Concave Compositions

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Dedicated to my friend Doron Zeilberger.

Abstract

Concave compositions are compositions (i.e. ordered partitions) of a number in which the parts decrease up to the middle summand(s) and increase thereafter. Perhaps the most surprising result is for even length, concave compositions where the generating function turns out to be the quotient of two instances of the pentagonal number theorem with variations of sign. The false theta function discoveries also lead to new facts about concatenatable, spiral, self-avoiding walks.

1 Introduction

Concave compositions are situated between ordinary integer partitions (where $3+3+1$ and $3+1+3$ are considered the same partition of 7) and integer compositions (where $3+3+1$ and $3+1+3$ are considered two different compositions of 7).

Concave compositions of even length (CCELs), $2m$, are sums of the form $\sum a_i + \sum b_i$ where

$$a_1 > a_2 > \cdots > a_m = b_m < b_{m-1} < \cdots < b_1$$

where $a_m \geqq 0$.

Concave compositions of odd length, $2m+1$, come in two flavors. First (CCOL1s) are sums of the form $\sum a_i + \sum b_i$

$$a_1 > a_2 > \cdots > a_{m+1} < b_m < b_{m-1} < \cdots < b_1,$$

where $a_{m+1} \geqq 0$.

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Second (CCOL2s)

$$a_1 > a_2 > \cdots > a_{m+1} \geqq b_m < b_{m-1} < \cdots < b_1,$$

where $a_{m+1} \geqq 0$.

We denote by $\mathrm{ce}(n)$, $\mathrm{co}_1(n)$ and $\mathrm{co}_2(n)$ the number of CCELs, CCOL1s, CCOL2s respectively that sum to n. The related generating functions are $\mathcal{CE}(q) = \sum_{n\geqq 0} \mathrm{ce}(n)q^n$, $\mathcal{CO}_1(q) = \sum_{n\geqq 0} \mathrm{co}_1(n)q^n$, and $\mathcal{CO}_2(q) = \sum_{n\geqq 0} \mathrm{co}_2(n)q^n$.

Theorem 1. *For* $|q| < 1$,

$$\begin{aligned}\mathcal{CE}(q) &= \frac{1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1-q^n)}{1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1+q^n)} \\ &= \frac{1 - q + q^2 - q^5 + q^7 - q^{12} + q^{15} - \cdots}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots} \\ &= 1 + 2q^2 + 2q^3 + 4q^4 + 4q^5 + 8q^6 + \cdots .\end{aligned}$$

By the way of illustrations we note that the eight CCELs of 6 are 5001, 1005, 4002, 2004, 3003, 33, 2112, 210012.

Theorem 2. *For* $|q| < 1$,

$$\begin{aligned}\mathcal{CO}_1(q) &= \frac{1 - \sum_{n=1}^{\infty} q^{2n(3n-1)}(1-q^{4n})}{1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1+q^n)} \\ &= \frac{1 - q^4 + q^8 - q^{20} + q^{28} - q^{48} + q^{60} - \cdots}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots} \\ &= 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + 9q^6 + \cdots .\end{aligned}$$

Here there are nine CCOL1s of 6: 6, 501, 105, 402, 204, 303, 312, 213, 21012.

Theorem 3. *For* $|q| < 1$,

$$\begin{aligned}\mathcal{CO}_2(q) &= \frac{1 + \sum_{n=1}^{\infty} q^{2n(3n-4)+3}(1-q^{4n-2})}{1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1+q^n)} \\ &= \frac{1 + q - q^3 + q^{11} - q^{17} + q^{33} - q^{43} + \cdots}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots} \\ &= 1 + 2q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + 15q^6 + \cdots .\end{aligned}$$

Here there are fifteen CCOL2s of 6, namely the nine listed after Theorem 2 plus the following six: 600, 411, 41001, 32001, 31002, 21003.

The impetus for this work lies in [6]. In that paper, the role of q-orthogonal polynomials in the study of Rogers-Ramanujan type identities was being extended. The results here in Section 3 are a natural outgrowth of that work. From these analytic results, concave compositions developed naturally.

In Section 2 we present q-series representations of $\mathcal{CE}(q)$, $\mathcal{CO}_1(q)$ and $\mathcal{CO}_2(q)$. Section 3 provides new q-series identities that are necessary to prove Theorems 1–3. Section 4 then proves these theorems, Section 5 is devoted to related partition identities. Perhaps the simplest of these can be stated using Frobenius symbol representation of partitions [3; p. 1] or [11; p. 60].

Theorem 4. *The number of partitons of n whose Frobenius symbol has no 0 on the top row equals the number of partitions of n in which the smallest number that is* not *a summand is odd.*

For example, there are eight such partitions of 7 with the required Frobenius symbol:

$$\binom{6}{0}, \binom{5}{1}, \binom{1}{5}, \binom{4}{2}, \binom{2}{4}, \binom{3}{3}, \binom{31}{10}, \binom{21}{20},$$

and the eight partitions of 7 with an odd the smallest non-summand are:

$7,\ 5+2,\ 4+3,\ 4+2+1,\ 3+2+2,\ 2+2+2+1,\ 2+2+1+1+1,\ 2+1+1+1+1+1+1.$

Section 6 looks at the implications for concatenatable SSAWS. Section 7 concludes with a plethora of questions raised by these discoveries.

2 Generating Functions

In this section we present three very similar theorems. The proofs are so much that same that we only include the first in detail.

Theorem 5.

$$\mathcal{CE}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2(1-q^{2n+2})},$$

where $(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$.

Proof. The CCELs with $2m+2$ parts have as generating function

$$\sum_{m=0}^{\infty} \sum_{\substack{a_1>\cdots>a_m>a_{m+1}=b_{m+1}<\cdots<b_1 \\ a_{m+1}=b_{m+1}\geqq 0}} q^{a_1+\cdots+a_{m+1}+b_{m+1}+\cdots+b_1}$$

$$= \sum_{m=0}^{\infty} \left(\frac{q^{\binom{m+1}{2}}}{(q)_m} \right)^2 \sum_{\substack{a_{m+1}=0 \\ a_{m+1}=b_{m+1}}}^{\infty} q^{ma_{m+1}+mb_{m+1}+a_{m+1}+b_{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m^2} \frac{1}{(1-q^{2m+2})}. \qquad \square$$

The penultimate line of the proof follows from the standard method for generating partitions into m distinct parts each larger than a_{m+1}.

Theorem 6.

$$\mathcal{CO}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2(1-q^{2n+1})}.$$

Proof. Everything is the same as the proof of Theorem 5 except b_{m+1} is missing. $\square$

Theorem 7.

$$\mathcal{CO}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2(1-q^{2n+1})}.$$

Proof. The previous arguments easily show that

$$\mathcal{CO}_2(q) - \mathcal{CO}_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_{n-1}(q)_n(1-q^{2n+1})},$$

and this together with Theorem 6 implies Theorem 7. $\square$

3 q-Series Identities

We begin with a new q-hypergeometric series sumation. We require the standard notation [8; p. 4]:

$${}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \ldots, a_r; q, t \\ b_1, \ldots, b_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n\cdots(a_r)_n t^n}{(q)_n(b_1)_n\cdots(b_r)_n}.$$

Theorem 8. *For integers* $0 \leqq m \leqq n$*,*

$${}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, x; q, q \\ aq^m, xq \end{matrix}\right) = \frac{x^n \left(\frac{a}{x}\right)_n (q)_n (a)_m}{\left(\frac{a}{x}\right)_m (xq)_n (a)_n}.$$

Remark. This result can be given a very elementary proof using Pfaff's method [5]. Our proof situates this identity in the q-hypergeometric literature.

Proof. Rewrite identity (III.11) from [8; p. 241] as

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, b, c; q, q \\ d, e \end{matrix}\right) = \frac{\left(\frac{bc}{d}\right)^n}{(e)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j \left(\frac{d}{b}\right)_j \left(\frac{d}{c}\right)_j q^j \left(\frac{deq^j}{bc}\right)_{n-j}}{(q)_j (d)_j}. \tag{3.1} $$

In this identity set $b = aq^n$, $d = aq^m$, $c = x$ and $e = xq$. Thus $d/b = q^{m-n}$, which means that each term of the sum vanishes for $j > n-m$. Also $deq^j/bc = q^{m+j+1-n}$, which means that each term of the sum vanishes if $j \leqq n-m-1$. Therefore the only non-vanishing term of the sum is $j = n-m$. Simplifying that one term yields the right-hand expression in Theorem 8. $\square$

Theorem 9.

$$ \sum_{n=0}^{\infty} \frac{q^{n^2} a^n (1-x)}{(q)_n (aq)_n (1-xq^n)} = \frac{1}{(aq;q)_\infty}\left(1 + \sum_{n=1}^{\infty} \frac{(-a)^n q^{n(3n-1)/2}(1-aq^{2n}) x^n \left(\frac{aq}{x}\right)_{n-1}}{(xq)_n}\right). $$

Proof. We begin by recalling the weak form of Bailey's Lemma [4; p. 27, eq.(3.33)]

$$ \sum_{j=0}^{\infty} q^{j^2} a^j \beta_j = \frac{1}{(aq)_\infty} \sum_{j=0}^{\infty} q^{j^2} a^j \alpha_j, \tag{3.2} $$

where

$$ \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} \tag{3.3} $$

or equivalently [2; p. 278, eq.(4.1)]

$$ \alpha_n = \frac{(1-aq^{2n})(-1)^n q^{\binom{n}{2}} (a)_n}{(1-a)(q)_n} \sum_{j=0}^{n} \left(q^{-n}\right)_j (aq^n)_j\, q^j \beta_j. \tag{3.4} $$

Now take

$$ \beta_n = \frac{(x)_n}{(aq)_n (xq)_n}, \tag{3.5} $$

and evaluate α_n by applying Theorem 8 with $m = 1$ (first noting that $\alpha_0 = \beta_0 = 1$), so for $n \geqq 1$

$$ \alpha_n = \frac{x^n \left(\frac{aq}{x}\right)_{n-1} (q)_n (1-a)}{(xq)_n (a)_n}. \tag{3.6} $$

Substituting (3.5) and (3.6) into (3.2) we obtain the identity stated in Theorem 9. $\square$

Theorem 10.

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n (1-a^2)}{(q)_n (a)_n\, (1-a^2q^{2n})} = \frac{1}{(aq;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{a^{2n} q^{n(3n-1)/2} (1-aq^{2n})(-q)_{n-1}}{(-aq)_n} \right).$$

Proof. We shall use the weak form of Bailey's Lemma as exemplified in (3.2) and (3.4) with now

$$\beta_n = \frac{(1-a^2)}{(q)_n (a)_n (1-a^2q^{2n})}. \tag{3.7}$$

Hence by (3.4), $\alpha_0 = \beta_0 = 1$, and for $n \geqq 1$

$$\begin{aligned} \alpha_n &= \frac{(1-aq^{2n})\,(-1)^n q^{\binom{n}{2}} (a)_n}{(1-a)(q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, -a; q, q \\ aq, -aq \end{matrix} \right) \\ &= \frac{(1-aq^{2n})\,(-1)^n q^{\binom{n}{2}} (a)_n}{(1-a)(q)_n} \cdot \frac{(-a)^n (-1)_n (q)_n (1-a)}{2(-aq)_n (a)_n} \\ &= \frac{a^n q^{\binom{n}{2}}\,(1-aq^{2n})\,(-q)_{n-1}}{(-aq)_n}. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.2), we obtain Theorem 9. □

4 Proofs of Theorems 1–3

Lemma 11.

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q)_n^2} = \frac{2}{(q)_\infty}.$$

Proof. We know [1; p. 20, set $z = q^{r+1}$ and multiply by $\frac{1}{(q)_r}$] (cf. [9; p. 872, eq.(1)] for case $r = 1$) that for each integer $r \geqq 0$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+rn}}{(q)_n (q)_{n+r}} = \frac{1}{(q)_\infty}. \tag{4.1}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q)_n^2} - \frac{1}{(q)_\infty} &= \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q)_n^2} - \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2} && \text{(by (4.1) with } r=0) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2-n}\,(1-q^n)}{(q)_n^2} = \sum_{n=1}^{\infty} \frac{q^{n^2-n}}{(q)_n (q)_{n-1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n (q)_{n+1}} = \frac{1}{(q)_\infty} && \text{(by (4.1) with } r=1), \end{aligned}$$

and comparing the extremes in the above string of equations we deduce Lemma 11. □

Proof of Theorem 1. If we set $a = 1$ in Theorem 10, we see that

$$1+2\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_n(q)_{n-1}(1-q^{2n})}=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}q^{n(3n-1)/2}(1-q^n)\right). \tag{4.2}$$

Also by Theorem 5

$$\begin{aligned}
1+2\sum_{n=1}^{\infty}&\frac{q^{n^2}}{(q)_n(q)_{n-1}(1-q^{2n})}+\mathcal{CE}(q)\\
&=1+2\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_n^2(1+q^n)}+\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n^2(1-q^{2n+2})}\\
&=\sum_{n=0}^{\infty}\frac{2q^{n^2}}{(q)_n^2(1+q^n)}+\sum_{n=1}^{\infty}\frac{q^{n^2-n}}{(q)_n(q)_{n-1}(1+q^n)}\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2-n}(2q^n+(1-q^n))}{(q)_n^2(1+q^n)}\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q)_n^2}=\frac{2}{(q)_\infty} \qquad \text{(by Lemma 11)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{CE}(q)&=\frac{2}{(q)_\infty}-\left(1+2\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q)_n^2(1+q^n)}\right)\\
&=\frac{2}{(q)_\infty}-\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}q^{n(3n-1)/2}(1-q^n)\right)\\
&=\frac{1}{(q)_\infty}\left(1-\sum_{n=1}^{\infty}q^{n(3n-1)/2}(1-q^n)\right),
\end{aligned}$$

which is Theorem 1. □

Proof of Theorems 2 and 3. We begin by noting that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n^2(1-q^{2n+1})}&=\sum_{n=0}^{\infty}\frac{q^{2n^2}(1+q^{2n+1})}{(q^2;q^2)_n^2(1-q^{4n+2})}\\
&=\mathcal{CO}_2(q^2)+q\mathcal{CO}_1(q^2),
\end{aligned} \tag{4.3}$$

by Theorems 6 and 7. Consequently, if we can prove that

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n^2(1-q^{2n+1})} = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n^2}\left(1+q^{2n+1}+q^{4n+2}+q^{6n+3}\right), \tag{4.4}$$

then Theorem 2 (resp. 3) will follow by extracting the odd (resp. even) parts of (4.4) and invoking (4.3).

To prove (4.4), we replace q by q^2 in Theorem 9; then we set $a = 1$ and $x = q$. Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n^2\,(1-q^{2n+1})} \\
&\qquad = \frac{1}{(q^2;q^2)_\infty\,(1-q)} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2}\,(1-q^{4n})\,(1-q)}{(1-q^{2n-1})\,(1-q^{2n+1})}\right) \\
&\qquad = \frac{1}{(q^2;q^2)_\infty} \left(\frac{1}{1-q} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2}\,(1-q^{2n-1}+q^{2n-1}\,(1-q^{2n-1}))}{(1-q^{2n-1})\,(1-q^{2n+1})}\right) \\
&\qquad = \frac{1}{(q^2;q^2)_\infty} \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{1-q^{2n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2+2n-1}}{1-q^{2n-1}}\right) \\
&\qquad = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}\,(1-q^{8n+4})}{1-q^{2n+1}}
\end{aligned}$$

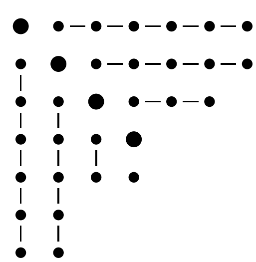
(replacing n by $n+1$ in the second sum)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{3n^2} \left(1 + q^{2n+1} + q^{4n+2} + q^{6n+3}\right),$$

which proves (4.4) and consequently Theorems 2 and 3. □

5 Partition Identities

We start this section with a result that is best phrased in terms of Frobenius symbols [3; p. 1]. We recall that the Frobenius symbol two equi-length rows of decreasing, non-negative integers. Usually one determines the Frobenius symbol from the Ferrers graph. For example, the partition $7+7+6+4+4+2+2$ (a partition of 32) has the Ferrers graph:



Now if one enumerates the entries in the 4 rows to the right of the diagonal for the top row, and the 4 columns below the diagonal for the bottom row, we obtain the Frobenius symbol of this partition:

$$\begin{pmatrix} 6 & 5 & 3 & 0 \\ 6 & 5 & 2 & 1 \end{pmatrix}.$$

Note that the number being partitioned is the sum of the entries in the Frobenius symbol plus the number of columns (i.e. $6+6+5+5+3+2+0+1+4=32$).

Lemma 12.

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

Remark. This result is effectively equivalent to an identity of Auluck [7; p. 681, eq.(10)]. We include a short proof for completeness.

Proof. In (III.2) of [8; p. 241] replace a and b each by q/τ, c by q^2, and z by τ^2. Then let $\tau \to 0$ and Lemma 12 follows. $\square$

Proof Theorem 4. If we let $F_\phi(n)$ denote the number of partitions of n whose Frobenius symbols do not have 0 in the top row, then

$$\begin{aligned}
\sum_{n=0}^{\infty} F_\phi(n) q^n &= 1 + \sum_{r=1}^{\infty} \sum_{\substack{a_1 > \cdots > a_r \geqq 1 \\ b_1 > \cdots > b_r \geqq 0}} q^{a_1 + \cdots a_r + b_1 + \cdots + b_r + r} \\
&= 1 + \sum_{r=1}^{\infty} \frac{q^{\binom{r+1}{2}}}{(q)_r} \cdot \frac{q^{\binom{r}{2}}}{(q)_r} \cdot q^r \\
&= 1 + \sum_{r=1}^{\infty} \frac{q^{r^2+r}}{(q)_r^2} \\
&= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} \qquad \text{(by Lemma 12)} \\
&= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} q^{n(2n+1)} \left(1 - q^{2n+1}\right) \\
&= \sum_{n=0}^{\infty} \frac{q^{1+2+3+\cdots+2n}}{\prod\limits_{\substack{j=1 \\ j \neq 2n+1}}^{\infty} \left(1 - q^j\right)}.
\end{aligned}$$

The n^{th} term of this latter sum is the generating function for partitions in which $2n+1$ is the smallest integer that is *not* a part. Summing over all n we find the latter sum is the generating function for all partitions in which the first non-summand is odd. $\square$

Corollary 13. *The number of partitions of n whose Frobenius symbol has a 0 on the top row equals the number of partitions of n in which the smallest number that is not a summand is even.*

Proof. The sets of partitions of n enumerated here are the respective complements of the sets of paritions of n listed in Theorem 4. $\square$

Theorems 1–3 each have interpretations of the same nature as Theorem 4. We state the interpretation explicitly for Theorem 1, and we leave the

Theorem 14. *$\mathcal{CE}(n)$ equals the number of partitions of n in which either: (i) 1 does not appear, or (ii) the smallest odd not appearing is $4m-3$ and $2m-1$ appears at least m times, or (iii) the smallest odd not appearing is $4m-1$ and $2m-1$ appears at least $m+1$ times.*

Proof. We clearly describe the way this result follows from Theorem 1.

$$\begin{aligned}
\mathcal{CE}(q) &= \frac{1}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}\left(1-q^{2n+1}\right)\\
&= \frac{(1-q)+(q^2-q^5)+(q^7-q^{12})+(q^{15}-q^{22})+(q^{26}-q^{35})+\cdots}{(q)_\infty}\\
&= \frac{1}{\prod\limits_{n=2}^{\infty}(1-q^n)} + \frac{q^{1+1}}{\prod\limits_{\substack{n=1\\ n\neq 3}}^{\infty}(1-q^n)} + \frac{q^{1+3+3}}{\prod\limits_{\substack{n=1\\ n\neq 5}}^{\infty}(1-q^n)} + \frac{q^{1+3+3+3+5}}{\prod\limits_{\substack{n=1\\ n\neq 7}}^{\infty}(1-q^n)}\\
&\quad + \frac{q^{1+3+5+5+5+7}}{\prod\limits_{\substack{n=1\\ n\neq 9}}^{\infty}(1-q^n)} + \frac{q^{1+3+5+5+5+5+7+9}}{\prod\limits_{\substack{n=1\\ n\neq 11}}^{\infty}(1-q^n)} + \frac{q^{1+3+5+7+7+7+7+9+11}}{\prod\limits_{\substack{n=1\\ n\neq 13}}^{\infty}(1-q^n)} + \cdots
\end{aligned}$$ $\square$

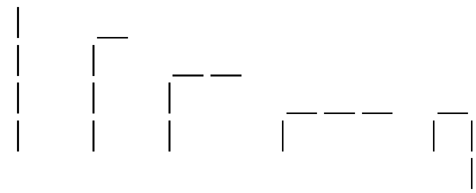
Similar results could be provided for $\mathcal{CO}_1(q)$ where now the second type of partitions deals with the smallest non-parts $\not\equiv 4 \pmod 8$. For $\mathcal{CO}_2(q) - \frac{1}{(q)_\infty}$, one would consider smallest non-parts $\equiv 2 \pmod 4$.

6 Concatenatable Spiral, Self-Avoiding Walks (= CSSAWs)

In [10], Guttmann and Hirschhorn reveal a bijection between the partitions of n, and the number of CSSAWs of length n.

Spiral, self-avoid walks are walks in the standard two-dimensional integral lattice with the requirement that (1) the walk starts along the positive y-axis, (2) each unit step is either straight ahead or a 90° turn to the right, (3) a possible right turn at the terminus of the walk could be of any length without intersecting the walk.

For example, the five CSSAWs of length 4 are



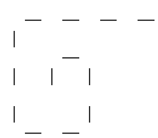
The Guttmann-Hirschhorn construction is effectively a mapping between the CSSAWs and the Frobenius symbols for the partitions of n. Namely, begin with a Frobenius symbol, say

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}.$$

Now add 1 to each entry in the bottom row to obtain

$$\begin{pmatrix} 3 & 2 & 1 \\ \downarrow\nwarrow & \downarrow\nwarrow & \downarrow \\ 4 & 2 & 1 \end{pmatrix}.$$

The arrows tell the successive lengths of segments in the CSSAW



When a positive integer terminates the top row, the CSSAW will of necessity have an odd number of turns. When zero terminates the top row originally, then there will be an even number of turns in the CSSAW.

Thus Theorem 4 is equivalent to the following:

Corollary 15. *The number of CSSAWs of length n with an odd number of turns equals the number of partitions of n in which the smallest integer* not *appearing is odd.*

In the case $n = 4$, we see three CSSAWs with an odd number of turns, and the related partitons are 4, $2+2$, $2+1+1$.

Corollary 13 is thus equivalent to the following:

Corollary 16. *The number of CSSAWs of length n with an even number of turns equals the number of partitions of n in which the smallest integer* not *appearing is even.*

In the case $n = 4$, we see two CSSAWs with an even number of turns, and the related partitions are $3+1$, $1+1+1+1$.

7 Conclusion and Open Problems

There are numerous other instances of

$$f(a,b,c) = (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(q)_n^2\,(1-q^{bn+c})}$$

that appear interesting. There are, of course, relationships of the form

$$f(a,b,c) - q^c f(a+b,b,c) = (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(q)_n^2}, \tag{7.1}$$

and

$$f(a,2b,2c) + q^c f(a+b,b,c) = f(a,b,c). \tag{7.2}$$

These indicate that some nice forms of $f(a,b,c)$ arise from linear combinations of simpler cases.

For example,

$$\begin{aligned} f(3,2,1) &= \frac{1}{q}\left(f(1,2,1) - (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2} \right) \\ &= \frac{1}{q}\left(1 - \sum_{n=1}^{\infty} q^{2n(3n-1)}\left(1-q^{4n}\right) - \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} \right) \end{aligned}$$

(by Theorems 2 and 6 and Lemma 12)

$$\begin{aligned} &= \sum_{n=1}^{\infty} \left(-q^{6n^2-2n-1} + q^{6n^2+2n-1} - (-1)^n q^{n(n+1)/2-1} \right) \\ &= 1 - q^2 - q^3 + q^5 + q^7 - q^9 + q^{14} - q^{15} - q^{20} + 2q^{27} - q^{35} + \cdots \end{aligned}$$

Obviously, it is natural to ask for combinatorial proof of Theorems 1, 2, 3, 4, and 14 (and Corollaries 13, 15 and 16).

This entire study had, as was mentioned in the introduction, its origins in bringing orthogonal polynomials especially the big q-Jacobi polynomials back into the study of q-series and partition identities [6]. This effort will be continued with further applications and examples.

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