)$ provided $i \geq j$.

We further find that $S(x, z) = \Phi_n(x, 2, v, 1, z)$ satisfies

$$S(x, z+2) = abq^{2n-x}S(x, z) + q^{-x}(q^x - abq^{2n})S(x+2, z).$$

Therefore, for any integers $i \geq j+k$ and $k \geq 0$, the quotient $\Phi_n(2j+1, 2, 2i, 1, 2k+2)/H_n$ is a rational function in q^n.

By a similar discussion, the sum $S(u, z) = \Phi_n(x, u, v, 1, z)$ satisfies

$$S(u-2, z) = \frac{aq^{2n+2}}{-q^u + aq^{2n+2}} S(u, z) - \frac{q^u}{-q^u + aq^{2n+2}} S(u, z+2).$$

Thus, for any integers $i \geq j + k + \ell$ and $k, \ell \geq 0$, the quotient $\Phi_n(2j+1, 2-2\ell, 2i, 1, 2k+2)/H_n$ is a rational function in q^n.

Finally, the sum $S(y, z) = \Phi_n(x, u, v, y, z)$ satisfies

$$S(y-2, z) = \frac{q^2}{q^2 - abq^y} S(y, z) - \frac{abq^y}{q^2 - abq^y} S(y, z+2),$$

which completes the proof. □

As examples, by Paule's symmetric technique and the q-Zeilberger algorithm, we derive that

$$\Phi_n(1,2,2,-1,2) = \frac{(a,b,-q;q)_n(ab;q^2)_n}{q^n(ab/q;q)_n(a,b;q^2)_n},$$

and

$$\Phi_n(-1,0,0,1,2) = \frac{(aq,bq,-q;q)_n(abq^2,q^2)_n}{(abq;q)_n(aq^2,bq^2;q^2)_n}.$$

The second equation coincides with the last formula in [5].

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