The (t,q)-Analogs of Secant and Tangent Numbers

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To Doron Zeilberger, with our warmest regards,
on the occasion of his sixtieth birthday.

Abstract. The secant and tangent numbers are given (t,q)-analogs with an explicit combinatorial interpretation. This extends, both analytically and combinatorially, the classical evaluations of the Eulerian and Roselle polynomials at $t=-1$.

1. Introduction

As is well-known (see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]), the coefficients T_{2n+1} of the Taylor expansion of $\tan u$, namely

$$\begin{aligned}(1.1)\qquad \tan u &= \sum_{n\ge 0}\frac{u^{2n+1}}{(2n+1)!}T_{2n+1}\\ &= \frac{u}{1!}1+\frac{u^3}{3!}2+\frac{u^5}{5!}16+\frac{u^7}{7!}272+\frac{u^9}{9!}7936+\frac{u^{11}}{11!}353792+\cdots\end{aligned}$$

are positive integral coefficients, usually called *tangent numbers*, while the *secant numbers* E_{2n}, also positive and integral, make their appearances in the Taylor expansion of $\sec u$:

$$\begin{aligned}(1.2)\qquad \sec u=\frac{1}{\cos u} &= 1+\sum_{n\ge 1}\frac{u^{2n}}{(2n)!}E_{2n}\\ &= 1+\frac{u^2}{2!}1+\frac{u^4}{4!}5+\frac{u^6}{6!}61+\frac{u^8}{8!}1385+\frac{u^{10}}{10!}50521+\cdots\end{aligned}$$

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On the other hand, the expansion

$$\text{(1.3)} \qquad \frac{1-s}{\exp(su)-s\exp(u)}\exp(Yu)=\sum_{n\ge 0}\frac{u^n}{n!}A_n(s,1,1,Y)$$

defines a sequence $(A_n(s,1,1,Y))$ $(n\ge 0)$ of polynomials with *Positive Integral Coefficients* [in short, PIC polynomials], whose specializations $(A_n(s,1,1,1))$ $(n\ge 0)$ for $Y=1$ are called *Eulerian* polynomials and go back to Euler himself [Eu55], while the version $A_n(s,1,1,0)$ $(n\ge 0)$ for $Y=0$ was introduced and combinatorially interpreted by Roselle [Ro68]. The two identities

$$\text{(1.4)} \qquad A_{2n}(-1,1,1,1)=0;\ \ (-1)^nA_{2n+1}(-1,1,1,1)=T_{2n+1}\ (n\ge 0);$$

$$\text{(1.5)} \qquad A_{2n+1}(-1,1,1,0)=0;\quad (-1)^nA_{2n}(-1,1,1,0)=E_{2n}\ (n\ge 0);$$

are due to Euler [Eu55] and Roselle [Ro68], respectively and a joint combinatorial proof of them can be found in [FS70], chap. 5.

The purpose of this paper is to prolong those two identities into a (t,q)-environment. Everybody is familiar with all successful attempts that have been made for finding q-analogs of the classical identities in analysis, using the now well-developed theory of q-series ([GR90], [AAR00]). The main feature in the present approach is the addition of another variable t, in such a way that properties that hold for positive integers or PIC polynomials initially considered, also hold, *mutatis mutandis*, for the polynomials having the further variables t and q.

The (t,q)-extensions of (1.4) and (1.5) will be obtained by the discoveries of three classes of *PIC polynomials* $(A_n(s,t,q,Y))$, $(T_{2n+1}(t,q))$, $(E_{2n}(t,q))$ $(n\ge 0)$ such that the following diagram holds

$$\begin{array}{ccc}
A_n(s,t,q,Y) & \xrightarrow{\ t=1,\,q=1\ } & A_n(s,1,1,Y)\\
\Big\downarrow{\scriptstyle s=-q^{-1}} & & {\scriptstyle s=-1}\Big\downarrow\\
A_n(-q^{-1},t,q,Y) & \xrightarrow{\ t=1,\,q=1\ } & A_n(-1,1,1,Y)
\end{array}$$

Fig. 1

together with the identities:

$$(1.4)_{tq} \quad A_{2n}(-q^{-1},t,q,1)=0;\ (-1)^nA_{2n+1}(-q^{-1},t,q,1)=T_{2n+1}(t,q);$$

$$(1.5)_{tq} \quad A_{2n+1}(-q^{-1},t,q,0)=0;\ (-1)^nA_{2n}(-q^{-1},t,q,0)=E_{2n}(t,q).$$

Note that the latter identities imply: $T_{2n+1}(1,1)=T_{2n+1}$ (the tangent number) and $E_{2n}(1,1)=E_{2n}$ (the secant number).

The sequence $((A_n(s,t,q,Y))$, further defined in (1.12), is a slight modification of a class $((A_n^*(s,t,q,Y))$ of polynomials (see (4.1)) that have been thoroughly studied and used in our previous paper [FH08]. However, the extensions $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$

of tangent and secant, as true PIC polynomials, are to be truly constructed. This is, indeed, the main goal of the paper.

Using the traditional q-ascending factorial $(t;q)_n := (1-t)(1-tq)\cdots(1-tq^{n-1})$ for $n \geq 1$ and $(t;q)_0 = 1$, Jackson [Ja04] (also see [GR90, p. 23]) introduced both *q-sine* "$\sin_q(u)$" and *q-cosine* "$\cos_q(u)$" as being the q-series:

$$\sin_q(u) := \sum_{n\geq 0}(-1)^n \frac{u^{2n+1}}{(q;q)_{2n+1}};$$

$$\cos_q(u) := \sum_{n\geq 0}(-1)^n \frac{u^{2n}}{(q;q)_{2n}};$$

so that the *q-tangent* "$\tan_q(u)$" and *q-secant* "$\sec_q(u)$" can be defined by the q-expansions:

$$(1.1)_q \qquad \tan_q(u) := \frac{\sin_q(u)}{\cos_q(u)} = \sum_{n\geq 0} \frac{u^{2n+1}}{(q;q)_{2n+1}} T_{2n+1}(q);$$

$$(1.2)_q \qquad \sec_q(u) := \frac{1}{\cos_q(u)} = \sum_{n\geq 0} \frac{u^{2n}}{(q;q)_{2n}} E_{2n}(q).$$

The coefficients $T_{2n+1}(q)$ and $E_{2n}(q)$ occurring in those expansions are called *q-tangent numbers* and *q-secant numbers*, respectively, and known to be PIC polynomials, such that $T_{2n+1}(1) = T_{2n+1}$, $E_{2n}(1) = E_{2n}$. See, e.g., [AG78], [AF80], [Fo81], [St97, p. 148-149].

For each $r \geq 0$ we introduce the q-series:

$$(1.6) \qquad \sin_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}} u^{2n+1};$$

$$(1.7) \qquad \cos_q^{(r)}(u) := \sum_{n\geq 0}(-1)^n \frac{(q^r;q)_{2n}}{(q;q)_{2n}} u^{2n};$$

$$(1.8) \qquad \tan_q^{(r)}(u) := \frac{\sin_q^{(r)}(u)}{\cos_q^{(r)}(u)};$$

$$(1.9) \qquad \sec_q^{(r)}(u) := \frac{1}{\cos_q^{(r)}(u)};$$

and define the (t,q)-*analogs of the tangent and secant numbers* as being the coefficients $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$, respectively, in the following two series:

$$(1.1)_{tq} \qquad \sum_{r\geq 0} t^r \tan_q^{(r)}(u) = \sum_{n\geq 0} \frac{u^{2n+1}}{(t;q)_{2n+2}} T_{2n+1}(t,q);$$

$$(1.2)_{tq} \qquad \sum_{r\geq 0} t^r \sec_q^{(r)}(u) = \sum_{n\geq 0} \frac{u^{2n}}{(t;q)_{2n+1}} E_{2n}(t,q).$$

Theorem 1.1. *The (t,q)-analogs $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$, defined in $(1.1)_{tq}$ and $(1.2)_{tq}$, have the following properties:*

(a) *they are* PIC *polynomials;*

(b) *furthermore,*

$$T_{2n+1}(1,q)=T_{2n+1}(q);\quad E_{2n}(1,q)=E_{2n}(q); \tag{1.10}$$

$$T_{2n+1}(1,1)=T_{2n+1};\qquad E_{2n}(1,1)=E_{2n}. \tag{1.11}$$

The first values of those PIC polynomials are next listed.

$$\begin{aligned}
&T_1(t,q)=t;\ T_3(t,q)=t^2q(1+q);\\
&T_5(t,q)=t^2q^2(1+q)(1+tq(1+2q+2q^2+q^3)+t^2q^6);\\
&T_7(t,q)=t^2q^3(1+q)(1+tq(2+5q+7q^2+7q^3+5q^4+2q^5)\\
&\qquad +t^2q^3(1+4q+10q^2+15q^3+18q^4+15q^5+10q^6+4q^7+q^8)\\
&\qquad +t^3q^8(2+5q+7q^2+7q^3+5q^4+2q^5)+t^4q^{14});
\end{aligned}$$

$$\begin{aligned}
&E_0(t,q)=1;\ E_2(t,q)=t;\ E_4(t,q)=t^2q(1+2q+q^2+tq^3);\\
&E_6(t,q)=t^2q^2(1+2q+q^2+tq(1+4q+8q^2+10q^3+8q^4+4q^5+q^6)\\
&\qquad +t^2q^5(2+5q+6q^2+5q^3+2q^4)+t^3q^{10});\\
&E_8(t,q)=t^2q^3(1+2q+q^2+tq(2+9q+20q^2+30q^3+34q^4+30q^5+20q^6\\
&\qquad +9q^7+2q^8)+t^2q^3(1+6q+21q^2+48q^3+81q^4+110q^5+122q^6\\
&\qquad +110q^7+81q^8+48q^9+21q^{10}+6q^{11}+q^{12})+t^3q^8(3+14q+35q^2\\
&\qquad +62q^3+86q^4+96q^5+86q^6+62q^7+35q^8+14q^9+3q^{10})\\
&\qquad +t^4q^{14}(3+9q+15q^2+18q^3+15q^4+9q^5+3q^6)+t^5q^{21}).
\end{aligned}$$

The proof of (a) is a consequence of Theorem 1.1a that follows. The proof of (b) will be fully given at the end of Section 3. It uses the following argument: as $\tan_q^{(r)}(u)$ (resp. $\sec_q^{(r)}(u)$) tends to $\tan_q(u)$ (resp. $\sec_q(u)$) when r tends to infinity (by using the topology of formal power series), we can multiply both $(1.1)_{tq}$ and $(1.2)_{tq}$ by $(1-t)$ and let $t=1$ (see, e.g., [FH04a], p. 163, the "$t=1$" Lemma) to obtain the identities

$$\tan_q(u)=\sum_{n\ge0}\frac{u^{2n+1}}{(q;q)_{2n+1}}T_{2n+1}(1,q);$$

$$\sec_q(u)=\sum_{n\ge0}\frac{u^{2n}}{(q;q)_{2n}}E_{2n}(1,q);$$

so that $T_{2n+1}(1,q)=T_{2n+1}(q)$ and $E_{2n}(1,q)=E_{2n}(q)$, by comparison with $(1.1)_q$ and $(1.2)_q$.

Now, let $(A_n(s,t,q,Y))$ $(n\ge0)$ be the sequence of coefficients occurring in the following factorial expansion:

$$\sum_{r\ge0}t^r\frac{1-sq}{\dfrac{1}{(usq;q)_r}-\dfrac{sq}{(u;q)_r}}\frac{1}{(uY;q)_r}=\sum_{n\ge0}A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}. \tag{1.12}$$

Theorem 1.2. *For each $n \geq 0$ the coefficient $A_n(s,t,q,Y)$ in (1.12) is a* PIC *polynomial. Furthermore, the diagram of Fig. 1 holds, together with identities $(1.4)_{tq}$ and $(1.5)_{tq}$.*

The fact that each $A_n(s,t,q,Y)$ is a PIC polynomial is a consequence of the further Theorem 1.2a, while the proofs of identities $(1.4)_{tq}$ and $(1.5)_{tq}$ are given in Section 5.

Several combinatorial methods have been developed in Special Functions for proving inequalities, essentially expressing finite or infinite sums as generating functions for well-defined finite structures by positive integral-valued statistics. See the pioneering works by Askey and his followers [AI76], [AIK78], [IT79]. Very soon, Zeilberger, following his mentor Gillis [EG76], has brought his decisive contribution to the subject [GZ83], [GRZ83], [FZ88].

The method of proof used in this paper is very much inspired by these papers. Both Theorems 1.1 and 1.2, of analytical nature, will get combinatorial counterparts, namely the next Theorems 1.1a and 1.2a, where all three families $(T_{2n+1}(t,q))$, $(E_{2n}(t,q))$ and $(A_n(s,t,q,Y))$ $(n \geq 0)$ will be shown to be generating polynomials for some classes of permutations by well-defined statistics. The underlying combinatorial set-up can be described as follows. As introduced by Désiré André [An79, An81], each permutation $\sigma = \sigma(1)\cdots\sigma(n)$ of $1\,2\cdots n$ is said to be *alternating* (resp. *falling alternating*) if the following properties hold: $\sigma(1) < \sigma(2), \sigma(2) > \sigma(3), \sigma(3) < \sigma(4)$, etc. (resp. $\sigma(1) > \sigma(2)$, $\sigma(2) < \sigma(3)$, $\sigma(3) > \sigma(4)$, etc.) in an alternating way. The set of alternating (resp. falling alternating) permutations of order n is denoted by $\mathfrak{T}_n$ (resp. by $\mathfrak{T}'_n$).

Désiré André's main result was to show that tangent and secant numbers were true enumerators for all alternating permutations: $\#\mathfrak{T}_{2n+1} = \#\mathfrak{T}'_{2n+1} = T_{2n+1}$ and $\#\mathfrak{T}_{2n} = \#\mathfrak{T}'_{2n} = E_{2n}$. It is remarkable that by counting those alternating permutations by the usual *number of inversions* "inv," the underlying generating polynomial $\sum_{\sigma\in\mathfrak{T}_n} q^{\mathrm{inv}\,\sigma}$ is equal to $T_n(q)$ (n odd) or $E_n(q)$ (n even) (see [AG78], [AF80], [Fo81], [St97, p. 148-149]). As "inv" is a traditional q-maker, it was tantalizing to pursue our t-extension with "inv," and add another suitable statistic counted by the variable t. In fact, it was far more convenient to continue with another q-maker having the same distribution over $\mathfrak{T}_n$ as "inv," as is now explained.

For each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ from the symmetric group $\mathfrak{S}_n$ let $\mathrm{IDES}\,\sigma$ (resp. $\mathrm{ides}\,\sigma$) denote the *set* (resp. the *number*) of all letters $\sigma(i)$ such that for some $j < i$ the equality $\sigma(j) = \sigma(i)+1$ holds and let $\mathrm{imaj}\,\sigma := \sum_{\sigma(i)\in\mathrm{IDES}\,\sigma}\sigma(i)$. It is known that "imaj" and "inv" are equally distributed on each set $\mathfrak{T}_n$, a result that can be proved by means of the so-called second fundamental transformation [FS78]. The most natural statistic that can be associated with "imaj" is then "ides." It is again remarkable that Désiré André's set-up will *also* provide the appropriate combinatorial model needed for our (t,q)-extension, as is now stated.

Theorem 1.1a. *The (t,q)-analogs $T_{2n+1}(t,q)$ and $E_{2n}(t,q)$ of the tangent and secant numbers defined by $(1.1)_{tq}$ and $(1.2)_{tq}$ have the following combinatorial interpretations:*

$$T_{2n+1}(t,q) = \sum_{\sigma\in\mathfrak{T}_{2n+1}} t^{1+\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}; \tag{1.13}$$

$$E_{2n}(t,q)=\sum_{\sigma\in\mathfrak{T}_{2n}} t^{1+\operatorname{ides}\sigma}q^{\operatorname{imaj}\sigma}. \tag{1.14}$$

In particular, they are PIC *polynomials.*

The combinatorial interpretations of the coefficients $A_n(s,t,q,Y)$ are based on the model introduced in our previous paper [FH08]. Each word $w=x_1x_2\cdots x_m$, of length m, whose letters are positive integers all different, is called a *hook* if $x_1>x_2$ and either $m=2$, or $m\ge 3$ and $x_2<x_3<\cdots<x_m$. As proved by Gessel [Ge91], each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2\cdots\tau_k$, where p is an *increasing* word and each factor $\tau_1,\tau_2,\ldots,\tau_k$ is a hook. Define $\operatorname{pix}\sigma$ to be the *length* of the factor p. Finally, for each i let $\operatorname{inv}\tau_i$ be the *number of inversions* of τ_i and define: $\operatorname{lec}\sigma:=\sum_{1\le i\le k}\operatorname{inv}\tau_i$.

Theorem 1.2a. *The coefficients $A_n(s,t,q,Y)$ $(n\ge 0)$ defined by identity* (1.12) *have the following combinatorial interpretations:*

$$A_n(s,t,q,Y)=\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{lec}\sigma}t^{\operatorname{ides}\sigma+\chi(\sigma(1)=1)}q^{\operatorname{imaj}\sigma}Y^{\operatorname{pix}\sigma}, \tag{1.15}$$

where $\chi(\sigma(1)=1)=1$ if $\sigma(1)=1$ and 0 otherwise. Accordingly, they are PIC *polynomials.*

In the next section we recall a result on permutation *lignes of routes* derived in a previous paper of ours [FH04], then we prove Theorem 1.1a in Section 3. For the proof of Theorem 1.2a, given in Section 4, we actually show that the factorial generating function for the *polynomials* defined by (1.15) satisfy identity (1.12). Identities $(1.4)_{tq}$ and $(1.5)_{tq}$ are derived in Section 5. We conclude the paper by indicating that besides (1.13) each polynomial $T_{2n+1}(t,q)$ may be given two other combinatorial interpretations involving a triple of statistics.

2. Lignes of route

Let $\mathcal{L}=\{\ell_1<\cdots<\ell_k\}$ be a subset of the interval $\{1,2,\ldots,n-1\}$. By convention, $\ell_0:=0$ and $\ell_{k+1}:=n$. Designate by $W_r(\mathcal{L},n)$ the set of all words $w=x_1x_2\cdots x_n$, of length n, whose letters are nonnegative integers satisfying the inequalities:

$$\begin{gathered} r\ge x_1\ge\cdots\ge x_{\ell_1}\ge 0;\quad r\ge x_{\ell_1+1}\ge\cdots\ge x_{\ell_2}\ge 0;\quad\cdots\\ r\ge x_{\ell_k+1}\ge\cdots\ge x_n\ge 0;\\ x_{\ell_1}<x_{\ell_1+1},\quad x_{\ell_2}<x_{\ell_2+1},\ \ldots,\ x_{\ell_k}<x_{\ell_k+1}. \end{gathered} \tag{2.1}$$

Say that the *ligne of route* of a permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ is equal to $\mathcal{L}$, and write $\operatorname{Ligne}\sigma=\mathcal{L}$, if and only if $\sigma(i)>\sigma(i+1)$ whenever $i\in\mathcal{L}$. Notice that $\operatorname{IDES}\sigma$ and $\operatorname{ides}\sigma$ are simply the ligne of route and the number of descents of the inverse permutation σ^{-1}, respectively.

The next identity requires some classical techniques on stardardizations of words. It is proved in the forementioned paper ([FH04] Propositions 8.1 and 8.2) and reads

$$\frac{\sum_{\sigma,\,\mathrm{Ligne}\,\sigma=\mathcal{L}} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \sum_{w\in W_r(\mathcal{L},n)} q^{\mathrm{tot}\,w} \qquad (n\ge 1), \tag{2.2}$$

where $\mathrm{tot}\,w$ stands for the sum of all letters of w.

When $\mathcal{L}=\{2,4,6,\dots\}$ the set of all permutations σ from $\mathfrak{S}_n$ such that $\mathrm{Ligne}\,\sigma=\mathcal{L}$ is the set $\mathfrak{T}$ of all *alternating* permutations. We then have the subsequent result.

Theorem 2.1. *With $\mathcal{L}=\{2,4,6,\dots\}$ the following identity holds:*

$$\frac{\sum_{\sigma\in\mathfrak{T}_n} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{n+1}} = \sum_{r\ge 0} t^r \sum_{w\in W_r(\mathcal{L},n)} q^{\mathrm{tot}\,w} \qquad (n\ge 1). \tag{2.3}$$

For each $r\ge 1$ and each $n\ge 1$ the set $V_r(\mathcal{L},n):=W_r(\mathcal{L},n)\setminus W_{r-1}(\mathcal{L},n)$ consists of all words $w=x_1x_2\cdots x_n$ such that (2.1) holds (in particular, for $\mathcal{L}=\{2,4,6,\dots\}$) with the further property that at least one of the letters x_1, x_{ℓ_1+1}, x_{ℓ_2+1}, $\dots$ is equal to r. Let $\max w$ the maximum letter in w. Then,

$$w\in V_r(\mathcal{L},n) \Longrightarrow \max w = r \quad\text{and}\quad \mathrm{tot}\,w-\max w\ge 0. \tag{2.4}$$

Note that the sets $V_r(\mathcal{L},n)$ are disjoint and

$$\sum_r V_r(\mathcal{L},n) = \sum_r W_r(\mathcal{L},n) =: W(\mathcal{L},n). \tag{2.5}$$

Proposition 2.2. *For each $n\ge 1$ we have*

$$(1-t)\frac{\sum_{\sigma\in\mathfrak{T}_n} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{n+1}}\Bigg|_{\{t=1\}} = \frac{\sum_{\sigma\in\mathfrak{T}_n} q^{\mathrm{imaj}\,\sigma}}{(q;q)_n}. \tag{2.6}$$

Proof. We have:

$$\begin{aligned}
(1-t)\frac{\sum_{\sigma\in\mathfrak{T}_n} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{n+1}} &= \frac{\sum_{\sigma\in\mathfrak{T}_n} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(tq;q)_n}\\
&= (1-t)\sum_{r\ge 0} t^r \sum_{w\in W_r(\mathcal{L},n)} q^{\mathrm{tot}\,w} && \text{[by (2.3)]}\\
&= \sum_{w\in W_0(\mathcal{L},n)} q^{\mathrm{tot}\,w} + \sum_{r\ge 1} t^r \sum_{w\in V_r(\mathcal{L},n)} q^{\mathrm{tot}\,w} && \text{[by definition of } V_r(\mathcal{L},n)\text{]}\\
&= 1+\sum_{w\in W(\mathcal{L},n)} t^{\max w} q^{\mathrm{tot}\,w} && \text{[by (2.4) and (2.5)]}\\
&= 1+\sum_{w\in W(\mathcal{L},n)} (qt)^{\max w} q^{\mathrm{tot}\,w-\max w}.
\end{aligned}$$

As $\operatorname{tot} w - \max w \geq 0$ for all $w \in W(\mathcal{L}, n)$ by (2.5), it makes sense to have the substitution $tq \leftarrow q$ in the last expression, that is, $1 \leftarrow t$ in $\sum_{\sigma \in \mathfrak{I}_n} t^{\operatorname{ides} \sigma} q^{\operatorname{imaj} \sigma}/(tq;q)_n$ to obtain $\sum_{\sigma \in \mathfrak{I}_n} q^{\operatorname{imaj} \sigma}/(q;q)_n$. $\square$

3. Proof of Theorem 1.1

For the proof of identity (1.14) we shall start with the definition of $\cos_q^{(r)}(u)$ given in (1.7), and express $\sec_q^{(r)}(u) = 1/\cos_q^{(r)}(u)$ as a generating series for a class of words with nonnegative integral letters. For this purpose we introduce the set $\mathrm{NIW}_n(r)$ of all monotonic nonincreasing words $c = c_1 c_2 \cdots c_n$, of length n, whose letters are nonnegative integers at most equal to r: $r \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Also, designate the *length* (resp. the *sum of all the letters*) of each word w by λw (resp. $\operatorname{tot} w$).

The next identity is classical (see, e.g., [An76, chap. 2]):

$$\frac{(q^r;q)_n}{(q;q)_n} = \sum_{w \in \mathrm{NIW}_n(r-1)} q^{\operatorname{tot} w}. \tag{3.1}$$

Using (3.1) we get:

$$\cos_q^{(r)}(u) = \sum_{m \geq 0} \frac{(q^r;q)_{2m}}{(q;q)_{2m}} (-1)^m u^{2m} = 1 - \sum_{m \geq 1} (-1)^{m-1} u^{2m} \sum_{w \in \mathrm{NIW}_{2m}(r-1)} q^{\operatorname{tot} w}.$$

Hence,

$$\frac{1}{\cos_q^{(r)}(u)} = 1 + \sum_{n \geq 1} u^{2n} \sum_{\substack{(m_1,\ldots,m_k) \\ (w_1,\ldots,w_k)}} (-1)^{m_1+\cdots+m_k-k} q^{\operatorname{tot}(w_1 \cdots w_k)}, \tag{3.2}$$

where the second sum is over all sequences $(m_1, \ldots, m_k)$ and $(w_1, \ldots, w_k)$ such that $m_1 + \cdots + m_k = n$ and $w_i \in \mathrm{NIW}_{2m_i}(r-1)$ $(i = 1, \ldots, k)$.

Each sequence $(w_1, \ldots, w_k)$ in the above sum is said to have a *decrease* at j if $1 \leq j \leq k-1$ and the last letter of w_j is greater than or equal to the first letter of w_{j+1} [in short, $L\, w_j \geq F\, w_{j+1}$]. If the sequence has no decrease and all the factors w_j are of length 2, then $k = n$. If it is not the case, let j be the integer with the following properties:

(i) $\lambda w_1 = \cdots = \lambda w_{j-1} = 2$;
(ii) no decrease at $1, 2, \ldots, j-1$;
(iii) either $\lambda w_j \geq 4$, or
(iv) $\lambda w_j = 2$ and there is a decrease at j.

Say that the sequence is of class C_j (resp. C'_j) if (i), (ii) and (iii) (resp. (i), (ii) and (iv)) hold. If the sequence is of class C_j, let $w_j = x_1 x_2 \cdots x_{2m}$ (remember that $r - 1 \geq x_1 \geq \cdots \geq x_{2m}$) and form the sequence

$$(w_1,\ \ldots,\ w_{j-1},\ x_1x_2,\ x_3 \cdots x_{2m},\ w_{j+1},\ \ldots,\ w_k)$$

having $(k+1)$ factors. As $L\, x_1 x_2 = x_2 \geq x_3 = F\, x_3 \cdots x_{2m}$, the j-th factor is of length 2 and there is a decrease at j. It then belongs to C'_j. This defines a sign-reversing involution on the set of those sequences. By applying the involution to the above sum, the remaining terms correspond to the sequences $(w_1, w_2, \ldots, w_n)$, such that $\lambda w_i \in \mathrm{NIW}_2(r-1)$ $(i = 1, 2, \ldots, n)$ and $L\,w_1 < F\,w_2,\ L\,w_2 < F\,w_3,\ \ldots\ ,\ L\,w_{n-1} < F\,w_n$. In particular, $k = n$, $m_1 = \cdots = m_n = 1$ and there is no more minus sign left on the right-hand side of (3.2).

Those sequences are in bijection with the set $W_{r-1}(\mathcal{L}, 2n)$, described in (2.1), when $\mathcal{L} = \{2, 4, \ldots, (2n-2)\}$. Referring to (3.2) we then have:

$$\sum_{\substack{(m_1,\ldots,m_k)\\(w_1,\ldots,w_k)}} (-1)^{m_1+\cdots+m_k-k} q^{\mathrm{tot}(w_1\cdots w_k)} = \sum_{w\in W_{r-1}(\mathcal{L},2n)} q^{\mathrm{tot}\,w},$$

so that

$$\frac{1}{\cos_q^{(r)}(u)} = 1 + \sum_{n\geq 1} u^{2n} \sum_{w\in W_{r-1}(\mathcal{L},2n)} q^{\mathrm{tot}\,w}; \tag{3.3}$$

and then by using (2.3)

$$\begin{aligned}
\sum_{r\geq 0} t^r \frac{1}{\cos_q^{(r)}(u)} &= 1 + \sum_{r\geq 1} t^r \frac{1}{\cos_q^{(r)}(u)} = 1 + \sum_{r\geq 1} t^r \Big(1 + \sum_{n\geq 1} u^{2n} \sum_{w\in W_{r-1}(\mathcal{L},2n)} q^{\mathrm{tot}\,w}\Big)\\
&= \frac{1}{1-t} + \sum_{n\geq 1} u^{2n} \sum_{r\geq 1} t^r \sum_{w\in W_{r-1}(\mathcal{L},2n)} q^{\mathrm{tot}\,w}\\
&= \frac{1}{1-t} + \sum_{n\geq 1} u^{2n} \frac{\sum_{\sigma\in\mathfrak{S}_{2n},\,\mathrm{Ligne}\,\sigma=\mathcal{L}} t^{1+\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{2n+1}}\\
&= \frac{1}{1-t} + \sum_{n\geq 1} u^{2n} \frac{\sum_{\sigma\in\mathfrak{T}_{2n}} t^{1+\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{2n+1}}
\end{aligned}$$

and this proves (1.14) with the convention $E_0(t, q) = 1$.

For the proof of (1.13) we use the same techniques, in particular identities (3.1) and (3.3). We have:

$$\frac{1}{\cos_q^{(r)}(u)} \sin_q^{(r)}(u) = \sum_{j\geq 0} u^{2j} \sum_{w\in W_{r-1}(\mathcal{L},2j)} q^{\mathrm{tot}\,w} \times \sum_{i\geq 0} (-1)^i u^{2i+1} \sum_{v\in\mathrm{NIW}_{2i+1}(r-1)} q^{\mathrm{tot}\,v},$$

making the convention that the first sum is equal to 1 for $j = 0$. Hence,

$$\frac{1}{\cos_q^{(r)}(u)} \sin_q^{(r)}(u) = \sum_{n\geq 0} u^{2n+1} \sum_{j+i=n} (-1)^i \sum_{\substack{w\in W_{r-1}(\mathcal{L},2j)\\ v\in\mathrm{NIW}_{2i+1}(r-1)}} q^{\mathrm{tot}\,wv}.$$

Say that the pair (w, v) is of class (D) (resp. class (D')) if $L\,w < F\,v$ and $\lambda v \geq 3$ (resp. $L\,w \geq F\,v$). If (w, v) is of class (D), write $v = v_1 v_2$ with $\lambda v_1 = 2$. Then, define $w' := wv_1$

and $v' := v_2$. As v is monotonic nonincreasing, we have $L\,w' = L\,v_1 \geq F\,v_2 = F\,v'$, so that the pair (w', v') is of class (D'). Moreover, if $i = (\lambda v - 1)/2$ and $i' = (\lambda v' - 1)/2$, we have: $i = i' + 1$, so that $(-1)^i q^{\mathrm{tot}\,wv} + (-1)^{i'} q^{\mathrm{tot}\,w'v'} = 0$. Consequently, the mapping $(w, v) \mapsto (w, v')$ is a sign-reversing involution. When the involution is applied to the above sum, only remain the pairs (w, v) such that $\lambda v = 1$ (one-letter word) and $L\,w < F\,v = v$. In particular, $v \leq r - 1$. The corresponding sign $(-1)^i$ is also equal to $(-1)^{(\lambda v - 1)/2} = 1$. We then get

$$\frac{1}{\cos_q^{(r)}(u)} \sin_q^{(r)}(u) = \sum_{n\geq 0} u^{2n+1} \sum_{w \in W_{r-1}(\mathcal{L}, 2n+1)} q^{\mathrm{tot}\,w},$$

with $\mathcal{L} = \{2, 4, 6, \ldots, 2n\}$. By using (2.3) we can then conclude:

$$\sum_{r\geq 0} t^r \tan_q^{(r)}(u) = \sum_{n\geq 0} u^{2n+1} \frac{\sum\limits_{\sigma \in \mathfrak{T}_{2n+1}} t^{1+\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma}}{(t;q)_{2n+2}}. \quad \square$$

To complete the proof of Theorem 1.1 (b) we proceed as follows. Let $a_r := \tan_q^{(r)}(u)$ (resp. $\sec_q^{(r)}(u)$) and $a := \tan_q(u)$ (resp. $\sec_q(u)$) and for each pair (i, j) let $a_r(i, j)$ (resp. $a(i, j)$) be the coefficient of $q^i u^j$ in a_r (resp. in a). A simple calculation shows that $a_r - a$ can be expressed as $q^r c$, where c is a formal series in q, u. Hence, $a_r(i, j) - a(i, j) = 0$ for all $r \geq i + 1$ and then $\lim_r a_r = a$. Let $b(t) = \sum_{r\geq 0} t^r b_r := (1 - t) \sum_{r\geq 0} t^r a_r$, so that $b_0 = a_0$ and $b_r = a_r - a_{r-1}$ for $r \geq 1$. For all $r \geq i + 2$ we then have $b_r(i, j) = a_r(i, j) - a_{r-1}(i, j) = a(i, j) - a(i, j) = 0$ and the finite sum $b_0(i, j) + b_1(i, j) + \cdots + b_r(i, j)$ is equal to $a_0(i, j) + (a_1(i, j) - a_0(i, j)) + \cdots + (a_{i+1}(i, j) - a_i(i, j)) = a_{i+1}(i, j) = a(i, j)$. This proves that the sum $\sum_r b_r$ is convergent and converges to a, that is, $b(1) = \sum_r b_r = a$. Thus, $(1-t) \sum\limits_{r\geq 0} t^r \tan_q^{(r)}(u) \Big|_{t=1} = \tan_q(u)$ and $(1-t) \sum\limits_{r\geq 0} t^r \sec_q^{(r)}(u) \Big|_{t=1} = \sec_q(u)$. This achieves the proof of Theorem 1.1 (b) in view of Proposition 2.2 and the combinatorial interpretations derived in Theorem 1.1a.

4. Proof of Theorem 1.2a

In our previous paper [FH08] we have calculated the factorial generating function for the polynomials

$$A_n^*(s, t, q, Y) = \sum_{\sigma \in \mathfrak{S}_n} s^{\mathrm{lec}\,\sigma} t^{\mathrm{ides}\,\sigma} q^{\mathrm{imaj}\,\sigma} Y^{\mathrm{pix}\,\sigma} \quad (n \geq 0), \tag{4.1}$$

and found

$$\sum_{n\geq 0} A_n^*(s, t, q, Y) \frac{u^n}{(t;q)_{n+1}} = \sum_{r\geq 0} t^r \frac{1 - sq}{\dfrac{1}{(usq;q)_r} - \dfrac{sq}{(u;q)_r}} \frac{1}{(uY;q)_{r+1}}. \tag{4.2}$$

Notice that the generating functions for the polynomials $A_n(s,t,q,Y)$ and $A_n^*(s,t,q,Y)$ differ only by the fraction $1/(uY;q)_r$ for the first one (see (1.12)) and $1/(uY;q)_{r+1}$ for the second. From (4.2) we can obtain the factorial generating function for the polynomials $A_n(s,t,q,Y)$ themselves in the following manner. Starting with definition (1.15) we can write:

$$A_n(s,t,q,Y) = \sum_{\substack{\sigma\in\mathfrak{S}_n,\\ \sigma(1)\neq 1}} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma} + \sum_{\substack{\sigma\in\mathfrak{S}_n,\\ \sigma(1)=1}} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma+1} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma}.$$

Now, for $n\geq 1$ the transformation

$$\sigma=\sigma(1)\sigma(2)\cdots\sigma(n-1)\mapsto \tau = 1\,(\sigma(1)+1)(\sigma(2)+2)\cdots(\sigma(n-1)+1)$$

is a bijection of $\mathfrak{S}_{n-1}$ onto the set of permutations from $\mathfrak{S}_n$ starting with 1 having the property

$$\operatorname{lec}\tau=\operatorname{lec}\sigma;\ \operatorname{ides}\tau=\operatorname{ides}\sigma;\ \operatorname{imaj}\tau=\operatorname{imaj}\sigma+\operatorname{ides}\sigma;\ \operatorname{pix}\tau=\operatorname{pix}\sigma+1.$$

Hence,

$$YA_{n-1}^*(s,tq,q,Y) = \sum_{\sigma\in\mathfrak{S}_n,\sigma(1)=1} s^{\operatorname{lec}\sigma} t^{\operatorname{ides}\sigma} q^{\operatorname{imaj}\sigma} Y^{\operatorname{pix}\sigma}, \tag{4.3}$$

so that, for $n\geq 1$,

$$A_n(s,t,q,Y) = A_n^*(s,t,q,Y) - YA_{n-1}^*(s,tq,q,Y) + tYA_{n-1}^*(s,tq,q,Y).$$

It then follows that

$$\begin{aligned}\sum_{n\geq 0} & A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}}\\ &= \frac{1}{1-t} + \sum_{n\geq 1}\Bigl(A_n^*(s,t,q,Y) - Y(1-t)A_{n-1}^*(s,tq,q,Y)\Bigr)\frac{u^n}{(t;q)_{n+1}}\\ &= \sum_{n\geq 0} A_n^*(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}} - uY\sum_{n\geq 0} A_n^*(s,tq,q,Y)\frac{u^n}{(tq;q)_{n+1}}.\end{aligned}$$

Making use of (4.2) we obtain:

$$\sum_{n\geq 0} A_n(s,t,q,Y)\frac{u^n}{(t;q)_{n+1}} = \sum_{r\geq 0} t^r \frac{1-sq}{\dfrac{1}{(usq;q)_r} - \dfrac{sq}{(u;q)_r}}\frac{1}{(uY;q)_{r+1}}(1-uq^rY), \tag{4.4}$$

which is identity (1.12). ∎

5. The identities $(1.4)_{\mathbf{tq}}$ and $(1.5)_{\mathbf{tq}}$

First, derive other expressions for $\cos_q^{(r)}(u)$ and $\sin_q^{(r)}(u)$ using the q-binomial theorem (see, e.g., [GR90], p. 9):

$$\frac{1}{(iu;q)_r}+\frac{1}{(-iu;q)_r}=\sum_{n\ge0}\Bigl(\frac{(q^r;q)_n}{(q;q)_n}(iu)^n+\frac{(q^r;q)_n}{(q;q)_n}(-iu)^n\Bigr)$$
$$=2\sum_{n\ge0}(-1)^n\frac{(q^r;q)_{2n}}{(q;q)_{2n}}u^{2n}=2\cos_q^{(r)}(u). \tag{5.1}$$

Also

$$\frac{1}{(iu;q)_r}-\frac{1}{(-iu;q)_r}=\sum_{n\ge0}\frac{(q^r;q)_n}{(q;q)_n}(iu)^n-\frac{(q^r;q)_n}{(q;q)_n}(-iu)^n$$
$$=2i\sum_{n\ge0}(-1)^n\frac{(q^r;q)_{2n+1}}{(q;q)_{2n+1}}u^{2n+1}=2i\sin_q^{(r)}(u), \tag{5.2}$$

so that

$$\tan_q^{(r)}(u)=\frac{-i}{\dfrac{1}{(iu;q)_r}+\dfrac{1}{(-iu;q)_r}}\Bigl(\frac{1}{(iu;q)_r}-\frac{1}{(-iu;q)_r}\Bigr). \tag{5.3}$$

Let $s\leftarrow -q^{-1}$, $u\leftarrow iu$ in (4.4). We get

$$\sum_{n\ge0}A_n(-q^{-1},t,q,Y)\frac{(iu)^n}{(t;q)_{n+1}}=\sum_{r\ge0}t^r\frac{2}{\dfrac{1}{(-iu;q)_r}+\dfrac{1}{(iu;q)_r}}\frac{1}{(iuY;q)_r}.$$

Hence, by (5.1)

$$\sum_{n\ge0}A_n(-q^{-1},t,q,0)\frac{(iu)^n}{(t;q)_{n+1}}=\sum_{r\ge0}t^r\frac{1}{\cos_q^{(r)}(u)}=\sum_{r\ge0}t^r\sec_q^{(r)}(u).$$

By definition of $\sec_q^{(r)}(u)$ given in $(1.2)_{tq}$ we deduce for $n\ge0$:

$$A_{2n}(-q^{-1},t,q,0)(-1)^n=E_{2n}(t,q);\quad A_{2n+1}(-q^{-1},t,q,0)=0.$$

With $Y\leftarrow 1$ we obtain

$$\sum_{n\ge0}A_n(-q^{-1},t,q,1)\frac{(iu)^n}{(t;q)_{n+1}}=\sum_{r\ge0}t^r\frac{2}{\dfrac{1}{(-iu;q)_r}+\dfrac{1}{(iu;q)_r}}\frac{1}{(iu;q)_r},$$

and with $Y\leftarrow -1$

$$\sum_{n\ge0}A_n(-q^{-1},t,q,-1)\frac{(iu)^n}{(t;q)_{n+1}}=\sum_{r\ge0}t^r\frac{2}{\dfrac{1}{(-iu;q)_r}+\dfrac{1}{(iu;q)_r}}\frac{1}{(-iu;q)_r}.$$

Hence,

$$\sum_{n\geq 0}\frac{1}{2}\Big(A_n(-q^{-1},t,q,1)+A_n(-q^{-1},t,q,-1)\Big)\frac{(iu)^n}{(t;q)_{n+1}}=\sum_{r\geq 0}t^r, \tag{5.4}$$

while

$$\begin{aligned}\sum_{n\geq 0}\frac{1}{2}\Big(A_n(-q^{-1},t,q,1)-A_n(-q^{-1},t,q,-1)\Big)\frac{(iu)^n}{(t;q)_{n+1}}\\ =\sum_{r\geq 0}t^r\frac{1}{\dfrac{1}{(-iu;q)_r}+\dfrac{1}{(iu;q)_r}}\Big(\frac{1}{(-iu;q)_r}-\frac{1}{(iu;q)_r}\Big)\\ =\sum_{r\geq 0}t^r\, i\tan_q^{(r)}(u).\end{aligned} \tag{5.5}$$

We conclude that $A_n(-q^{-1},t,q,1)+A_n(-q^{-1},t,q,-1)=0$ for all $n\geq 1$, and $A_n(-q^{-1},t,q,1)-A_n(-q^{-1},t,q,-1)=0$ for all $n\geq 1$ *even*. Also $(A_{2n+1}(-q^{-1},t,q,1)-A_{2n+1}(-q^{-1},t,q,-1))(-1)^n=T_{2n+1}(t,q)$ for all $n\geq 0$. This proves $(1.4)_{tq}$ and $(1.5)_{tq}$.

6. Concluding remarks

Recall that the *number of excedances*, "$\operatorname{exc}\sigma$," of a permutation $\sigma=\sigma(1)\cdots\sigma(n)$ from $\mathfrak{S}_n$ is defined by $\operatorname{exc}\sigma:=\#\{i : 1\leq i\leq n, \sigma(i)>i\}$, while the *number of descents*, "$\operatorname{des}\sigma$" (resp. the *major index*, "$\operatorname{maj}\sigma$") is the number (resp. the sum) of all elements in $\operatorname{Ligne}\sigma$. Also, let $\operatorname{iexc}\sigma:=\operatorname{exc}\sigma^{-1}$ and let $\operatorname{fix}\sigma$ be the number of fixed points of σ. As shown in our previous paper [FH08], the three quadruples $(\operatorname{exc},\operatorname{des},\operatorname{maj},\operatorname{fix})$, $(\operatorname{lec},\operatorname{ides},\operatorname{imaj},\operatorname{pix})$, $(\operatorname{iexc},\operatorname{ides},\operatorname{imaj},\operatorname{fix})$ are equally distributed on $\mathfrak{S}_n$. It then follows that $(1.4)_{tq}$ implies the identity:

$$\sum_{\sigma\in\mathfrak{T}_{2n}}t^{1+\operatorname{ides}\sigma}q^{\operatorname{imaj}\sigma}=(-1)^n\sum_{\substack{\sigma\in\mathfrak{S}_{2n},\\ \operatorname{fix}\sigma=0}}(-q^{-1})^{\operatorname{iexc}\sigma}t^{\operatorname{ides}\sigma}q^{\operatorname{imaj}\sigma}.$$

As "imaj" and "inv" are equally distributed on each set $\mathfrak{T}_n$, we also have

$$T_{2n+1}(1,q)=\sum_{\sigma\in\mathfrak{T}_{2n+1}}q^{\operatorname{inv}\sigma},\quad E_{2n}(1,q)=\sum_{\sigma\in\mathfrak{T}_{2n}}q^{\operatorname{inv}\sigma}, \tag{6.1}$$

which are the traditional combinatorial interpretations of the q-tangent $T_{2n+1}(q)$ and q-secant $E_{2n}(q)$ numbers. Now, let $t=1$ in identities $(1.4)_{tq}$–$(1.5)_{tq}$. Taking (6.1) into account we get:

$$E_{2n}(q)=(-1)^n\sum_{\substack{\sigma\in\mathfrak{S}_{2n},\\ \operatorname{pix}\sigma=0}}(-q^{-1})^{\operatorname{lec}\sigma}q^{\operatorname{imaj}\sigma};$$

$$T_{2n+1}(q) = (-1)^n \sum_{\sigma \in \mathfrak{S}_{2n+1}} (-q^{-1})^{\operatorname{lec}\sigma} q^{\operatorname{imaj}\sigma};$$

$$0 = \sum_{\substack{\sigma \in \mathfrak{S}_{2n+1},\\ \operatorname{pix}\sigma = 0}} (-q^{-1})^{\operatorname{lec}\sigma} q^{\operatorname{imaj}\sigma};$$

and for $n \geq 1$

$$0 = \sum_{\sigma \in \mathfrak{S}_{2n}} (-q^{-1})^{\operatorname{lec}\sigma} q^{\operatorname{imaj}\sigma}.$$

But, as the triples (lec, imaj, pix) and (lec, inv, pix) and (exc, maj, fix) are all equidistributed on each $\mathfrak{S}_n$ [FH08], the previous identities can be rewritten as:

$$E_{2n}(q) = (-1)^n \sum_{\substack{\sigma \in \mathfrak{S}_{2n},\\ \operatorname{fix}\sigma = 0}} (-q^{-1})^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma};$$

$$T_{2n+1}(q) = (-1)^n \sum_{\sigma \in \mathfrak{S}_{2n+1}} (-q^{-1})^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma};$$

$$0 = \sum_{\substack{\sigma \in \mathfrak{S}_{2n+1},\\ \operatorname{fix}\sigma = 0}} (-q^{-1})^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma};$$

and for $n \geq 1$

$$0 = \sum_{\sigma \in \mathfrak{S}_{2n}} (-q^{-1})^{\operatorname{exc}\sigma} q^{\operatorname{maj}\sigma},$$

four identities that were previously derived in [FH10].

The polynomials $T_{2n+1}(t, q)$, $E_{2n}(t, q)$ $(n \geq 0)$ introduced in this paper have been referred to as being the (t, q)-*analogs* of the tangent and secant numbers, respectively. They may be regarded as the *graded forms* of the traditional q-tangent and q-secant numbers $T_{2n+1}(q)$, $E_n(q)$ defined in $(1.1)_q$ and $(1.2)_q$. The order of the variables t, q matters, as other authors have spoken of (q, t)-analogs, in particular Reiner and Stanton [RS09] in their extensions of the binomial coefficients, in connection with their study of Hilbert series from the invariant theory of $GL_n(F_q)$. Other studies of (q, t)-analogs are due to Garsia, Haglund, Haiman [GH96, GH02] in their works on (q, t)-Catalan numbers, and to Haiman and Woo [HW07] in enumeration problems occurring in Geometric Combinatorics.

At the $Z = 60$ conference in honor of Doron Zeilberger the attention of the first author has been drawn by Sergei Suslov to the study of q-trigonometric functions occurring in a new theory of basic Fourier series, based on another basic analog of the exponential function (see [Su98], [Su03]). Several classical functions and identities have elegant counterparts in this new q-world. For the time being, it remains to be seen whether combinatorial techniques could bring a new light to this theory.

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