Flag f-vectors of three-colored complexes

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Abstract

The flag f-vectors of three-colored complexes are characterized. This also characterizes the flag h-vectors of balanced Cohen-Macaulay complexes of dimension two, as well as the flag h-vectors of balanced shellable complexes of dimension two.

1 Introduction

In the late 1970s, Stanley [6] showed that characterizing the flag f-vectors of colored simplicial complexes is equivalent to characterizing the flag h-vectors of a different class of complexes. Several years later, Björner, Frankl, and Stanley [1] showed that two additional classes of simplicial complexes shared this same characterization. Unfortunately, no one has a characterization for any of these classes of simplicial complexes, but we only know that characterizing one would suffice for all four. There are some known cases that are trivial. In this paper, we solve one of the simplest non-trivial cases by characterizing the flag f-vectors of 3-colored simplicial complexes.

Recall that a *simplicial complex* Δ on a vertex set W is a collection of subsets of W such that (i) for every $v \in W$, $\{v\} \in \Delta$ and (ii) for every $B \in \Delta$, if $A \subseteq B$, then $A \in \Delta$. The elements of Δ are called *faces*. A face on i vertices is said to have *dimension* $i-1$, while the dimension of a complex is the maximum dimension of a face of the complex.

The *i-th f-number* of a simplicial complex Δ, $f^{i-1}(\Delta)$, is the number of faces of Δ on i vertices. The *f-vector* of Δ lists the f-numbers of Δ. One interesting question to ask is which integer vectors can arise as f-vectors of simplicial complexes. Much work has been done toward answering this for various classes of simplicial complexes. For example, the Kruskal-Katona theorem [5, 4] characterizes the f-vectors of all simplicial complexes.

In this paper, we wish to deal with colored complexes, where the coloring provides additional data. A *coloring* of a simplicial complex is a labeling of the vertices of the complex with colors such that no two vertices in the same face receive the same color. Because any two vertices in a face are connected by an edge, this is equivalent to requiring that any two adjacent vertices be assigned different colors. If there are n colors, we refer them as $1, 2, \ldots, n$ and the set of colors is denoted by $[n] = \{1, 2, \ldots, n\}$. The *color set* of a face is the subset of $[n]$ consisting of the colors of the vertices of the face.

If an (uncolored) complex can be colored with n colors, we call it n-colorable. This is equivalent to its 1-skeleton taken as a graph having chromatic number at most n. The Frankl-Füredi-Kalai [2] theorem characterizes the f-vectors of n-colorable complexes. If an n-colorable complex has dimension $n-1$, we say that Δ is *balanced*, following Stanley [6].

We wish to use a refinement of the usual notion of f-vectors. Let Δ be an n-colored simplicial complex. For any $S \subseteq [n]$, the *flag f-number* $f_S(\Delta)$ counts the number of faces of Δ with color set S. The *flag f-vector* of Δ, $f(\Delta)$, is the collection of the flag f-numbers of Δ for all subsets $S \subseteq [n]$.

For simplicity, we sometimes refer to colors by their numbers and drop the brackets when we do so. For example, $f_3(\Delta)$ is the number of vertices of Δ of color 3. Similarly, $f_{12}(\Delta)$ is the number of edges of Δ with one vertex of color 1 and one vertex of color 2.

The relation between the f-numbers and the flag f-numbers is that the former ignores the colors. The f-numbers can be computed from the flag f-numbers by

$$f^{i-1}(\Delta) = \sum_{|S|=i} f_S(\Delta).$$

The f-numbers of a complex are usually written with the number as a subscript, not a superscript. We do not do this because this paper is mainly interested in flag f-numbers, and we wish to be able to drop the brackets and write, for example, $f_{12}(\Delta)$ rather than $f_{\{1,2\}}(\Delta)$ without it being mistaken for the number of twelve-dimensional faces of Δ.

One can ask which nonnegative integer vectors can arise as the flag f-vectors of colored simplicial complexes. It can help to define the flag h-vector of a complex by

$$h_S(\Delta) = \sum_{T \subseteq S} f_T(\Delta)(-1)^{|S|-|T|}.$$

The flag h-vector of a complex contains the same information as the flag f-vector, and it is sometimes more convenient to work with the flag h-vector. If given the flag h-vector of a colored complex Δ, we can recover its flag f-vector by

$$f_S(\Delta) = \sum_{T \subseteq S} h_T(\Delta).$$

A simplicial complex of dimension $d-1$ is *balanced* if it is d-colorable. For a simplicial complex to be *shellable* or *Cohen-Macaulay* are technical conditions that appear in the following theorem, but we do not need them directly, so we do not define them in this paper. For our purposes, we only need that all shellable complexes are Cohen-Macaulay.

Theorem 1.1 (Stanley [6], Björner-Frankl-Stanley [1]) *Let n be a fixed positive integer. The following are equivalent for a vector $\vec{x} = (x_S)_{S \subseteq [n]}$.*

1. *$\vec{x}$ is the flag f-vector of an n-colored simplicial complex,*

2. *$\vec{x}$ is the flag h-vector of a balanced Cohen-Macaulay complex of dimension $n-1$,*

3. *$\vec{x}$ is the flag h-vector of a balanced shellable complex of dimension $n-1$,*

4. *$\vec{x}$ is the flag f-vector of an n-colored, color-shifted complex.*

Stanley [6] showed that the first two conditions are equivalent. Björner, Frankl, and Stanley [1] showed that the last two conditions are also equivalent to the first two. The problem here is that while four different classes of complexes have equivalent characterizations, none of them have a known characterization.

One might hope that stronger local restrictions than what Björner, Frankl, and Stanley found could be placed upon the complexes without changing the characterization of the flag f-vectors, and work toward a solution that way. For example, the Kruskal-Katona theorem says that to characterize the f-vectors of simplicial complexes, we can restrict to family of the shifted complexes. As there is only one possible shifted complex for a given f-vector, this effectively solves the problem. Frankl, Füredi, and Kalai [2] did something similar to characterize the f-vectors of colored complexes.

Indeed, the paper of Björner, Frankl, and Stanley already did impose stronger restrictions to some extent. Every color-shifted colored complex is, in particular, a colored complex, so they showed that in order to characterize the flag f-vectors of colored complexes, it sufficed to consider only color-shifted complexes. Likewise, every shellable complex is Cohen-Macaulay, so they showed that to characterize the flag h-vectors of balanced Cohen-Macaulay complexes, it suffices to consider only the balanced shellable complexes. However, another paper of the author [3] showed that one cannot impose stronger local conditions than color-shifting in a certain sense.

Another approach is to try to impose some bounds. Walker [7] showed that the only linear inequalities on the flag f-numbers of simplicial complexes are the trivial ones, namely, that all flag f-numbers are non-negative. In the same paper, he computed all linear inequalities on the logarithms of the flag f-numbers of a simplicial complex. These give inequalities on the products of flag f-numbers. For example, the most trivial case is that $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta)$, as any edge whose color set is $\{1,2\}$ must use a vertex of color 1 and a vertex of color 2, and the ways to pick these vertices are $f_1(\Delta)$ and $f_2(\Delta)$, respectively. While an interesting result, Walker's result is shy of a full characterization of the flag f-vectors of simplicial complexes in multiple ways.

First, it avoids dealing with discreteness issues. For example, if $f_{12}(\Delta) = f_{13}(\Delta) = f_{23}(\Delta) = 3$ and $f_1(\Delta) = f_2(\Delta) = f_3(\Delta) = 2$, then the only possible way to arrange the edges in a color-shifted manner is for two vertices of distinct colors to be adjacent unless both are the second vertex of their color. This arrangement forces $f_{123}(\Delta) \leqslant 4$. However, the smallest of Walker's bounds gives $f_{123}(\Delta) \leqslant 3\sqrt{3} \approx 5.2$. Since flag f-numbers must be integers, this immediately gives that $f_{123}(\Delta) \leqslant 5$. But Walker's bounds are unable to produce the real upper bound of $f_{123}(\Delta) \leqslant 4$.

What happens here is that discreteness gets in the way to make Walker's bounds not sharp. Otherwise, we could take a complete tripartite graph on $\sqrt{3}$ vertices of each color and hit his bound exactly.

Another issue is that sharp bounds on flag f-numbers of certain color sets in terms of their flag f-numbers on proper subsets can sometimes conflict with each other. For example, suppose that $f_{12}(\Delta) = 1$, $f_{13}(\Delta) = 2$, $f_{23}(\Delta) = 2$, $f_{24}(\Delta) = 2$, and $f_{34}(\Delta) = 1$. Walker's bounds give that $f_{123}(\Delta) \leqslant 2$ and $f_{234}(\Delta) \leqslant 2$. It is possible to obtain either one of these. The former is a complete tripartite graph on two vertices of color 3 and one vertex each of colors 1 and 2. The latter is a complete tripartite graph on two vertices of color 2 and one vertex each of colors 3 and 4.

It is not possible to make $f_{123}(\Delta) = 2$ and $f_{234}(\Delta) = 2$ simultaneously, however. The former requires that the edges with color set $\{2, 3\}$ have a common vertex of color 2, while the latter requires that the edges have a common vertex of color 3.

Walker's bounds are enough to settle the case of two colors. A proposed nonnegative integer flag f-vector corresponds to a non-empty two-colored simplical complex if and only if $f_\emptyset(\Delta) = 1$ and $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta)$. The problem remains open for more colors, however.

In this paper, we give a characterization of the flag f-vectors of 3-colored complexes in Theorem 3.17 in a way that accounts for discreteness issues. Given any prospective flag f-vector for a 3-colored complex, we can either give a complex that has the desired flag f-vector or show that no such complex exists. The problem of different higher dimensional flag f-numbers forcing different configurations on lower dimensional faces only appears when there are at least four colors.

We begin by dispensing with some trivial cases. We can assume that $f_\emptyset(\Delta) = 1$, as this is true for every non-empty complex. We also assume that $f_i(\Delta)f_j(\Delta) \geqslant f_{ij}(\Delta)$, as if not, then it does not correspond to any complex. Finally, we assume that $f_S(\Delta) > 0$ for all $S \subseteq [3]$, as if not, the problem is trivial. This allows us to refer to two-dimensional faces as *facets*, if we follow the convention that facets are faces of the same dimension as the complex, rather than the convention that facets are all faces that are maximal under inclusion.

For simplicity, we refer to an edge of color set $\{1, 2\}$ as being an edge of color 12, and similarly for edges of color 13 or 23. With only three colors, this cannot lead to ambiguity about the color set intended.

Our approach to the problem is to find the maximum number of facets possible, given the rest of the flag f-vector. Throughout this paper, we define complexes by specifying the edges. The facets are assumed to be all possible facets for which all three edges are present in the complex. This means that the complexes are flag complexes. If we wish to have fewer facets than the maximum number allowed, we can take a construction with more facets and discard some facets.

Doing the computations to determine the maximum number of facets allowed by the rest of the flag f-vector is practical. For an arbitrary integer k, if the number of edges of each color set is chosen independently and uniformly at randomly from $[k] = \{1, 2, \ldots, k\}$, then the expected number of complexes we must check to find the one that maximizes the number of facets is less than 15, independent of k and regardless of how many vertices of each color are allowed. In the worst possible case, we check on the order of $k^{\frac{1}{4}}$ complexes.

In Section 2, we give a construction with five parameters and show that it suffices to consider only this construction. In Section 3, we give our characterization of the flag f-vectors of 3-colored complexes. In Section 4 we explain how to give an explicit complex for any of the equivalent conditions of Theorem 1.1 with a specified flag f-vector or flag h-vector, if such a complex exists. In Section 5, we give proofs of the lemmas stated and used in Section 3. In Section 6, we give some examples of computations to characterize the flag f-vectors of 3-colored complexes. These computations demonstrate why there shouldn't be a much nicer characterization. In Section 7, we discuss the analogous problem for more than three colors.

2 Some parameters

We start this section by giving our construction in Definition 2.1. The key results of this section are Propositions 2.3 and 2.4, which ensure that there is a complex as defined in Definition 2.1 that maximizes the number of facets.

We can place an arbitrary order on the vertices of each color. We label the j-th vertex of color i as v^i_j, so that the vertices of color i are $v^i_1, v^i_2, \ldots, v^i_{f_i(\Delta)}$.

Definition 2.1 Let Δ be a 3-colored complex. Let $\mathcal{A}(\Delta)$ be the set of 3-colored complexes constructed as follows. Choose nonnegative integers g_1, g_2, and g_3 such that $g_i \leqslant f_i(\Delta)$ and $g_i g_j \leqslant f_{ij}(\Delta)$ for all $i, j \in [3]$ with $i \neq j$. Also choose distinct $p, q \in [3]$. Define $r = 6 - p - q$; this means that p and q are two of the numbers in [3], and r is the third.

Start with a complete tripartite graph Γ_1 on g_1 vertices of color 1, g_2 vertices of color 2, and g_3 vertices of color 3. Next, define a new complex Γ_2. The vertices of Γ_2 are the same as those of Γ_1, except that if $f_p(\Delta) > g_p$, Γ_2 has an extra vertex v of color p. The edges of Γ_2 not including v are the same as those of Γ_1. The vertex v should be adjacent to precisely the first $\min\{f_{pq}(\Delta) - f_{pq}(\Gamma_1), f_q(\Gamma_1)\}$ vertices of color q and the first $\min\{f_{pr}(\Delta) - f_{pr}(\Gamma_1), f_r(\Gamma_1)\}$ vertices of color r.

After this, define a new complex Γ_3. The vertices of Γ_3 are the same as those of Γ_2, except that if $f_q(\Delta) > g_q$, then Γ_3 has an extra vertex w of color q. The edges of Γ_3 not including w are the same as those of Γ_2. The vertex w should be adjacent to precisely the first $\min\{f_{pq}(\Delta) - f_{pq}(\Gamma_2), f_p(\Gamma_1)\}$ vertices of color q and the first $\min\{f_{qr}(\Delta) - f_{qr}(\Gamma_1), f_r(\Gamma_1)\}$ vertices of color r. A complex is in $\mathcal{A}(\Delta)$ exactly if it can be obtained as Γ_3 from suitable constants g_1, g_2, g_3, p, and q.

By construction, for every $\Gamma \in \mathcal{A}(\Delta)$, $f_S(\Gamma) \leqslant f_S(\Delta)$ for all $S \neq [3]$.

Definition 2.2 Let Δ be a 3-colored complex. We define

$$\begin{aligned} m(\Delta) &= \max\{f_{123}(\Gamma) \mid \Gamma \in \mathcal{A}(\Delta)\}, \\ \mathcal{B}(\Delta) &= \{\Gamma \in \mathcal{A}(\Delta) \mid f_{123}(\Gamma) = m(\Delta)\}, \end{aligned}$$

$$\begin{aligned} n(\Delta) &= \max\{f_{12}(\Gamma)+f_{13}(\Gamma)+f_{23}(\Gamma) \mid \Gamma \in \mathcal{B}(\Delta)\}, \qquad \text{and} \\ \mathcal{C}(\Delta) &= \{\Gamma \in \mathcal{B}(\Delta) \mid f_{12}(\Gamma)+f_{13}(\Gamma)+f_{23}(\Gamma) = n(\Delta)\}. \end{aligned}$$

It is immediate from the definition that $\mathcal{A}(\Delta) \supseteq \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$.

The goal of this section is to prove the following propositions, which greatly restrict the arrangements of edges that we must consider.

Proposition 2.3 *For every 3-colored complex Δ, there is a $\Gamma \in \mathcal{C}(\Delta)$ such that $f_{123}(\Gamma) \geqslant f_{123}(\Delta)$.*

Proposition 2.4 *Let Θ and Δ be simplicial complexes such that $f_S(\Theta) \leqslant f_S(\Delta)$ for every proper subset $S \subset [3]$. Then $f_{123}(\Theta) \leqslant m(\Delta)$. Furthermore, if Σ is a complex such that $f_{123}(\Sigma) < f_{123}(\Theta)$, then $\Sigma \notin \mathcal{B}(\Delta)$.*

The consequence of these propositions is that in order to maximize $f_{123}(\Delta)$ given the rest of the flag f-vector, it suffices to consider only the complexes in $\mathcal{C}(\Delta)$. The reason for this is that if we have a 3-colored complex Θ with $f_S(\Theta) = f_S(\Delta)$ for all proper subsets $S \subset [3]$, then we are guaranteed that there is a complex $\Gamma \in \mathcal{C}(\Delta)$ such that $f_{123}(\Gamma) = m(\Delta) \geqslant f_{123}(\Theta)$. While the complex Γ may have fewer vertices and edges than Δ, adding more vertices and edges to a flag complex does not decrease the number of facets.

This reduces the problem to finding the values of g_1, g_2, g_3, p, and q that maximize the number of facets of the construction in Definition 2.1. Furthermore, we only need to consider values that produce a complex in $\mathcal{C}(\Delta)$. As all complexes in $\mathcal{C}(\Delta)$ have $m(\Delta)$ facets, the problem of maximizing the number of facets is equivalent to finding $m(\Delta)$.

The proof of these propositions is delayed until the end of this section, as we need some other results first.

Definition 2.5 Let Δ be an n-colored simplicial complex. We say that Δ is *color-shifted* if, for all $b_1 \leqslant a_1, b_2 \leqslant a_2, \ldots, b_j \leqslant a_j$, $\{v^{i_1}_{a_1}, v^{i_2}_{a_2}, \ldots v^{i_j}_{a_j}\} \in \Delta$ implies $\{v^{i_1}_{b_1}, v^{i_2}_{b_2}, \ldots v^{i_j}_{b_j}\} \in \Delta$.

Theorem 2.6 (Björner-Frankl-Stanley [1]) *Let Δ be an n-colored simplicial complex. Then there is an n-colored, color-shifted simplicial complex Γ such that $f_S(\Delta) = f_S(\Gamma)$ for all $S \subseteq [n]$.*

This is the portion of Theorem 1.1 that we need to use. Björner, Frankl, and Stanley called the concept "compressed" rather than color-shifted. Furthermore, their proof allowed for a more general notion of coloring where, for example, one could have three colors, but allow a face to have up to 3 vertices of color 1, up to 5 vertices of color 2, and up to 2 vertices of color 3. In this paper, we focus on the case where only one vertex of each color is allowed in a face.

Lemma 2.7 *Let Δ be a 3-colored simplicial complex. Then there is a simplicial complex Γ, together with positive integers c_1, c_2, c_3, c_4, c_5, and c_6 such that*

1. *Γ is 3-colored,*

2. *Γ is color-shifted,*

3. *$f_S(\Gamma) = f_S(\Delta)$ for all $S \subset [3]$ with $S \neq [3]$,*

4. *$f_{123}(\Gamma) \geqslant f_{123}(\Delta)$,*

5. *$\{v^1_{a_1}, v^2_{a_2}\} \in \Gamma$ for all $a_1 \leqslant c_1$ and $a_2 \leqslant c_2$,*

6. *Γ has at most one vertex other than $\{v^1_1, \ldots, v^1_{c_1}, v^2_1, \ldots, v^2_{c_2}\}$ contained in an edge of Γ of color 12.*

7. *$\{v^1_{a_3}, v^3_{a_4}\} \in \Gamma$ for all $a_3 \leqslant c_3$ and $a_4 \leqslant c_4$,*

8. *Γ has at most one vertex other than $\{v^1_1, \ldots, v^1_{c_3}, v^3_1, \ldots, v^3_{c_4}\}$ contained in an edge of Γ of color 13.*

9. *$\{v^2_{a_5}, v^3_{a_6}\} \in \Gamma$ for all $a_5 \leqslant c_5$ and $a_6 \leqslant c_6$, and*

10. *Γ has at most one vertex other than $\{v^2_1, \ldots, v^2_{c_5}, v^3_1, \ldots, v^3_{c_6}\}$ contained in an edge of Γ of color 23.*

Proof: Theorem 2.6 ensures that there is a complex Δ_0 that satisfies properties (1) through (4), though it could fail the rest. We start with this complex and rearrange edges of one color at a time such that after each rearrangement, the complex satisfies the two properties relevant to that color, while retaining any numbered properties that it held before the rearrangement. After rearranging edges of all three colors, we have the complex Γ.

Suppose that it is not possible to choose c_1 and c_2 and create a complex Δ_1 satisfying properties (1) through (6) with the edges of Δ_1 exactly the same as the edges of Δ_0 except for those of color 12. Taking $\Delta_1 = \Delta_0$ and $c_1 = c_2 = 1$ would leave the necessary faces unchanged and satisfy properties (1) through (5), so the only obstruction here is property (6).

Let Σ be a complex that minimizes the number of vertices contained in an edge of color 12 among all rearrangements of the edges of Δ_0 of color 12 that satisfy conditions (1) through (5). Let the vertices of Σ contained in an edge of color 12 be $\{v^1_1, \ldots, v^1_{d_1}, v^2_1, \ldots, v^2_{d_2}\}$. Let the number of vertices of color 1 adjacent to the vertex v^3_i be p_i and the number of vertices of color 2 adjacent to v^3_i be q_i. Since Σ is color-shifted, we must have $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_{f_3(\Delta)}$ and $q_1 \geqslant q_2 \geqslant \ldots \geqslant q_{f_3(\Delta)}$.

Suppose first that $p_{f_3(\Delta)} \geqslant d_1$ and $q_{f_3(\Delta)} \geqslant d_2$. In this case, every edge of color 12 together with a vertex of color 3 forms a facet of Σ. Furthermore, rearranging edges of Σ of color 12 does not change the number of facets provided that all edges only use the vertices $\{v^1_1, \ldots, v^1_{d_1}, v^2_1, \ldots, v^2_{d_2}\}$. Let $w = \frac{f_{12}(\Delta)}{d_2}$. Rearrange the edges of Σ of color 12 such that the vertices $\{v^1_1, \ldots, v^1_{\lfloor w \rfloor}\}$ are each adjacent to all of $\{v^2_1, \ldots, v^2_{d_2}\}$ and the vertex

$v^1_{\lfloor w\rfloor+1}$ is adjacent to $\{v^2_1,\ldots,v^2_{d_2(w-\lfloor w\rfloor)}\}$. Take $c_1=\lfloor w\rfloor$ and $c_2=d_2$. It is easy to check that this new Σ satisfies all of the conditions necessary for Δ_1.

Otherwise, let $m=\min\{i \mid \text{either } p_i<d_1 \text{ or } q_i<d_2\}$. Suppose without loss of generality that $p_m<d_1$; exactly the same argument applies if $q_m<d_2$. If $f_{12}(\Delta)\leqslant(d_1-1)d_2$, then we can move the edges of Σ of color 12 that contain $v^1_{d_1}$ to instead use one vertex in $\{v^1_1,\ldots,v^1_{d_1-1}\}$ (without $v^1_{d_1}$) and one vertex in $\{v^2_1,\ldots,v^2_{d_2}\}$, with the new edges chosen so as to keep the new complex color-shifted. For $i<m$, this does not change the number of facets of Σ containing v^3_i because every edge of Σ of color 12 together with v^3_i forms a facet of Σ both before and after the rearrangement. It also cannot decrease the number of facets of Σ containing v^3_i for $i\geqslant m$, as v^3_i together with the vertices of a removed edge did not form a facet as $\{v^3_i,v^1_{d_1}\}\notin\Sigma$. As such, the new Σ satisfies conditions (1) through (5) while having one vertex fewer contained in an edge of color 12. This contradicts the choice of Σ.

The other possibility is that $f_{12}(\Delta)>(d_1-1)d_2$. In this case, rearrange the edges of Σ of color 12 such that the edges are every possible combination of one vertex in $\{v^1_1,\ldots,v^1_{d_1-1}\}$ (without $v^1_{d_1}$) and one vertex in $\{v^2_1,\ldots,v^2_{d_2}\}$, as well as that $v^1_{d_1}$ forms an edge with each vertex of $\{v^2_1,\ldots,v^2_{f_{12}(\Delta)-(d_1-1)d_2}\}$. As in the previous paragraph, this cannot decrease the number of facets of Σ containing vertex v^3_i with $i<m$ because v^3_i would form a facet with the edge both before and after it is moved. It cannot decrease the number of facets of Σ containing v^3_i for $i\geqslant m$, as v^3_i together with the vertices of a removed edge did not form a facet as $\{v^3_i,v^1_{d_1}\}\notin\Sigma$. We can take $c_1=d_1-1$, $c_2=d_2$, and $\Delta_1=\Sigma$ and satisfy conditions (1) through (6), as $v^1_{d_1}$ is the only extra vertex.

Now we repeat the process by rearranging the edges of other colors. We can create Δ_2 from Δ_1 by rearranging the edges with color 13 in the same manner as how Δ_1 was created. This retains properties (1) through (4) for the same reasons that Δ_1 did and makes properties (7) and (8) hold if $\Gamma=\Delta_2$ for the same reasons that properties (5) and (6) hold if $\Gamma=\Delta_1$. Since the edges of color 12 are unchanged, properties (5) and (6) hold for Δ_2 because they hold for Δ_1.

Finally, we create Δ_3 from Δ_2 by rearranging edges of color 23. This inherits properties (5) through (8) from Δ_2. It retains properties (1) through (4) and adds properties (9) and (10) for the same reasons as happened analogously with Δ_1 and Δ_2. Taking $\Gamma=\Delta_3$ completes the proof. $\square$

Lemma 2.8 *Let Δ be a 3-colored simplicial complex. There is a $\Gamma\in\mathcal{A}(\Delta)$ with $f_{123}(\Gamma)\geqslant f_{123}(\Delta)$.*

Proof: Let Δ_1 be a simplicial complex satisfying the conditions of Lemma 2.7. Suppose first that the extra vertices of conditions (6), (8), and (10) from Lemma 2.7 are of at most two different colors.

We construct Γ from Δ_1 as follows. Let $g_1=\min\{c_1,c_3\}$, $g_2=\min\{c_2,c_5\}$, and $g_3=\min\{c_4,c_6\}$. If there are no extra vertices, then choose p and q arbitrarily. If there is only one color with an extra vertex, then let that color be p, and choose q arbitrarily.

If there are two colors with an extra vertex, then let them be p and q. To choose which color is p and which is q, consult the following table. The first column gives the possibilities for the set $\{p, q\}$. There is only one extra vertex in condition (6), (8), or (10) corresponding to this color set. The second column gives the possibilities of the color of this extra vertex. If there isn't an extra vertex for the corresponding condition in Lemma 2.7, then we can choose p and q arbitrarily. The third column gives the conditions for which we choose p to be less than q. For example, if $\{p, q\} = \{1, 2\}$ and the condition is satisfied, then we pick $p = 1$ and $q = 2$; if not, we pick $p = 2$ and $q = 1$.

$\{p,q\}$	extra	condition
$\{1,2\}$	1	$c_2 \leqslant c_5$
$\{1,2\}$	2	$c_1 > c_3$
$\{1,3\}$	1	$c_4 \leqslant c_6$
$\{1,3\}$	3	$c_3 > c_1$
$\{2,3\}$	2	$c_6 \leqslant c_4$
$\{2,3\}$	3	$c_5 > c_2$

With these constants, construct Γ as described in Definition 2.1. The complex Γ of the definition is a subcomplex of Δ_1, so it has few enough edges that the construction works, and $\Gamma \in \mathcal{A}(\Delta)$. Furthermore, all facets of Δ_1 are also in Γ, so $f_{123}(\Gamma) \geqslant f_{123}(\Delta_1) \geqslant f_{123}(\Delta)$.

Otherwise, the extra vertices are forced upon us by Δ_1 and one extra vertex is of each color. One possibility is that the extra vertex for an edge of color set $\{i, i+1\}$ (modulo 3) is of color i for all $i \in [3]$; the other possibility is that the extra vertex is always of color $i + 1$. By symmetry, it suffices to consider only the former case.

Suppose that $c_1 \geqslant c_3$. The extra vertex of color 1 is then not adjacent to any vertices of color 3. As such, if we remove all edges containing it, we do not lose any facets. This means that there is no longer an extra vertex corresponding to condition (6) of Lemma 2.7, so we are back in a previous case. The same analysis applies if $c_5 \geqslant c_2$ or $c_4 \geqslant c_6$. This leaves only the case where $c_1 < c_3$, $c_5 < c_2$, and $c_4 < c_6$. These conditions imply that no two of the extra vertices are adjacent.

Construct a simplicial complex Δ_2 from Δ_1 by discarding all vertices and edges not contained in a facet of Δ_1. In the new complex Δ_2, let $v^1_{g_1+1}$ be adjacent to d_{12} vertices of color 2 and d_{13} vertices of color 3, let $v^2_{g_2+1}$ be adjacent to d_{21} vertices of color 1 and d_{23} vertices of color 3, and let $v^3_{g_3+1}$ be adjacent to d_{31} vertices of color 1 and d_{32} vertices of color 2. Because no two extra vertices are adjacent, any two vertices of distinct colors adjacent to one of these extra vertices must be adjacent to each other. As such, $v^1_{g_1+1}$ is contained in $d_{12}d_{13}$ facets, $v^2_{g_2+1}$ is contained in $d_{21}d_{23}$ facets, and $v^3_{g_3+1}$ is contained in $d_{31}d_{32}$ facets. Our construction gives that $d_{21} = g_1$, $d_{32} = g_2$, and $d_{13} = g_3$. No two of the extra vertices are adjacent to each other, so if we add the facets not containing any of the extra vertices, we can compute

$$f_{123}(\Delta_2) = g_1g_2g_3 + d_{12}d_{13} + d_{21}d_{23} + d_{31}d_{32} = g_1g_2g_3 + d_{12}g_3 + d_{23}g_1 + d_{31}g_2.$$

Suppose that $g_1 \geqslant g_2$. We wish to compute $f_{12}(\Delta_2)$. There are $g_1 g_2$ edges containing neither of the extra vertices. There are d_{12} edges containing the extra vertex of color 1. There are $d_{21} = g_1$ edges containing the extra vertex of color 2. Add the edges containing neither of the extra vertices and we get

$$f_{12}(\Delta_2) = g_1 g_2 + d_{12} + d_{21} = g_1 g_2 + d_{12} + g_1.$$

We create Γ from Δ_2 by rearranging the edges of color 12 as follows. Remove the edges containing $v^1_{g_1+1}$ or $v^2_{g_2+1}$. In their place, make $v^1_{g_1+1}$ adjacent to the first g_2 vertices of color 2. Use the remaining edges to make $v^2_{g_2+1}$ adjacent to the first $g_1 + d_{12} - g_2$ edges of color 1. We can do this because $g_1 \geqslant g_2$ gives that $g_1 + d_{12} - g_2 \geqslant d_{12} > 0$ and $d_{12} \leqslant g_2$ gives that $g_1 + d_{12} - g_2 \leqslant g_1$, so there are enough vertices of color 1. This keeps all flag f-numbers of Γ the same as of Δ_2 except possibly the number of facets. If we plug the new values of d_{12} and d_{21} into the above formula, we get

$$f_{123}(\Gamma) = g_1 g_2 g_3 + g_2 g_3 + d_{23}(g_1 + d_{12} - g_2) + d_{31} g_2.$$

Now we can compute

$$\begin{aligned} f_{123}(\Gamma) - f_{123}(\Delta_2) &= g_1 g_2 g_3 + g_2 g_3 + d_{23}(g_1 + d_{12} - g_2) + d_{31} g_2 \\ &\qquad -(g_1 g_2 g_3 + d_{12} g_3 + d_{23} g_1 + d_{31} g_2) \\ &= (g_2 - d_{12}) g_3 + d_{23}(d_{12} - g_2) \\ &= (g_2 - d_{12})(g_3 - d_{23}) \geqslant 0 \end{aligned}$$

The last line follows because both factors are nonnegative by construction.

We must check that Γ satisfies all of the needed conditions. First, for every proper subset $S \subset [3]$, we have $f_S(\Gamma) = f_S(\Delta_2) \leqslant f_S(\Delta_1) = f_S(\Delta)$. For condition 2, we have just shown $f_{123}(\Gamma) \geqslant f_{123}(\Delta_2) = f_{123}(\Delta_1) \geqslant f_{123}(\Delta)$. For condition 4, the vertex $v^1_{g_1+1}$ is now adjacent to the first g_2 vertices of color 2 and the first g_3 vertices of color 3. As such, we it is no longer an extra vertex, and we can increase g_1 by 1. This leaves only the other two vertices as extra vertices. Condition 3 is clear. Finally, Γ comes from the specified construction with the new value of g_1, $p = 3$, and $q = 2$.

If $g_2 \geqslant g_3$, we can do the same procedure as before, this time rearranging edges of color 23 to make $v^2_{g_2+1}$ no longer an extra vertex. Likewise, if $g_3 \geqslant g_1$, we rearrange edges of color 13 to make $v^3_{g_3+1}$ no longer an extra vertex. This leaves only the case where $g_1 < g_2 < g_3 < g_1$, which is impossible. □

Proof of Proposition 2.3: By Lemma 2.8, there is a $\Gamma_1 \in \mathcal{A}(\Delta)$ with $f_{123}(\Gamma_1) \geqslant f_{123}(\Delta)$. It is immediate from the definitions that there is a $\Gamma_2 \in \mathcal{C}(\Delta)$. Furthermore, because $\Gamma_2 \in \mathcal{C}(\Delta)$, we get $f_{123}(\Gamma_2) = m(\Delta) \geqslant f_{123}(\Gamma_1) \geqslant f_{123}(\Delta)$. □

Proof of Proposition 2.4: Let $\Gamma \in \mathcal{C}(\Theta)$. By Proposition 2.3, $f_{123}(\Gamma) \geqslant f_{123}(\Theta)$. By construction, $f_S(\Gamma) \leqslant f_S(\Theta) \leqslant f_S(\Delta)$ for all proper subsets $S \subset [3]$, and so $\Gamma \in \mathcal{A}(\Delta)$. Therefore, $m(\Delta) \geqslant f_{123}(\Gamma) \geqslant f_{123}(\Theta)$. The second claim follows immediately from $f_{123}(\Sigma) < f_{123}(\Theta) \leqslant m(\Delta)$. □

3 Determining the optimal constants

In this section, we characterize the flag f-vectors of three-colored complexes. Proposition 2.3 reduced the problem to finding the choices of g_1, g_2, g_3, p and q that maximize the number of facets when we construct a complex using the method of Definition 2.1. In this section, we give a series of lemmas that greatly restrict which choices of these constants it is necessary to check. Theorem 3.17 assembles the lemmas into an algorithm that is guaranteed to find the maximum number of facets, and usually does so very quickly.

Some of the lemmas have proofs that are rather lengthy and unenlightening. As such, the proofs are deferred until the next section.

If the number of vertices of each color is a meaningful restriction, then Lemma 3.2 usually solves the problem. The more difficult case is when the numbers of edges are the only important restrictions.

The intuition behind the characterization is to start by ignoring discreteness. One might guess that the complex that maximizes the number of facets is the complete tripartite complex on c_1 vertices of color 1, c_2 vertices of color 2, and c_3 vertices of color 3, for suitable constants c_1, c_2, and c_3. The relevant equations are $f_{12}(\Delta) = c_1c_2$, $f_{13}(\Delta) = c_1c_3$, and $f_{23}(\Delta) = c_2c_3$. We can solve for the constants to get $c_1 = \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}$, $c_2 = \sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}}$, and $c_3 = \sqrt{\frac{f_{13}(\Delta)f_{23}(\Delta)}{f_{12}(\Delta)}}$.

This would give a complex Γ with $f_{123}(\Gamma) = c_1c_2c_3 = \sqrt{f_{12}(\Delta)f_{13}(\Delta)f_{23}(\Delta)}$. This is one of Walker's [7] upper bounds. One might expect that when we impose the discreteness of flag complexes, the complex that maximizes the number of facets still looks similar to the answer in the non-discrete case. With this in mind, we make the following definition.

Definition 3.1 Let Δ be a 3-colored simplicial complex with flag f-vector $f(\Delta)$. Define

$$\begin{aligned} b_1(\Delta) &= \left\lfloor \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}} \right\rfloor, \\ b_2(\Delta) &= \left\lfloor \sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}} \right\rfloor, \qquad \text{and} \\ b_3(\Delta) &= \left\lfloor \sqrt{\frac{f_{13}(\Delta)f_{23}(\Delta)}{f_{12}(\Delta)}} \right\rfloor. \end{aligned}$$

In subsequent lemmas, we sometimes have more than one complex constructed as Γ was in Definition 2.1, except using different constants. To avoid confusion, we refer to the constants associated to a particular complex as $g_1(\Gamma), p(\Gamma)$, and so forth.

The next lemma solves the problem in certain special cases.

Lemma 3.2 *Let Δ be a 3-colored simplicial complex and let $\Gamma \in \mathcal{C}(\Delta)$.*

1. $f_{123}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$ *if and only if* $\left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant f_{23}(\Delta)$.

2. $f_{123}(\Gamma) = f_2(\Delta)f_{13}(\Delta)$ *if and only if* $\left\lfloor \frac{f_{12}(\Delta)}{f_2(\Delta)} \right\rfloor \left\lfloor \frac{f_{23}(\Delta)}{f_2(\Delta)} \right\rfloor \geqslant f_{13}(\Delta)$.

3. $f_{123}(\Gamma) = f_3(\Delta)f_{12}(\Delta)$ *if and only if* $\left\lfloor \frac{f_{13}(\Delta)}{f_3(\Delta)} \right\rfloor \left\lfloor \frac{f_{23}(\Delta)}{f_3(\Delta)} \right\rfloor \geqslant f_{12}(\Delta)$.

Walker's bounds in [7] include $f_{123}(\Gamma) \leqslant f_1(\Delta)f_{23}(\Delta)$, $f_{123}(\Gamma) \leqslant f_2(\Delta)f_{13}(\Delta)$, and $f_{123}(\Gamma) \leqslant f_3(\Delta)f_{12}(\Delta)$.

If Lemma 3.2 gives $m(\Delta)$ by asserting that $f_{123}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$, then the facets of Γ consist of all ways to choose a vertex of color 1 together with an edge of color 23. Something analogous happens with the other parts of the lemma.

If Lemma 3.2 does not solve the problem, then this next lemma allows some far stronger restrictions on the complexes that we must consider, as it means that the fourth condition must hold.

Lemma 3.3 *Let Δ be a 3-colored simplicial complex. At least one of the following holds for every $\Gamma \in \mathcal{C}(\Delta)$.*

1. $f_{123}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$;
2. $f_{123}(\Gamma) = f_2(\Delta)f_{13}(\Delta)$;
3. $f_{123}(\Gamma) = f_3(\Delta)f_{12}(\Delta)$; *or*
4. $f_{12}(\Gamma) = f_{12}(\Delta)$, $f_{13}(\Gamma) = f_{13}(\Delta)$, *and* $f_{23}(\Gamma) = f_{23}(\Delta)$.

We define some notation to describe complexes that must satisfy the fourth condition of Lemma 3.3.

Definition 3.4 Let Δ be a 3-colored simplicial complex. Define

$$\mathcal{D}(\Delta) = \{\Gamma \in \mathcal{C}(\Delta) \mid f_{123}(\Gamma) < \min\{f_1(\Delta)f_{23}(\Delta), f_2(\Delta)f_{13}(\Delta), f_3(\Delta)f_{12}(\Delta)\}\}.$$

If Lemma 3.2 gives $m(\Delta)$, then $\mathcal{D}(\Delta) = \emptyset$. Otherwise, $\mathcal{D}(\Delta) = \mathcal{C}(\Delta)$. Either way, we now have $\mathcal{A}(\Delta) \supseteq \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta) \supseteq \mathcal{D}(\Delta)$.

Definition 3.5 Let Δ be a 3-colored simplicial complex and let $\Gamma \in \mathcal{A}(\Delta)$. Define

$$\begin{aligned} j_1(\Gamma) &= f_{23}(\Delta) - g_2(\Gamma)g_3(\Gamma) \\ j_2(\Gamma) &= f_{13}(\Delta) - g_1(\Gamma)g_3(\Gamma) \\ j_3(\Gamma) &= f_{12}(\Delta) - g_1(\Gamma)g_2(\Gamma). \end{aligned}$$

The $j's$ are the number of edges left over of a given color set before adding the additional vertices for $p(\Gamma)$ and $q(\Gamma)$.

This next lemma solves an easy case, and is used mainly to avoid division by zero in some later proofs.

Lemma 3.6 *If Δ is a 3-colored simplicial complex such that $b_1(\Delta) = 0$ and $\mathcal{D}(\Delta) \neq \emptyset$, then $m(\Delta) = f_{12}(\Delta) f_{13}(\Delta)$.*

If we know r and $g_r(\Gamma)$, this next proposition gives us an explicit construction that maximizes the number of facets. This reduces the problem to finding r and $g_r(\Gamma)$, rather than needing to know all of the parameters. This works for other choices of p and q as well by relabeling the colors.

Proposition 3.7 *Let Δ be a 3-colored simplicial complex. Suppose that $\Gamma \in \mathcal{D}(\Delta)$, $p(\Gamma) = 1$, and $q(\Gamma) = 2$. Define Γ_1 by $g_3(\Gamma_1) = g_3(\Gamma)$,*

$$
\begin{aligned}
g_1(\Gamma_1) &= \begin{cases} \left\lfloor \frac{f_{13}(\Delta)}{g_3(\Gamma)} \right\rfloor & \text{if } \frac{f_{13}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z} \\ \frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1 & \text{if } \frac{f_{13}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z} \text{ and } \frac{f_{13}(\Delta)}{g_3(\Gamma)} \left\lceil \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1 \right\rceil > f_{12}(\Delta) \\ \frac{f_{13}(\Delta)}{g_3(\Gamma)} & \text{if } \frac{f_{13}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z} \text{ and } \frac{f_{13}(\Delta)}{g_3(\Gamma)} \left\lceil \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1 \right\rceil \leqslant f_{12}(\Delta) \end{cases}, \\
g_2(\Gamma_1) &= \begin{cases} \left\lfloor \frac{f_{23}(\Delta)}{g_3(\Gamma)} \right\rfloor & \text{if } \frac{f_{23}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z} \\ \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1 & \text{if } \frac{f_{23}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z} \text{ and } \frac{f_{23}(\Delta)}{g_3(\Gamma)} g_1(\Gamma_1) > f_{12}(\Delta) \\ \frac{f_{23}(\Delta)}{g_3(\Gamma)} & \text{if } \frac{f_{23}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z} \text{ and } \frac{f_{23}(\Delta)}{g_3(\Gamma)} g_1(\Gamma_1) \leqslant f_{12}(\Delta) \end{cases}, \\
p(\Gamma_1) &= \begin{cases} 1 & \text{if } j_2(\Gamma_1) \geqslant j_1(\Gamma_1) \\ 2 & \text{if } j_2(\Gamma_1) < j_1(\Gamma_1) \end{cases}, \qquad \text{and} \\
q(\Gamma_1) &= 3 - p(\Gamma_1).
\end{aligned}
$$

Then $\Gamma_1 \in \mathcal{D}(\Delta)$.

Proposition 3.7 gives the main way that we construct complexes in the more difficult cases. Unlike Lemmas 3.2 and 3.6, this one doesn't give an easy formula for the number of facets. Rather, it is easier to give the constants used in Definition 2.1, construct the complex, and then count the facets.

It is convenient to assume without loss of generality that $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$. When constructing complexes as in Proposition 3.7, this typically allows $g_3(\Gamma)$ to vary more than $g_1(\Gamma)$ or $g_2(\Gamma)$ and still give a complex that we would have to check. Thus, it would be convenient to mostly avoid having to consider the case when $r(\Gamma) = 3$, as this will greatly reduce the number of complexes that we need to check.

Definition 3.8 Define $\mathcal{E}(\Delta)$ by $\Gamma \in \mathcal{E}(\Delta)$ exactly if $\Gamma \in \mathcal{D}(\Delta)$ and $g_i(\Gamma) = b_i(\Delta)$ for some $i \in [3]$.

It is immediate from this definition that $\mathcal{E}(\Delta) \subseteq \mathcal{D}(\Delta)$. Thus, $\mathcal{A}(\Delta) \supseteq \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta) \supseteq \mathcal{D}(\Delta) \supseteq \mathcal{E}(\Delta)$.

This next lemma means that either we know one value of $g_i(\Gamma)$ immediately, which greatly restricts the rest of the complex, or else we can find a complex $\Gamma \in \mathcal{E}(\Delta)$ with $r(\Gamma) \neq 3$.

Lemma 3.9 *Let Δ be a 3-colored simplicial complex. Either $\mathcal{D}(\Delta) = \mathcal{E}(\Delta)$ or else there are two complexes $\Gamma_1, \Gamma_2 \in \mathcal{E}(\Delta)$ with $r(\Gamma_1) \neq r(\Gamma_2)$.*

In particular, Lemma 3.9 asserts that $\mathcal{E}(\Delta) \neq \emptyset$ unless Lemma 3.3 settled the problem. With Lemma 3.9, the basic idea is to find a complex $\Gamma \in \mathcal{E}(\Delta)$ by breaking the problem into nine cases, defined by three choices for the value of i for which $g_i(\Gamma) = b_i(\Delta)$ and three choices for the value of $r(\Gamma)$. Proposition 3.7 settles three of these nine cases, and this next lemma settles three more.

Lemma 3.10 *Let Δ be a 3-colored simplicial complex and let $\Gamma \in \mathcal{E}(\Delta)$. If $g_i(\Gamma) = b_i(\Delta) \geqslant b_{r(\Gamma)}(\Delta)$ and $r(\Gamma) \neq i$, then $\left\lceil \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} \right\rceil \leqslant \left\lfloor \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)} \right\rfloor$. Furthermore, either $g_{r(\Gamma)}(\Gamma) = \left\lceil \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} \right\rceil$ or else $g_{r(\Gamma)}(\Gamma) = \left\lfloor \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)} \right\rfloor$.*

Definition 3.11 Let Δ be a 3-colored simplicial complex. Define $\mathcal{F}(\Delta)$ by $\Gamma \in \mathcal{F}(\Delta)$ exactly if

1. $\Gamma \in \mathcal{A}(\Delta)$,

2. Γ has exactly as many edges as Δ,

3. $\mathcal{D}(\Delta) \neq \emptyset$, and

4. $g_i(\Gamma) = b_i(\Delta)$ for some $i \in [3]$.

It follows from the definition that $\mathcal{E}(\Delta) \subseteq \mathcal{F}(\Delta) \subseteq \mathcal{A}(\Delta)$. However, $\mathcal{F}(\Delta)$ does not have to be a subset or superset of the other sets of complexes that we have defined. In particular, there can be complexes in $\mathcal{F}(\Delta)$ with relatively few facets.

This next lemma says that to check the cases where $r(\Gamma) = 3$, it suffices to check only the complexes where $g_3(\Gamma) = f_3(\Delta)$.

Lemma 3.12 *Let Δ be a 3-colored simplicial complex such that $f_{12}(\Delta) \leqslant f_{23}(\Delta)$ and $f_{13}(\Delta) \leqslant f_{23}(\Delta)$. If $\Gamma_0 \in \mathcal{F}(\Delta)$, $r(\Gamma_0) = 3$, and $f_3(\Delta) > g_3(\Gamma_0)$, then at least one of the following holds:*

1. there is some $\Gamma \in \mathcal{F}(\Delta)$ with $r(\Gamma) \neq 3$ and $f_{123}(\Gamma) \geqslant f_{123}(\Gamma_0)$;

2. there is some $\Gamma \in \mathcal{A}(\Delta)$ with $f_{123}(\Gamma) > f_{123}(\Gamma_0)$; or

3. there is some $\Gamma \in \mathcal{F}(\Delta)$ with $r(\Gamma) = 3$, $f_{123}(\Gamma) \geqslant f_{123}(\Gamma_0)$, and $g_3(\Gamma) = f_3(\Delta)$.

While we now have three lemmas that each handle three of the nine cases, the case where $g_3(\Gamma) = b_3(\Delta)$ and $r(\Gamma) = 3$ has been covered twice. The one remaining case is when $r(\Gamma) = 2$ and $g_1(\Gamma) = b_1(\Delta)$. This next lemma restricts the values of $g_2(\Gamma)$ for which constructing Γ as in Proposition 3.7 will give a well-defined complex.

Lemma 3.13 *Let Δ be a 3-colored simplicial complex with $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$. Suppose further that $\mathcal{D}(\Delta) \neq \emptyset$ and that there is a complex $\Gamma \in \mathcal{F}(\Delta)$ such that $g_1(\Gamma) = b_1(\Delta)$ and $r(\Gamma) = 2$. This guarantees that $f_1(\Delta) \geqslant b_1(\Delta)$. Furthermore, if $f_1(\Delta) = b_1(\Delta)$, then $g_2(\Gamma) = \frac{f_{12}(\Delta)}{f_1(\Delta)}$. If $f_1(\Delta) > b_1(\Delta)$, then*

1. $f_{13}(\Delta) \leqslant f_3(\Delta)(b_1(\Delta)+1)$,
2. $g_2(\Gamma) \geqslant \frac{f_{12}(\Delta)}{f_1(\Delta)}$,
3. $g_2(\Gamma) \geqslant \frac{f_{12}(\Delta)}{b_1(\Delta)+1}$,
4. $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{\lfloor \frac{f_{13}(\Delta)}{b_1(\Delta)} \rfloor + 1}$,
5. $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{f_3(\Delta)}$,
6. $g_2(\Gamma) \leqslant \frac{f_{12}(\Delta)}{b_1(\Delta)}$,
7. $g_2(\Gamma) \leqslant \frac{f_{23}(\Delta)}{\lceil \frac{f_{13}(\Delta)}{b_1(\Delta)+1} \rceil - 1}$, *and*
8. $g_2(\Gamma) \leqslant f_2(\Delta)$.

It is possible to prove more in the above lemma, but it isn't necessary for our purposes, as we are mainly interested in restricting how many cases there are to check. More precisely, if $f_1(\Delta) = b_1(\Delta)$, then there is a complex $\Gamma \in \mathcal{F}(\Delta)$ such that $g_1(\Gamma) = b_1(\Delta)$ and $r(\Gamma) = 2$ if and only if $g_2(\Gamma) = \frac{f_{12}(\Delta)}{f_1(\Delta)}$, $g_2(\Gamma) \leqslant f_2(\Delta)$, $\max\left\{\frac{f_{23}(\Delta)}{z}, \frac{f_{13}(\Delta)}{f_1(\Delta)}\right\} \leqslant f_3(\Delta)$, and

$$\max\left\{\left\lceil \frac{f_{23}(\Delta)}{z} \right\rceil - 1, \left\lceil \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rceil - 1\right\} \leqslant \min\left\{\left\lfloor \frac{f_{23}(\Delta)}{z} \right\rfloor, \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor\right\}.$$

In addition, if $f_1(\Delta) > b_1(\Delta)$, then the converse of the above lemma holds as well. That is, if z is an integer that satisfies all eight of the listed conditions for $g_2(\Gamma)$, then there is a complex $\Gamma \in \mathcal{F}(\Delta)$ such that $g_1(\Gamma) = b_1(\Delta)$, $r(\Gamma) = 2$, and $g_2(\Gamma) = z$.

There could still be many different values of $g_2(\Gamma)$ that give a well-defined complex. Next, we want to get an upper bound on $f_{123}(\Gamma)$ as a function of $g_2(\Gamma)$.

Definition 3.14 Let Δ be a 3-colored simplicial complex. Define

$$\begin{aligned} v(\Delta, t) &= b_1(\Delta)f_{23}(\Delta) + (f_{12}(\Delta) - b_1(\Delta)t)\Big(f_{13}(\Delta) - \frac{b_1(\Delta)f_{23}(\Delta)}{t}\Big) \text{ and} \\ s(\Delta) &= \sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}}. \end{aligned}$$

Note that it is immediate from the definition that $b_2(\Delta) = \lfloor s(\Delta) \rfloor$.

Lemma 3.15 *Let Δ be a 3-colored simplicial complex and let $\Gamma \in \mathcal{F}(\Delta)$ with $g_1(\Gamma) = b_1(\Delta)$. If $r(\Gamma) = 2$, then $f_{123}(\Gamma) \leqslant v(\Delta, g_2(\Gamma))$.*

Any complex that we can construct gives us a lower bound on $m(\Delta)$. It suffices to consider the values of $g_2(\Gamma)$ for which the upper bound of Lemma 3.15 is greater than the number of facets in any other complex that we have already constructed.

The bound of Lemma 3.15 depends only on $g_2(\Gamma)$ and the flag f-vector of Δ. If we multiply out the bound, the coefficients on the $g_2(\Gamma)$ and $\frac{1}{g_2(\Gamma)}$ terms are both negative. Thus, for sufficiently large or small $g_2(\Gamma)$, the bound is small. This restricts the values of $g_2(\Gamma)$ that we must check to an interval, and often a much smaller interval than that of Lemma 3.13. This next lemma says that there usually are not very many possible values of $g_2(\Gamma)$ in that interval.

Lemma 3.16 *Let Δ be a 3-colored simplicial complex with $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$ and $b_1(\Delta) \geqslant 1$. Suppose that there is some $\Gamma_0 \in \mathcal{D}(\Delta)$ with $g_1(\Gamma_0) = b_1(\Delta)$ and $r(\Gamma_0) = 2$. Then we can find some $\Gamma \in \mathcal{D}(\Delta)$ by checking fewer than $6 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ potential values of $g_2(\Gamma)$ and applying Proposition 3.7 to each potential value of $g_2(\Gamma)$ and $r(\Gamma) = 2$.*

Finally, we reach the main theorem. This basically summarizes the lemmas of this section, and gives a method guaranteed to produce a complex with the maximal number of facets subject to the restrictions on the number of vertices and edges of various color sets.

Theorem 3.17 *Given positive integers $f_1(\Delta)$, $f_2(\Delta)$, $f_3(\Delta)$, $f_{12}(\Delta)$, $f_{13}(\Delta)$, and $f_{23}(\Delta)$, the following procedure will suffice to compute $m(\Delta)$.*

1. *Check whether the inequalities $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta)$, $f_1(\Delta)f_3(\Delta) \geqslant f_{13}(\Delta)$, and $f_2(\Delta)f_3(\Delta) \geqslant f_{23}(\Delta)$ all hold. If not, then there is no Δ with the desired flag f-numbers, so stop.*

2. *Check the inequalities of Lemma 3.2. If any of them hold, then the lemma gives $m(\Delta)$, so stop.*

3. *Relabel the colors if necessary to ensure that $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$.*

4. *Compute $b_1(\Delta)$, $b_2(\Delta)$, and $b_3(\Delta)$. If $b_1(\Delta) = 0$, then Lemma 3.6 gives $m(\Delta)$, so stop.*

5. *Attempt to construct complexes such that $g_{r(\Gamma)}(\Gamma) = b_{r(\Gamma)}(\Delta)$ for each of $r(\Gamma) = 1$ and $r(\Gamma) = 2$ as described in Proposition 3.7. Compute $f_{123}(\Gamma)$ for each such complex that is well-defined.*

6. *Attempt to construct complexes such that $g_3(\Gamma) = b_3(\Delta)$ and $r(\Gamma) = 2$ as explained in Lemma 3.10. Compute $f_{123}(\Gamma)$ for each such complex that is well-defined.*

7. *Repeat the previous step using $r(\Gamma) = 1$.*

8. *Repeat the previous step using* $g_2(\Gamma) = b_2(\Delta)$ *(and* $r(\Gamma) = 1$*).*

9. *Attempt to construct a complex* Γ *with* $r(\Gamma) = 3$ *and* $g_3(\Gamma) = f_3(\Delta)$ *as explained in Proposition 3.7. Compute* $f_{123}(\Gamma)$ *if the complex is well-defined.*

10. *Use Lemma 3.13 to compute the maximum and minimum possible values of* $g_2(\Gamma)$ *if* $g_1(\Gamma) = b_1(\Delta)$ *and* $r(\Gamma) \neq 1$*.*

11. *If it is possible to have* $g_2(\Gamma) = b_2(\Delta)$*, then construct such a complex as explained in Lemma 3.16. Decrease* $g_2(\Gamma)$ *by 1 and construct the complexes again repeatedly until either it is not possible to construct complexes or Lemma 3.15 says that decreasing* $g_2(\Gamma)$ *further will necessarily give no more facets than an already known complex. Likewise, try* $g_2(\Gamma) = b_2(\Delta) + 1$ *and increase* $g_2(\Gamma)$ *by 1 and construct complexes repeatedly until they are not defined or the lemma says that increasing* $g_2(\Gamma)$ *further will necessarily give no more facets than an already known complex.*

12. *If Lemma 3.13 gives a lower bound on* $g_2(\Gamma)$ *that is greater than* $b_2(\Delta)$*, then try setting* $g_2(\Gamma)$ *to this lower bound and construct a complex as explained in Lemma 3.16. Increase* $g_2(\Gamma)$ *by 1 and construct complexes again repeatedly until we stop as in the previous step.*

13. *If Lemma 3.13 gives a upper bound on* $g_2(\Gamma)$ *that is less than* $b_2(\Delta)$*, then try setting* $g_2(\Gamma)$ *to this upper bound and construct a complex as explained in Lemma 3.16. Decrease* $g_2(\Gamma)$ *by 1 and construct complexes again repeatedly until we stop as in the previous step.*

14. *Compare the values of* $f_{123}(\Gamma)$ *for the various complexes constructed. The largest such value is* $m(\Delta)$*.*

Furthermore, this process requires computing the number of facets of fewer than $15 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ *complexes.*

Proof: If the inequalities in point (1) hold, then we can easily construct Δ by picking arbitrary subsets of the appropriate sizes of the possible edges of each color set. In this case, it is clear from the definitions that $\mathcal{C}(\Delta) \neq \emptyset$. By Lemma 3.3, either $\mathcal{D}(\Delta) \neq \emptyset$ or else Lemma 3.2 completes the problem in step (2). In the former case, Lemma 3.9 gives that $\mathcal{E}(\Delta) \neq \emptyset$.

There are three ways to pick a value of i such that $g_i(\Gamma) = b_i(\Delta)$ and three ways to pick a value of $r(\Gamma)$, for nine possibilities in all. Part five handles two of these nine cases, and parts six through eight each handle one. If $b_1(\Delta) = 0$, then Lemma 3.6 solves the problem. Otherwise, parts (10) through (13) handle a sixth case.

If $\mathcal{D}(\Delta) = \mathcal{E}(\Delta)$, then Lemma 3.12 says that either step (9) finds a complex in $\mathcal{E}(\Delta)$ or else one of the other six cases has such a complex. If $\mathcal{E}(\Delta)$ is a proper subset of $\mathcal{D}(\Delta)$, then Lemma 3.9 ensures that one of the other six cases produces a complex in $\mathcal{E}(\Delta)$. Therefore, we are guaranteed to find a complex in $\mathcal{E}(\Delta)$ by this procedure if there is one.

The following table summarizes the nine cases and says which lemmas give the upper bounds on how many complexes it could be necessary to construct for that particular case.

$g_1(\Gamma) = b_1(\Delta)$	$r(\Gamma) = 1$	Proposition 3.7	1
$g_1(\Gamma) = b_1(\Delta)$	$r(\Gamma) = 2$	Lemma 3.16	$< 6 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$
$g_2(\Gamma) = b_2(\Delta)$	$r(\Gamma) = 1$	Lemma 3.10	2
$g_2(\Gamma) = b_2(\Delta)$	$r(\Gamma) = 2$	Proposition 3.7	1
$g_3(\Gamma) = b_3(\Delta)$	$r(\Gamma) = 1$	Lemma 3.10	2
$g_3(\Gamma) = b_3(\Delta)$	$r(\Gamma) = 2$	Lemma 3.10	2
	$r(\Gamma) = 3$	Lemma 3.12	1

Add up all of the cases to get fewer than $15 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ complexes to check in total. $\square$

The bound on how many complexes we have to check is a worst-case scenario, and the number we must actually construct by this procedure can be much smaller than the given bound. We give some examples of this in Section 5. Even so, this bound is good enough to guarantee that a computer program to determine whether or not there is a 3-colored complex with a specified flag f-vector can give an answer almost instantly unless the number of edges is very large. In contrast, a naive brute force search of trying all possible ways to arrange the vertices and edges is completely impractical for merely hundreds of edges.

It is also worthwhile to note that the quantity $\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ is rarely large. For a large integer k, if one picks $f_{12}(\Delta)$, $f_{13}(\Delta)$, and $f_{23}(\Delta)$ uniformly at random from $[k]$ and then sorts them to make $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$, an easy triple integral approximation finds that the expected value of $\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ is essentially $\frac{8}{9}$. Therefore, the expected number of complexes that one must check by the method of Theorem 3.17 is less than 18.

We can actually do better than that. If we use the line $\sqrt{24 + 8\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)^2}}$ from the proof of Lemma 3.16, this has an average value of $4\sqrt{2} < 6$. If we use this rather than 9 as the approximation for the average upper bound on the number of complexes to check in steps (10) through (13), then on average, we have to check fewer than 15 complexes.

Furthermore, even in the most pathological cases, the number of edges must be very large to make the use of Theorem 3.17 impractical. The algorithm is simple to apply if $b_1(\Delta) = 0$, as we stop at step (4). Otherwise, we must have $b_1(\Delta) \geqslant 1$, which means that $f_{12}(\Delta)f_{13}(\Delta) \geqslant f_{23}(\Delta)$. Because we have sorted to get $f_{23}(\Delta) \geqslant f_{13}(\Delta) \geqslant f_{12}(\Delta)$, we get $f_{13}(\Delta)^2 \geqslant f_{12}(\Delta)f_{13}(\Delta) \geqslant f_{23}(\Delta)$, from which

$$\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)} \leqslant \frac{\sqrt{f_{13}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)} = \sqrt{\frac{f_{23}(\Delta)}{f_{13}(\Delta)}} \leqslant \sqrt{\frac{f_{23}(\Delta)}{\sqrt{f_{23}(\Delta)}}} = \sqrt[4]{f_{23}(\Delta)}.$$

As $f_{23}(\Delta)$ is at least one third of the total number of edges, this means that at worst, the number of complexes to check is on the order of the fourth root of the number of edges.

4 Explicit constructions

Theorem 1.1 says that four problems are equivalent. If there is no three-colorable complex of the relevant class corresponding to the proposed flag f-vector or flag h-vector, then Theorem 3.17 will tell us this. If such a complex does exist, then this section explains how to find an explicit complex.

If we have a proposed flag f-vector $f(\Delta)$, then Theorem 3.17 usually explains how to construct a 3-colored complex Γ with $f_{123}(\Gamma) \geqslant f_{123}(\Delta)$ and $f_S(\Gamma) \leqslant f_S(\Delta)$ for all proper subsets $S \subset [3]$. The exceptions are if step (2) or (4) gives $m(\Delta)$.

If step (2) solves the problem, then suppose that condition (1) of Lemma 3.2 is the one that gives $m(\Delta)$. We define a complex Γ by $g_1(\Gamma) = f_1(\Delta)$, $g_2(\Gamma) = \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor$, $g_3(\Gamma) = \left\lfloor \frac{f_{23}(\Delta)}{f_2(\Delta)} \right\rfloor$, $p(\Gamma) = 3$, and $q(\Gamma) = 2$. That this complex is well-defined and has the desired flag f-numbers is shown in the proof of Lemma 3.2. If a different condition of the lemma gives $m(\Delta)$, we can permute the colors of this construction to get the corresponding Γ.

If step (4) solves the problem, then define Γ by $g_1(\Gamma) = 1$, $g_2(\Gamma) = f_{12}(\Delta)$, $g_3(\Gamma) = f_{13}(\Delta)$. The colors of the extra vertices do not matter, as they cannot add any additional facets. The proof of Lemma 3.6 shows that this complex is well-defined and has the desired flag f-numbers.

In order to construct Σ with $f(\Sigma) = f(\Delta)$, we start with Γ and then delete some facets and add some vertices and edges arbitrarily, in order to have exactly the right flag f-vector. This is easy to do, and gives us an explicit construction for statement (1) of Theorem 1.1.

Next, we observe that a complex obtained from Definition 2.1 is color-shifted by definition. Since all constructions in Theorem 3.17 come from this construction, Γ is color-shifted. If we want Σ to be color-shifted, then rather than deleting facets and adding edges arbitrarily, we must choose them in a manner to make Σ color-shifted. This is easy to do. Such a Σ is an explicit construction for statement (4) of Theorem 1.1.

In order to construct a shellable complex with a desired flag h-vector, we start by constructing a complex Σ with the desired flag f-vector. We then construct Θ from Σ by adding one vertex of each color. A set of vertices in Θ forms a face in Θ exactly if they are of distinct colors and the restriction of the set to the vertices in Σ forms a face in Σ. Björner, Frankl, and Stanley [1, Section 5] showed that $h(\Theta) = f(\Sigma)$ and that Θ is shellable. They proved this in a more general context, but this is their construction in the case dealt with in this paper. This gives an explicit construction for statement (3) of Theorem 1.1.

Recall that all shellable complexes are Cohen-Macaulay. Therefore, the construction that works for statement (3) of the theorem also works for statement (2).

5 Some proofs

In this section, we prove the lemmas that we stated without proof in the previous section. We also add some additional lemmas that are useful to prove the lemmas that were stated in Section 3.

Proof of Lemma 3.3: Let $\Gamma \in \mathcal{C}(\Delta)$.

Case I: All three of the equalities in condition (4) of the lemma fail.

This means we can add an extra edge of each color set and still have $f_S(\Gamma) \leqslant f_S(\Delta)$ for all $S \subset [3]$.

Case I A: Some pair of vertices of distinct colors is not adjacent.

We can add an edge to connect this pair of vertices, and make both vertices adjacent to some vertex of the third color by adding an edge if necessary. This adds another facet, so by Proposition 2.4, $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$, a contradiction.

Case I B: Every pair of vertices of distinct colors is adjacent.

Because $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta) > f_{12}(\Gamma) = f_1(\Gamma)f_2(\Gamma)$, either $f_1(\Delta) > f_1(\Gamma)$ or $f_2(\Delta) > f_2(\Gamma)$. Assume without loss of generality that $f_1(\Delta) > f_1(\Gamma)$. Add another vertex of color 1 and make it adjacent to a vertex of each other color to add a facet. Hence, Proposition 2.4 gives $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$, a contradiction.

Case II: Exactly two of the equalities in condition (4) of the lemma fail.

Assume without loss of generality that $f_{12}(\Gamma) < f_{12}(\Delta)$ and $f_{13}(\Gamma) < f_{13}(\Delta)$.

Case II A: $f_1(\Gamma) < f_1(\Delta)$

We can construct a new complex Γ_1 from Γ by adding another vertex of color 1 to Γ and making it adjacent to at least one vertex of each of the other colors. This increases the number of facets, so $f_{123}(\Gamma_1) > f_{123}(\Gamma)$. By Proposition 2.4, $m(\Delta) \geqslant f_{123}(\Gamma_1) > f_{123}(\Gamma)$, so $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$, a contradiction.

Case II B: $f_1(\Gamma) = f_1(\Delta)$

An edge of color 23 can be contained in at most $f_1(\Delta)$ facets of Γ, as a facet is uniquely determined by the choice of an edge of color 23 and a vertex of color 1.

Case II B 1: Every edge of color 23 is contained in exactly $f_1(\Delta)$ facets.

That only two of the equalities of condition (4) fail means that $f_{23}(\Gamma) = f_{23}(\Delta)$. Since $f_{123}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$, option (1) in the lemma holds.

Case II B 2: There is some edge of color 23 contained in fewer than $f_1(\Delta)$ facets of Γ.

Let the edge in question be $\{v_i^2, v_j^3\}$. Since Γ is color-shifted, these two vertices together with $v^1_{f_1(\Delta)}$ do not form a facet of Γ. Add edges as necessary to make v_i^2 and v_j^3 adjacent to $v^1_{f_1(\Delta)}$. This adds an extra facet, and we had a spare edge available of both of the relevant color sets. Thus, by Proposition 2.4, $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$, a contradiction.

Case III: Exactly one of the equalities in condition (4) of the lemma fails.

Assume without loss of generality that $f_{12}(\Gamma) < f_{12}(\Delta)$.

Case III A: $q(\Gamma) = 3$

Assume without loss of generality that $p(\Gamma) = 1$.

Case III A 1: $g_1(\Gamma) = f_1(\Delta)$

This means that applying the $p(\Gamma) = 1$ step doesn't add any faces, as a vertex can't be added. Since $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta) > f_{12}(\Gamma) = f_1(\Gamma)f_2(\Gamma)$, we have $f_2(\Gamma) < f_2(\Delta)$. Define Γ_1 by $g_i(\Gamma_1) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 2$. After adding the first extra vertex, we have Γ exactly. The second extra vertex uses at least one additional edge of color 12, so $f_{12}(\Gamma) + f_{13}(\Gamma) + f_{23}(\Gamma) < f_{12}(\Gamma_1) + f_{13}(\Gamma_1) + f_{23}(\Gamma_1) \leqslant n(\Delta)$. Therefore, $\Gamma \notin \mathcal{C}(\Delta)$, a contradiction.

Case III A 2: $g_1(\Gamma) < f_1(\Delta)$

Case III A 2 a: $f_{13}(\Gamma) \geqslant (g_1(\Gamma) + 1)g_3(\Gamma)$

The extra vertex of color 1 is adjacent to all $g_2(\Gamma)$ vertices of color 2 (since we must have leftover edges of color 12) and at least $g_3(\Gamma)$ vertices of color 3, as the construction requires making $v^1_{g_1(\Gamma)+1}$ adjacent to as many other vertices as the restrictions on edges allow.

Case III A 2 a i: $g_1(\Gamma) + 1 < f_1(\Delta)$

We can create a new complex Γ_1 as in the construction of Lemma 2.8 using $g_1(\Gamma_1) = g_1(\Gamma)+1$, $g_2(\Gamma_1) = g_2(\Gamma)$, $g_3(\Gamma_1) = g_3(\Gamma)$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 1$. After adding the first extra vertex, we have the complex Γ exactly. Adding the second extra vertex uses at least one additional edge of color 12, while $f_{123}(\Gamma_1) = f_{123}(\Gamma)$. Thus, $f_{12}(\Gamma)+f_{13}(\Gamma)+f_{23}(\Gamma) < f_{12}(\Gamma_1) + f_{13}(\Gamma_1) + f_{23}(\Gamma_1) \leqslant n(\Delta)$, so $\Gamma \notin \mathcal{C}(\Delta)$, a contradiction.

Case III A 2 a ii: $g_1(\Gamma) + 1 = f_1(\Delta)$

We can create a new complex Γ_1 as in the construction of Lemma 2.8 using $g_1(\Gamma_1) = g_1(\Gamma) + 1$, $g_2(\Gamma_1) = g_2(\Gamma)$, $g_3(\Gamma_1) = g_3(\Gamma)$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 2$. After adding the first extra vertex, we have the complex Γ exactly. Since $f_1(\Delta)f_2(\Delta) \geqslant f_{12}(\Delta) > f_{12}(\Gamma) = f_1(\Gamma)f_2(\Gamma)$ and $f_1(\Gamma) = g_1(\Gamma) + 1 = f_1(\Delta)$, we have $f_2(\Gamma) < f_2(\Delta)$. Adding the second extra vertex uses at least one additional edge of color 12, while $f_{123}(\Gamma_1) = f_{123}(\Gamma)$. Thus, $f_{12}(\Gamma)+f_{13}(\Gamma)+f_{23}(\Gamma) < f_{12}(\Gamma_1)+f_{13}(\Gamma_1)+f_{23}(\Gamma_1) \leqslant n(\Delta)$, so $\Gamma \notin \mathcal{C}(\Delta)$, a contradiction.

Case III A 2 b: $f_{13}(\Gamma) < (g_1(\Gamma) + 1)g_3(\Gamma)$

Adding the vertex $v^1_{g_1(\Gamma)+1}$ because $p(\Gamma) = 1$ uses up all of the remaining edges of color 13. Hence, $v^3_{g_3(\Gamma)+1}$, which is added because $q(\Gamma) = 3$, cannot be adjacent to any vertices of color 1, and therefore is not in any facets.

Case III A 2 b i: $f_2(\Delta) = g_2(\Gamma)$

All edges of color 13 are of the form $\{v^1_i, v^3_j\}$ with $i \leqslant g_1(\Gamma) + 1$ and $j \leqslant g_3(\Gamma)$. Every vertex of color 2 is adjacent to all of the vertices in $\{v^1_1, \ldots, v^1_{g_1(\Gamma)+1}, v^3_1, \ldots, v^3_{g_3(\Gamma)}\}$. Thus, every choice of an edge of color 13 and a vertex of color 2 forms a facet. Therefore, $f_{123}(\Gamma) = f_2(\Gamma)f_{13}(\Gamma) = f_2(\Delta)f_{13}(\Delta)$, which is option (2) in the lemma.

Case III A 2 b ii: $f_2(\Delta) > g_2(\Gamma)$

Define a complex Γ_1 by $g_i(\Gamma_1) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_1) = 1$, and $q(\Gamma_1) = 2$.

Case III A 2 b ii (1): $f_{23}(\Delta) > g_2(\Gamma)g_3(\Gamma)$

The difference in facets between Γ and Γ_1 is the number added by the second extra vertex. This vertex is not contained in any facets of Γ. The new vertex $v^2_{g_2(\Gamma)+1}$ is adjacent in Γ_1 to vertices of both color 1 and color 3, and is thus contained in at least one facet. Therefore, by Proposition 2.4, $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$.

Case III A 2 b ii (2): $f_{23}(\Delta) = g_2(\Gamma)g_3(\Gamma)$

All edges of color 23 were in Γ before adding either extra vertex. All edges of color 13 were in Γ after adding the first extra vertex (for $p(\Gamma) = 1$) and before adding the second. As such, adding the extra vertex of color 3 in Γ does not use any more edges, nor does it add any facets. The second extra vertex of Γ_1 does at least use some additional edges of color 12. As such, $f_{123}(\Gamma) = f_{123}(\Gamma_1)$ and $f_{12}(\Gamma)+f_{13}(\Gamma)+f_{23}(\Gamma) < f_{12}(\Gamma_1)+f_{13}(\Gamma_1)+f_{23}(\Gamma_1) \leqslant n(\Delta)$, so $\Gamma \notin \mathcal{C}(\Delta)$.

Case III B: $p(\Gamma) = 3$

Assume without loss of generality that $q(\Gamma) = 1$.

Case III B 1: $g_1(\Gamma) = f_1(\Delta)$

Making the second extra vertex of color 2 rather than color 1 increases the number of edges used by the same argument as in Case III A 1, so $\Gamma \notin \mathcal{C}(\Delta)$.

Case III B 2: $g_1(\Gamma) < f_1(\Delta)$

Define Γ_1 by $g_i(\Gamma_1) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_1) = 1$, and $q(\Gamma_1) = 3$. The only edges that can differ between Γ and Γ_1 are that Γ has some extra edges containing $v^3_{g_3(\Gamma)+1}$ but not $v^1_{g_1(\Gamma)+1}$, while Γ_1 has the same number of extra edges containing $v^1_{g_1(\Gamma)+1}$ but not $v^3_{g_3(\Gamma)+1}$. The only vertex that may not be adjacent to all $g_2(\Gamma)$ vertices of color 2 is $v^3_{g_3(\Gamma)+1}$. Thus, any edge in Γ_1 but not Γ is in at least as many facets as each edge in Γ but not Γ_1. Therefore, $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$. Since the rest of their flag f-vectors are the same, if $\Gamma \in \mathcal{C}(\Delta)$, then $\Gamma_1 \in \mathcal{C}(\Delta)$. We now apply Case III A to Γ_1.

Case III C: $r(\Gamma) = 3$

Because $f_{12}(\Delta) > f_{12}(\Gamma)$, all vertices of color 1 are adjacent to all vertices of color 2, including the extra vertex of each color. Thus, Γ is exactly the same complex regardless of whether $p(\Gamma) = 1$ and $q(\Gamma) = 2$ or vice versa.

Note that using all edges of colors 13 and 23 means that $f_{13}(\Gamma) \leqslant (g_1(\Gamma) + 1)g_3(\Gamma)$ and $f_{23}(\Gamma) \leqslant (g_2(\Gamma) + 1)g_3(\Gamma)$.

Case III C 1: $f_1(\Delta) = g_1(\Gamma)$ or $f_2(\Delta) = g_2(\Gamma)$

Assume without loss of generality that $f_1(\Delta) = g_1(\Gamma)$. Since $f_{13}(\Delta) = f_{13}(\Gamma) \geqslant g_1(\Gamma)g_3(\Gamma)$ and there are leftover edges of color 12, every vertex of color 1 is adjacent to every vertex of color 2 or 3. Therefore, $f_{123}(\Gamma) = f_1(\Gamma)f_{23}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$, which is option (1) of the lemma.

Case III C 2: $f_1(\Delta) > g_1(\Gamma)$ and $f_2(\Delta) > g_2(\Gamma)$

Case III C 2 a: $f_{13}(\Gamma) = (g_1(\Gamma) + 1)g_3(\Gamma)$ or $f_{23}(\Gamma) = (g_2(\Gamma) + 1)g_3(\Gamma)$

Assume without loss of generality that $f_{13}(\Gamma) = (g_1(\Gamma) + 1)g_3(\Gamma)$. As noted above, we can assume that $p(\Gamma) = 1$.

Case III C 2 a i: $f_1(\Gamma) = f_1(\Delta)$

Every vertex of color 1 is adjacent to every vertex of color 2 or 3, so it forms a facet together with every edge of color 23. Therefore, $f_{123}(\Gamma) = f_1(\Gamma)f_{23}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$, which is option (1) of the lemma.

Case III C 2 a ii: $f_1(\Gamma) < f_1(\Delta)$

Define Γ_1 by $g_1(\Gamma_1) = g_1(\Gamma) + 1$, $g_2(\Gamma_1) = g_2(\Gamma)$, $g_3(\Gamma_1) = g_3(\Gamma)$, $p(\Gamma_1) = 2$, and $q(\Gamma_1) = 1$. In this case, Γ_1 is obtained from Γ by adding an extra vertex of color 1 and some edges of color 12. Thus, Γ_1 has all of the facets of Γ and more edges, so $\Gamma \notin \mathcal{C}(\Delta)$.

Case III C 2 b: $f_{13}(\Gamma) < (g_1(\Gamma) + 1)g_3(\Gamma)$, $f_{23}(\Gamma) < (g_2(\Gamma) + 1)g_3(\Gamma)$, and either $f_{13}(\Gamma) \geqslant (g_1(\Gamma) + 1)(g_3(\Gamma) - 1)$ or $f_{23}(\Gamma) \geqslant (g_2(\Gamma) + 1)(g_3(\Gamma) - 1)$

Assume without loss of generality that $f_{13}(\Gamma) \geqslant (g_1(\Gamma)+1)(g_3(\Gamma)-1)$. Again, we can assume that $p(\Gamma) = 1$. Define Γ_1 by $g_1(\Gamma_1) = g_1(\Gamma)+1$, $g_2(\Gamma_1) = g_2(\Gamma)$, $g_3(\Gamma_1) = g_3(\Gamma)-1$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 2$. It is easy to check that Γ_1 has exactly the same edges of colors 12 and 23 as Γ. The only possible difference in edges is that Γ_1 may have some extra edges containing $v^1_{g_1(\Gamma)+1}$ but not $v^3_{g_3(\Gamma)}$, while Γ may have some extra edges containing $v^3_{g_3(\Gamma)}$ but not $v^1_{g_1(\Gamma)+1}$. An edge of the former type is contained in at least $g_2(\Gamma)$ facets, as the first $g_2(\Gamma)$ vertices of color 2 are adjacent to all vertices of other colors. An edge of the latter type is contained in at most $g_2(\Gamma)$ facets because $v^3_{g_3(\Gamma)}$ is adjacent to only $g_2(\Gamma)$ vertices of color 2, as $f_{23}(\Gamma) < (g_2(\Gamma)+1)g_3(\Gamma)$. Therefore $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$. Since Γ_1 has the same number of edges as Γ, we get $\Gamma_1 \in \mathcal{C}(\Delta)$. We now apply Case III B to Γ_1.

Case III C 2 c: $f_{13}(\Gamma) < (g_1(\Gamma)+1)(g_3(\Gamma)-1)$ and $f_{23}(\Gamma) < (g_2(\Gamma)+1)(g_3(\Gamma)-1)$

Define Γ_1 by $g_1(\Gamma_1) = g_1(\Gamma)$, $g_2(\Gamma_1) = g_2(\Gamma)$, $g_3(\Gamma_1) = g_3(\Gamma) - 1$, $p(\Gamma_1) = 1$, and $q(\Gamma_1) = 2$. The facets of Γ not in Γ_1 are those containing $v^3_{g_3(\Gamma)}$. There are $g_1(\Gamma)g_2(\Gamma)$ such facets. The facets of Γ_1 not in Γ are those containing the edges of Γ_1 but not Γ. Both of the vertices of each such edge of color 13 are adjacent to at least the first $g_2(\Gamma)$ vertices of color 2. Each of the $g_1(\Gamma)$ such edges adds at least $g_2(\Gamma)$ facets. Likewise, the vertices of a new edge of color 23 are adjacent to at least the first $g_1(\Gamma)$ vertices of color 1, so the edge is contained in at least $g_1(\Gamma)$ facets. Thus, Γ_1 contains at least $2g_1(\Gamma)g_2(\Gamma)$ facets that Γ_2 does not. Therefore, $f_{123}(\Gamma_1) > f_{123}(\Gamma)$, so $\Gamma \notin \mathcal{B}(\Delta) \supseteq \mathcal{C}(\Delta)$.

Case IV: All of the equalities of condition (4) of the lemma hold.

This makes condition (4) of the lemma true. □

Proof of Lemma 3.2: It suffices to prove one of the statements, as the others follow by relabeling the colors. Suppose first that $\left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant f_{23}(\Delta)$. Let Γ_1 be defined by $g_1(\Gamma_1) = f_1(\Delta)$, $g_2(\Gamma_1) = \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor$, $g_3(\Gamma_1) = \left\lceil \frac{f_{23}(\Delta)}{g_2(\Gamma_1)} \right\rceil$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 2$. It follows that $g_3(\Gamma_1) > \frac{f_{23}(\Delta)}{g_2(\Gamma_1)} - 1$, from which $(g_3(\Gamma_1)+1)g_2(\Gamma_1) > f_{23}(\Delta)$. Thus, all edges of color 23 have their vertices in the set $\{v^2_1, \ldots, v^2_{g_2(\Gamma_1)}, v^3_1, \ldots, v^3_{g_3(\Gamma_1)+1}\}$.

Since $\left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant f_{23}(\Delta)$, we get $g_2(\Gamma_1) \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant f_{23}(\Delta)$, and so $\left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant \frac{f_{23}(\Delta)}{g_2(\Gamma_1)}$. If the right side is not an integer, then $\left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor \geqslant \left\lfloor \frac{f_{23}(\Delta)}{g_2(\Gamma_1)} \right\rfloor + 1 = g_3(\Gamma_1) + 1$, so all vertices contained in edges of color 23 are adjacent to all vertices of color 1. If $\frac{f_{23}(\Delta)}{g_2(\Gamma_1)}$ is an integer, then $v^3_{g_3(\Gamma_1)+1}$ is not contained in any edges of color 23, and again, all vertices contained in edges of color 23 are adjacent to all vertices of color 1. Either way, every edge of color 23 forms a facet together with each vertex of color 1, so $f_{123}(\Gamma_1) = f_1(\Gamma_1)f_{23}(\Gamma_1) = f_1(\Delta)f_{23}(\Delta)$. By Proposition 2.4, for any $\Gamma \in \mathcal{C}(\Delta)$, $f_{123}(\Gamma) \geqslant f_{123}(\Gamma_1) = f_1(\Delta)f_{23}(\Delta)$. Since we also have $f_{123}(\Gamma) \leqslant f_1(\Gamma)f_{23}(\Gamma) \leqslant f_1(\Delta)f_{23}(\Delta)$, the result follows.

Conversely, suppose that $f_{123}(\Gamma) = f_1(\Delta)f_{23}(\Delta)$. We must have $f_{23}(\Gamma) = f_{23}(\Delta)$, and every edge of Γ of color 23 must form a facet with each of the $f_1(\Delta)$ vertices of color 1. Thus, every vertex of an edge of color 23 must be adjacent to every vertex of color 1. If there are d_2 such vertices of color 2 and d_3 such vertices of color 3, then the number of required edges is $f_1(\Delta)d_2$ of color 12 and $f_1(\Delta)d_3$ of color 13. Since we are only allowed so many edges of each color set, we have $f_1(\Delta)d_2 \leqslant f_{12}(\Delta)$ and $f_1(\Delta)d_3 \leqslant f_{13}(\Delta)$.

These yield $d_2 \leqslant \frac{f_{12}(\Delta)}{f_1(\Delta)}$ and $d_3 \leqslant \frac{f_{12}(\Delta)}{f_1(\Delta)}$, respectively. Since d_2 and d_3 must be integers, the inequality still holds if we take the integer parts of the right sides. This yields $d_2 \leqslant \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor$ and $d_3 \leqslant \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor$, respectively. The number of edges of color 23 on d_2 vertices of color 2 and d_3 vertices of color 3 is at most $d_2 d_3$, so we have $f_{23}(\Delta) \leqslant d_2 d_3 \leqslant \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor \left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor$, as desired. $\square$

This next lemma tells us when swapping $p(\Gamma)$ with $q(\Gamma)$ is beneficial. Intuitively, this means switching the order in which the two extra vertices are added.

Lemma 5.1 *Let Δ be a 3-colored simplicial complex and let $\Gamma \in \mathcal{A}(\Delta)$ with $j_{p(\Gamma)}(\Gamma) \geqslant j_{q(\Gamma)}(\Gamma)$. Define Γ_1 by $g_i(\Gamma_1) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_1) = q(\Gamma)$, and $q(\Gamma_1) = p(\Gamma)$. Then $\Gamma_1 \in \mathcal{A}(\Delta)$ and $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$. Furthermore, if $\Gamma \in \mathcal{D}(\Delta)$, then $\Gamma_1 \in \mathcal{D}(\Delta)$.*

Proof: All of the edges that $g_1(\Gamma_1)$, $g_2(\Gamma_1)$, and $g_3(\Gamma_1)$ require Γ_1 to have are in Γ, so Δ has enough edges available for Γ_1 to be well-defined. That $\Gamma_1 \in \mathcal{A}(\Delta)$ is immediate from the construction. If $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$, then $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma) = m(\Delta)$, so $\Gamma_1 \in \mathcal{B}(\Delta)$. Because Γ_1 has the same number of vertices of each color as Γ, it uses just as many edges of each color set as Γ. If $\Gamma \in \mathcal{D}(\Delta)$, then $\Gamma \in \mathcal{C}(\Delta)$, so we have $\Gamma_1 \in \mathcal{C}(\Delta)$. If $\Gamma \in \mathcal{D}(\Delta)$, then $\mathcal{D}(\Delta) \neq \emptyset$, and so $\mathcal{D}(\Delta) = \mathcal{C}(\Delta)$. Thus, $\Gamma_1 \in \mathcal{D}(\Delta)$. Therefore, it suffices to show that $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$.

It is immediate from the construction that both complexes have exactly the same edges of color sets $\{p(\Gamma), r(\Gamma)\}$ and $\{q(\Gamma), r(\Gamma)\}$. All that can differ is the edges of color set $\{p(\Gamma), q(\Gamma)\}$. Among these, all that can differ is that Γ may have some extra edges containing $v^{p(\Gamma)}_{g_{p(\Gamma)}+1}$ but not $v^{q(\Gamma)}_{g_{q(\Gamma)}+1}$ while Γ_1 may have some extra edges containing $v^{q(\Gamma)}_{g_{q(\Gamma)}+1}$ but not $v^{p(\Gamma)}_{g_{p(\Gamma)}+1}$. Any vertex of color $p(\Gamma)$ or $q(\Gamma)$ other than the two extra vertices is adjacent to exactly $g_{r(\Gamma)}(\Gamma)$ vertices of color $r(\Gamma)$ in both Γ and Γ_1. The vertex $v^{p(\Gamma)}_{g_{p(\Gamma)}+1}$ is adjacent to $j_{q(\Gamma)}(\Gamma) \leqslant g_{r(\Gamma)}(\Gamma)$ vertices of color $r(\Gamma)$. The vertex $v^{q(\Gamma)}_{g_{q(\Gamma)}+1}$ is adjacent to $j_{p(\Gamma)}(\Gamma) \leqslant g_{r(\Gamma)}(\Gamma)$ vertices of color $r(\Gamma)$. Hence, an extra edge of Γ is contained in $j_{q(\Gamma)}(\Gamma)$ facets, while an extra edge of Γ_1 is contained in $j_{p(\Gamma)}(\Gamma)$ facets. Since $j_{p(\Gamma)}(\Gamma) \geqslant j_{q(\Gamma)}(\Gamma)$, we have $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma)$. $\square$

Proof of Proposition 3.7: We have that $g_3(\Gamma_1) = g_3(\Gamma)$. If $g_1(\Gamma_1) = g_1(\Gamma)$ and $g_2(\Gamma_1) = g_2(\Gamma)$, then Lemma 5.1 promises that $\Gamma_1 \in \mathcal{D}(\Delta)$. Thus, for the lemma to be false, we must have either $g_1(\Gamma_1) \neq g_1(\Gamma)$ or $g_2(\Gamma_1) \neq g_2(\Gamma)$.

Since Γ does not have an extra vertex of color 3, we must have $f_1(\Gamma) \geqslant \frac{f_{13}(\Gamma)}{f_3(\Gamma)} = \frac{f_{13}(\Delta)}{g_3(\Gamma)}$ and $f_2(\Gamma) \geqslant \frac{f_{23}(\Gamma)}{f_3(\Gamma)} = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$. One can check the various cases in the definition to easily see that $g_1(\Gamma_1) \leqslant f_1(\Gamma) \leqslant f_1(\Delta)$ and $g_2(\Gamma_1) \leqslant f_2(\Gamma) \leqslant f_2(\Delta)$. We also have that $g_3(\Gamma_1) = g_3(\Gamma) \leqslant f_3(\Delta)$, so there are enough vertices for Γ_1 to be well-defined.

Because Γ uses at least $\frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1$ vertices of color 1 and at least $\frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1$ vertices of color 2 before adding extra vertices, there are enough edges of color 12 to do this. From the definition, Γ_1 does not require more vertices than this of either color unless there are enough edges. Thus, Γ_1 is well-defined. It is immediate from the definition that $\Gamma_1 \in \mathcal{A}(\Delta)$.

As the only extra vertex of color 13 that can contain edges is of color 1, we have $j_2(\Gamma) \leqslant g_3(\Gamma)$. Since $\Gamma \in \mathcal{D}(\Delta)$, it must use all edges of this color set, so we have $g_3(\Gamma)g_1(\Gamma) \leqslant f_{13}(\Gamma) \leqslant g_3(\Gamma)g_1(\Gamma) + g_3(\Gamma) = g_3(\Gamma)(g_1(\Gamma)+1)$. Divide by $g_3(\Gamma)$ and we have $g_1(\Gamma) \leqslant \frac{f_{13}(\Delta)}{g_3(\Gamma)} \leqslant g_1(\Gamma)+1$. This can be rearranged as $\frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1 \leqslant g_1(\Gamma) \leqslant \frac{f_{13}(\Delta)}{g_3(\Gamma)}$. If $\frac{f_{13}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$, then this forces $g_1(\Gamma) = \left\lfloor \frac{f_{13}(\Delta)}{g_3(\Gamma)} \right\rfloor = g_1(\Gamma_1)$. Similarly, if $\frac{f_{23}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$, we get $g_2(\Gamma) = g_2(\Gamma_1)$.

Case I: $\frac{f_{13}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$

As seen above, we have $g_1(\Gamma) = g_1(\Gamma_1)$.

Case I A: $\frac{f_{23}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$

This case has $g_1(\Gamma_1) = g_1(\Gamma)$ and $g_2(\Gamma_1) = g_2(\Gamma)$.

Case I B: $\frac{f_{23}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z}$

That Γ is well-defined and uses all edges of color 13 corresponds to the inequalities $g_3(\Gamma)g_2(\Gamma) \leqslant f_{23}(\Gamma) \leqslant g_3(\Gamma)(g_2(\Gamma)+1)$, which force either $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$ or $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1$.

Case I B 1: $f_{12}(\Delta) < \frac{f_{23}(\Delta)}{g_3(\Gamma)} g_1(\Gamma_1)$

There are not enough edges of color 12 to have $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$ or $g_2(\Gamma_1) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$. This means that $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1 = g_2(\Gamma_1)$, and we are done.

Case I B 2: $f_{12}(\Delta) \geqslant \frac{f_{23}(\Delta)}{g_3(\Gamma)} g_1(\Gamma_1)$

The definition of Γ_1 gives $g_2(\Gamma_1) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$. If we also have $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$, then we are done. Otherwise, $g_2(\Gamma) = \frac{f_{23}(\Delta)}{g_3(\Gamma)} - 1$. If this happens, we can compute $j_1(\Gamma) = g_3(\Gamma) > j_2(\Gamma) = j_2(\Gamma_1) > 0 = j_1(\Gamma_1)$ (with the strict inequalities because $\frac{f_{13}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$). This means that $p(\Gamma_1) = 1$. From this, Lemma 5.1 asserts that if we define Γ_2 and by $g_i(\Gamma_2) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_2) = 2$, and $q(\Gamma_2) = 1$, then $f_{123}(\Gamma_2) \geqslant f_{123}(\Gamma)$ and $\Gamma_2 \in \mathcal{D}(\Delta)$.

Note that Γ_1 is merely Γ_2 with possibly an extra isolated vertex added. Thus, $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma_2)$, so $\Gamma_1 \in \mathcal{B}(\Delta)$. Furthermore, since Γ_2 uses all available edges, so does Γ_1, and so $\Gamma_1 \in \mathcal{C}(\Delta)$. Since $\Gamma_2 \in \mathcal{D}(\Delta)$, we get $\mathcal{D}(\Delta) = \mathcal{C}(\Delta)$, and so $\Gamma_1 \in \mathcal{D}(\Delta)$.

Case II: $\frac{f_{13}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z}$

That Γ is well-defined and uses all edges of color 13 corresponds to the inequalities $g_3(\Gamma)g_1(\Gamma) \leqslant f_{13}(\Gamma) \leqslant g_3(\Gamma)(g_1(\Gamma)+1)$, which force either $g_1(\Gamma) = \frac{f_{13}(\Delta)}{g_3(\Gamma)}$ or $g_1(\Gamma) = \frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1$.

Case II A: $\frac{f_{23}(\Delta)}{g_3(\Gamma)} \notin \mathbb{Z}$

As we have seen, this gives $g_2(\Gamma) = g_2(\Gamma_1) = \left\lfloor \frac{f_{23}(\Delta)}{g_3(\Gamma)} \right\rfloor$.

Case II A 1: $f_{12}(\Delta) < \frac{f_{13}(\Delta)}{g_3(\Gamma)} g_2(\Gamma)$

This gives $g_1(\Gamma_1) = \frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1$. Since there are not enough edges for Γ_1 to have $g_1(\Gamma) = \frac{f_{13}(\Delta)}{g_3(\Gamma)}$, we must have $g_1(\Gamma) = \frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1 = g_1(\Gamma_1)$.

Case II A 2: $f_{12}(\Delta) \geqslant \frac{f_{13}(\Delta)}{g_3(\Gamma)} g_2(\Gamma)$

This time, the definition gives $g_1(\Gamma_1) = \frac{f_{13}(\Delta)}{g_3(\Gamma)}$. In order to have $g_1(\Gamma_1) \neq g_1(\Gamma)$, we must have $g_1(\Gamma) = \frac{f_{13}(\Delta)}{g_3(\Gamma)} - 1$. This only swaps colors 1 and 2 from Case I B 2, so $\Gamma_1 \in \mathcal{D}(\Delta)$ by the same argument as there.

Case II B: $\frac{f_{23}(\Delta)}{g_3(\Gamma)} \in \mathbb{Z}$

Case II B 1: $f_{12}(\Delta) \leqslant \frac{f_{13}(\Delta)}{g_3(\Gamma)} \frac{f_{23}(\Delta)}{g_3(\Gamma)}$

From the definition, it is clear that once the extra vertices are added, Γ_1 has at least $\frac{f_{13}(\Delta)}{g_3(\Gamma)}$ vertices of color 1 and $\frac{f_{23}(\Delta)}{g_3(\Gamma)}$ vertices of color 2. Furthermore, the order of the extra vertices dictates that Γ_1 must have at least this many of each color before any more vertices of either color are added. Thus, the $f_{12}(\Delta)$ edges of color 12 in Γ_1 all have both vertices among the first $\frac{f_{13}(\Delta)}{g_3(\Gamma)}$ vertices of color 1 and the first $\frac{f_{23}(\Delta)}{g_3(\Gamma)}$ of color 2. All of these vertices are adjacent to all vertices of color 3, so every edge of color 12 in Γ_1 is contained in $g_3(\Gamma)$ facets. Therefore,

$$f_{123}(\Gamma_1) = g_3(\Gamma) f_{12}(\Delta) = f_3(\Gamma) f_{12}(\Gamma) \geqslant f_{123}(\Gamma).$$

As such, since $\Gamma \in \mathcal{B}(\Delta)$, we have $\Gamma_1 \in \mathcal{B}(\Delta)$. Since Γ_1 has at least $\frac{f_{13}(\Delta)}{g_3(\Gamma)}$ vertices of color 1 and $\frac{f_{23}(\Delta)}{g_3(\Gamma)}$ vertices of color 2, it uses all available edges, and so $\Gamma_1 \in \mathcal{C}(\Delta) = \mathcal{D}(\Delta)$.

Case II B 2: $f_{12}(\Delta) > \frac{f_{13}(\Delta)}{g_3(\Gamma)} \frac{f_{23}(\Delta)}{g_3(\Gamma)}$

We get $g_1(\Gamma_1) = \frac{f_{13}(\Delta)}{g_3(\Gamma)}$ and $g_2(\Gamma_1) = \frac{f_{23}(\Delta)}{g_3(\Gamma)}$. As Γ has only $\frac{f_{13}(\Delta)}{g_3(\Gamma)}$ vertices of color 1 and $\frac{f_{23}(\Delta)}{g_3(\Gamma)}$ vertices of color 2 adjacent to any vertices of color 3, any edge of color 12 contained in any facets must have its vertices among the first $\frac{f_{13}(\Delta)}{g_3(\Gamma)}$ of color 1 and the first $\frac{f_{23}(\Delta)}{g_3(\Gamma)}$ of color 2. There are $\frac{f_{13}(\Delta)}{g_3(\Gamma)} \frac{f_{23}(\Delta)}{g_3(\Gamma)}$ such edges possible, each of which is contained in $g_3(\Gamma)$ facets, so we have $f_{123}(\Gamma) \leqslant \frac{f_{13}(\Delta)}{g_3(\Gamma)} \frac{f_{23}(\Delta)}{g_3(\Gamma)} g_3(\Gamma) = g_1(\Gamma_1) g_2(\Gamma_1) g_3(\Gamma_1) \leqslant f_{123}(\Gamma_1)$. Since Γ_1 has at least as many edges of each color set as Γ and $\Gamma \in \mathcal{D}(\Delta)$, we get $\Gamma_1 \in \mathcal{D}(\Delta)$. ☐

Proof of Lemma 3.9: In order for the first option of the lemma not to hold, there must be some $\Gamma \in \mathcal{D}(\Delta)$ with $\Gamma \notin \mathcal{E}(\Delta)$. We must either have $g_i(\Gamma) > b_i(\Delta)$ for at least two values of $i \in [3]$ or else $g_i(\Gamma) < b_i(\Delta)$ for at least two values of $i \in [3]$. Suppose that it is the former. Assume without loss of generality that $g_1(\Gamma) > b_1(\Delta)$ and $g_2(\Gamma) > b_2(\Delta)$. Since these are all integers, $g_1(\Gamma) \geqslant b_1(\Delta) + 1$ and $g_2(\Gamma) \geqslant b_2(\Delta) + 1$. We can compute

$$\begin{aligned} f_{12}(\Gamma) &\geqslant g_1(\Gamma) g_2(\Gamma) \\ &\geqslant (b_1(\Delta)+1)(b_2(\Delta)+1) \\ &= \left(\left\lfloor \sqrt{\frac{f_{12}(\Delta) f_{13}(\Delta)}{f_{23}(\Delta)}} \right\rfloor + 1 \right) \left(\left\lfloor \sqrt{\frac{f_{12}(\Delta) f_{23}(\Delta)}{f_{13}(\Delta)}} \right\rfloor + 1 \right) \\ &> \sqrt{\frac{f_{12}(\Delta) f_{13}(\Delta)}{f_{23}(\Delta)}} \sqrt{\frac{f_{12}(\Delta) f_{23}(\Delta)}{f_{13}(\Delta)}} \\ &= f_{12}(\Delta) \\ &= f_{12}(\Gamma). \end{aligned}$$

This is obviously impossible.

Otherwise, we must have $g_i(\Gamma) < b_i(\Delta)$ for at least two values of $i \in [3]$. Assume without loss of generality that $g_1(\Gamma) < b_1(\Delta)$ and $g_2(\Gamma) < b_2(\Delta)$. Since these are all integers, $g_1(\Gamma) \leqslant b_1(\Delta) - 1$ and $g_2(\Gamma) \leqslant b_2(\Delta) - 1$. We compute

$$\begin{aligned}
f_{12}(\Gamma) &\leqslant (g_1(\Gamma)+1)(g_2(\Gamma)+1) \\
&\leqslant b_1(\Delta)b_2(\Delta) \\
&= \left\lfloor\sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}\right\rfloor\left\lfloor\sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}}\right\rfloor \\
&\leqslant \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}\sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}} \\
&= f_{12}(\Delta) \\
&= f_{12}(\Gamma).
\end{aligned}$$

Because the opposite ends of the chain of inequalities are equal, equality must hold throughout. For the first inequality to be an equality, we must have $\{p(\Gamma), q(\Gamma)\} = \{1, 2\}$. We can assume without loss of generality that $p(\Gamma) = 1$ and $q(\Gamma) = 2$. The second inequality means that $b_1(\Delta) = g_1(\Gamma) + 1$ and $b_2(\Delta) = g_2(\Gamma) + 1$. The third gives that $\sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}$ and $\sqrt{\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)}}$ are integers, so taking their floors does not change them.

If $g_3(\Gamma) = b_3(\Delta)$, then $\Gamma \in \mathcal{E}(\Delta)$, which contradicts the choice of Γ. If $g_3(\Gamma) < b_3(\Delta)$, then by an argument analogous to the previous paragraph, we get

$$f_{13}(\Gamma) \leqslant (g_1(\Gamma)+1)(g_3(\Gamma)+1) \leqslant f_{13}(\Gamma),$$

from which $\{p(\Gamma), q(\Gamma)\} = \{1, 3\}$. This contradicts $\{p(\Gamma), q(\Gamma)\} = \{1, 2\}$.

The only other possibility is if $g_3(\Gamma) > b_3(\Delta)$. In this case, we compute

$$\begin{aligned}
f_{13}(\Delta) &= \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}\sqrt{\frac{f_{13}(\Delta)f_{23}(\Delta)}{f_{12}(\Delta)}} \\
&< \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}}\left(\left\lfloor\sqrt{\frac{f_{13}(\Delta)f_{23}(\Delta)}{f_{12}(\Delta)}}\right\rfloor + 1\right) \\
&= b_1(\Delta)(b_3(\Delta)+1).
\end{aligned}$$

Likewise, we can compute that $f_{23}(\Delta) < b_2(\Delta)(b_3(\Delta)+1)$.

Suppose that $g_3(\Gamma) \geqslant b_3(\Delta) + 2$. We get that

$$\begin{aligned}
j_2(\Gamma) &= f_{13}(\Delta) - g_1(\Gamma)g_3(\Gamma) \\
&< b_1(\Delta)(b_3(\Delta)+1) - (b_1(\Delta)-1)(b_3(\Delta)+2) \\
&= b_1(\Delta)b_3(\Delta) + b_1(\Delta) - b_1(\Delta)b_3(\Delta) + b_3(\Delta) - 2b_1(\Delta) + 2 \\
&= b_3(\Delta) - b_1(\Delta) + 2.
\end{aligned}$$

This gives $g_3(\Gamma) - 1 \geqslant b_3(\Delta) + 1 \geqslant b_1(\Delta) + j_2(\Gamma)$. By the same argument, $g_3(\Gamma) - 1 \geqslant b_2(\Delta) + j_1(\Gamma)$. Thus, we can define Γ_1 by $p(\Gamma_1) = 1$, $q(\Gamma_1) = 2$, $g_3(\Gamma_1) = g_3(\Gamma) - 1$, $g_1(\Gamma_1) = g_1(\Gamma)$, and $g_2(\Gamma_1) = g_2(\Gamma)$ and have Γ_1 use all available edges.

We can compute that Γ has $g_1(\Gamma)g_2(\Gamma)g_3(\Gamma)$ facets containing neither extra vertex, $g_2(\Gamma)j_2(\Gamma)$ facets containing $v^1_{b_1(\Delta)}$ but not $v^2_{b_2(\Delta)}$, $g_1(\Gamma)j_1(\Gamma)$ facets containing $v^2_{b_2(\Delta)}$ but not $v^1_{b_1(\Delta)}$, and $\min\{j_1(\Gamma), j_2(\Gamma)\}$ facets containing both extra vertices. Similarly, we compute that Γ_1 has $g_1(\Gamma)g_2(\Gamma)(g_3(\Gamma) - 1)$ facets containing neither extra vertex, $g_2(\Gamma)(j_2(\Gamma) + g_1(\Gamma))$ facets containing $v^1_{b_1(\Delta)}$ but not $v^2_{b_2(\Delta)}$, $g_1(\Gamma)(j_1(\Gamma) + g_2(\Gamma))$ facets containing $v^2_{b_2(\Delta)}$ but not $v^1_{b_1(\Delta)}$, and $\min\{j_1(\Gamma) + g_2(\Gamma), j_2(\Gamma) + g_1(\Gamma)\}$ facets containing both extra vertices. Thus,

$$\begin{aligned}
f_{123}(\Gamma_1) &= g_1(\Gamma)g_2(\Gamma)(g_3(\Gamma) - 1) + g_2(\Gamma)(j_2(\Gamma) + g_1(\Gamma)) \\
&\quad + g_1(\Gamma)(j_1(\Gamma) + g_2(\Gamma)) + \min\{j_1(\Gamma) + g_2(\Gamma), j_2(\Gamma) + g_1(\Gamma)\} \\
&= g_1(\Gamma)g_2(\Gamma)g_3(\Gamma) + g_1(\Gamma)j_1(\Gamma) + g_2(\Gamma)j_2(\Gamma) \\
&\quad + g_1(\Gamma)g_2(\Gamma) + \min\{j_1(\Gamma) + g_2(\Gamma), j_2(\Gamma) + g_1(\Gamma)\} \\
&> g_1(\Gamma)g_2(\Gamma)g_3(\Gamma) + g_1(\Gamma)j_1(\Gamma) + g_2(\Gamma)j_2(\Gamma) + \min\{j_1(\Gamma), j_2(\Gamma)\} \\
&= f_{123}(\Gamma).
\end{aligned}$$

Therefore, by Proposition 2.4, $\Gamma \notin \mathcal{B}(\Delta) \supset \mathcal{D}(\Delta)$, a contradiction.

Otherwise, $g_3(\Gamma) = b_3(\Delta) + 1$. In this case, we define Γ_1 by $p(\Gamma_1) = 3$, $q(\Gamma_1) = 1$, and $g_i(\Gamma_1) = b_i(\Gamma)$ for all $i \in [3]$. Since $f_{13}(\Delta) < b_1(\Delta)(b_3(\Delta)+1)$ and $f_{23}(\Delta) < b_1(\Delta)(b_3(\Delta)+1)$, the first extra vertex of Γ_1 uses up all remaining edges, and the second extra vertex is not contained in an edge, so it doesn't matter if there is another vertex of color 1 available. We also find that $f_{13}(\Delta) = (b_1(\Delta)-1)(b_3(\Delta)+1)+j_2(\Gamma) < b_1(\Delta)(b_3(\Delta)+1)$, from which $j_2(\Gamma) < b_3(\Delta) + 1$, and so $j_2(\Gamma) \leqslant b_3(\Delta)$. Similarly, $j_1(\Gamma) \leqslant b_3(\Delta)$.

We compute $j_2(\Gamma_1) = j_2(\Gamma)+b_1(\Delta)-1-b_3(\Delta)$ and $j_1(\Gamma_1) = j_1(\Gamma)+b_2(\Delta)-1-b_3(\Delta)$. The first extra vertex of Γ_1 contains $(j_2(\Gamma)+b_1(\Delta)-1-b_3(\Delta))(j_1(\Gamma)+b_2(\Delta)-1-b_3(\Delta))$ facets. We get that

$$\begin{aligned}
f_{123}(\Gamma_1) &= b_1(\Delta)b_2(\Delta)b_3(\Delta) \\
&\quad + (j_2(\Gamma) + b_1(\Delta) - 1 - b_3(\Delta))(j_1(\Gamma) + b_2(\Delta) - 1 - b_3(\Delta)) \\
&= b_1(\Delta)b_2(\Delta)b_3(\Delta) + (b_1(\Delta) - 1)(b_2(\Delta) - 1) - (b_1(\Delta) - 1)b_3(\Delta) \\
&\quad - (b_2(\Delta) - 1)b_3(\Delta) + (b_1(\Delta) - 1)j_1(\Gamma) + (b_2(\Delta) - 1)j_2(\Gamma) \\
&\quad + (j_1(\Gamma) - b_3(\Delta))(j_2(\Gamma) - b_3(\Delta)) \\
&= (b_1(\Delta) - 1)(b_2(\Delta) - 1)(b_3(\Delta) + 1) + (b_1(\Delta) - 1)j_1(\Gamma) \\
&\quad + (b_2(\Delta) - 1)j_2(\Gamma) + (b_3(\Delta) - j_1(\Gamma))(b_3(\Delta) - j_2(\Gamma)) + b_3(\Delta) \\
&\geqslant (b_1(\Delta) - 1)(b_2(\Delta) - 1)(b_3(\Delta) + 1) + (b_1(\Delta) - 1)j_1(\Gamma) \\
&\quad + (b_2(\Delta) - 1)j_2(\Gamma) + \min\{j_1(\Gamma), j_2(\Gamma)\} \\
&= f_{123}(\Gamma).
\end{aligned}$$

The inequality comes because $j_1(\Gamma) \leqslant b_3(\Delta)$ and $j_2(\Gamma) \leqslant b_3(\Delta)$.

Since $\Gamma \in \mathcal{B}(\Delta)$, we get $\Gamma_1 \in \mathcal{B}(\Delta)$. As we have already seen that Γ_1 uses all available edges, $\Gamma_1 \in \mathcal{C}(\Delta)$. Since $\Gamma \in \mathcal{D}(\Delta)$, we know that $\mathcal{D}(\Delta) \neq \emptyset$. The alternative is $\mathcal{D}(\Delta) = \mathcal{C}(\Delta)$, and so $\Gamma_1 \in \mathcal{D}(\Delta)$. Since $b_3(\Delta) = g_3(\Gamma_1)$, we get $\Gamma_1 \in \mathcal{E}(\Delta)$.

There was nothing special about choosing $q(\Gamma_1) = 1$, as the second extra vertex was not used at all. If we defined Γ_2 in exactly the same way as Γ_1 except that $q(\Gamma_2) = 2$, then $\Gamma_2 \in \mathcal{E}(\Delta)$ by the same argument as Γ_1. This completes the proof because $r(\Gamma_2) = 1 \neq 2 = r(\Gamma_1)$, which is the second option of the lemma. $\square$

Proof of Lemma 3.12: We have that $\{p(\Gamma_0), q(\Gamma_0)\} = \{1, 2\}$. Assume without loss of generality that $p(\Gamma_0) = 1$.

We break the proof into several cases. Each time that we construct a complex Γ, we need to check that it is well-defined, in $\mathcal{A}(\Delta)$, and if we might have $f_{123}(\Gamma) = f_{123}(\Gamma_0)$, also that $\Gamma \in \mathcal{F}(\Delta)$. To show that Γ is well-defined, it suffices to show that there are enough edges and vertices available to construct the complex. That $\Gamma \in \mathcal{A}(\Delta)$ follows from the definition. To check that $\Gamma \in \mathcal{F}(\Delta)$, the first and fourth conditions are true by construction and the third part holds because it is a property of Δ and must hold to get $\Gamma_0 \in \mathcal{F}(\Delta)$. It thus suffices to check the second condition.

In constructions where $g_i(\Gamma) \leqslant g_i(\Gamma_0)$ for all $i \in [3]$, there are enough vertices because Γ uses at most as many vertices of each color as Γ_0, except that Γ could use one additional vertex of color 3, which is available because $f_3(\Delta) > g_3(\Gamma_0)$. This extra vertex does not force Γ to contain any additional edges, so Γ is well-defined because Γ_0 is.

Suppose that $f_1(\Delta) = g_1(\Gamma_0)$. This means that Γ_0 does not have an extra vertex of color 1. Thus, we can set $p(\Gamma) = 2$ and $q(\Gamma) = 3$, and we have $\Gamma_0 \subseteq \Gamma$. Therefore, we can assume that $f_1(\Delta) > g_1(\Gamma_0)$. By an analogous argument, we can assume that $f_2(\Delta) > g_2(\Gamma_0)$. We can also assume that $g_3(\Gamma_0) < f_3(\Delta)$, as otherwise, we can take $\Gamma = \Gamma_0$ and meet the third option of the lemma.

We note that $j_1(\Gamma_0) \leqslant g_3(\Gamma_0)$ and $j_2(\Gamma_0) \leqslant g_3(\Gamma_0)$, as this is necessary for Γ_0 to use all edges of colors 23 and 13, respectively, as it does not have an extra vertex of color 3.

Case I: $j_1(\Gamma_0) \leqslant j_2(\Gamma_0)$

Case I A: $j_3(\Gamma_0) \leqslant g_2(\Gamma_0)$

This case means that the first extra vertex of Γ_0 uses all available edges of color 12. In particular, this means that the second extra vertex does not add any additional facets. We can compute that the first extra vertex of Γ_0 adds $j_3(\Gamma_0)j_2(\Gamma_0)$ facets, so

$$f_{123}(\Gamma_0) = g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_3(\Gamma_0)j_2(\Gamma_0).$$

Case I A 1: $j_1(\Gamma_0) \leqslant g_2(\Gamma_0)$

Define Γ by $p(\Gamma) = 1$, $q(\Gamma) = 3$, and $g_i(\Gamma) = g_i(\Gamma_0)$ for all $i \in [3]$. Because the second extra vertex of Γ_0 does not add any additional facets, every facet of Γ_0 is also in Γ, so $f_{123}(\Gamma) \geqslant f_{123}(\Gamma_0)$. The first extra vertex of Γ uses all edges of colors 12 and 13 because it also does so in Γ_0. Finally, because $j_1(\Gamma_0) \leqslant g_2(\Gamma_0)$, the second extra vertex of Γ uses all remaining edges of color 23. Hence, $\Gamma \in \mathcal{F}(\Delta)$, and we have shown that the first option of the lemma holds.

Case I A 2: $j_1(\Gamma_0) > g_2(\Gamma_0)$

Case I A 2 a: $j_2(\Gamma_0) < g_1(\Gamma_0)$

Define Γ by $q(\Gamma) = 2$, $p(\Gamma) = 3$, and $g_i(\Gamma) = g_i(\Gamma_0)$ for all $i \in [3]$. We can chain the inequalities of this case to get

$$g_1(\Gamma_0) > j_2(\Gamma_0) \geqslant j_1(\Gamma_0) > g_2(\Gamma_0) \geqslant j_3(\Gamma_0).$$

In particular, $g_1(\Gamma_0) > j_3(\Gamma_0)$, so the second extra vertex of Γ uses all available edges of color 12. That $j_2(\Gamma_0) < g_1(\Gamma_0)$ means that the first extra vertex of Γ uses all available edges of color 13. The second extra vertex of Γ uses any leftover edges of color 23 because it does in Γ_0. Thus, $\Gamma \in \mathcal{F}(\Delta)$.

We can compute that the first extra vertex of Γ adds $j_2(\Gamma_0)g_2(\Gamma_0)$ facets, and the second one adds $j_3(\Gamma_0)(j_1(\Gamma_0) - g_2(\Gamma_0))$ facets. This allows us to compute

$$\begin{aligned}
f_{123}(\Gamma) &= g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_2(\Gamma_0)g_2(\Gamma_0) + j_3(\Gamma_0)(j_1(\Gamma_0) - g_2(\Gamma_0)) \\
&\geqslant g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_2(\Gamma_0)g_2(\Gamma_0) \\
&\geqslant g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_2(\Gamma_0)j_3(\Gamma_0) \\
&= f_{123}(\Gamma_0),
\end{aligned}$$

which yields the first option in the lemma.

Case I A 2 b: $j_2(\Gamma_0) \geqslant g_1(\Gamma_0)$

Let $w = \min\{\lfloor \frac{j_2(\Gamma_0)}{g_1(\Gamma_0)} \rfloor, \lfloor \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)} \rfloor, f_3(\Delta) - g_3(\Gamma_0)\}$. Note that $w \geqslant 1$ because $j_2(\Gamma_0) \geqslant g_1(\Gamma_0)$, $j_1(\Gamma_0) > g_2(\Gamma_0)$, and $f_3(\Delta) > g_3(\Gamma_0)$. Define Γ_1 by $p(\Gamma_1) = 1$, $q(\Gamma_1) = 2$, $g_1(\Gamma_1) = g_1(\Gamma_0)$, $g_2(\Gamma_1) = g_2(\Gamma_0)$, and $g_3(\Gamma_1) = g_3(\Gamma_0) + w$. There are enough edges to do this because the first two terms assert that w is small enough not to use more edges than allowed of color sets 13 or 23, respectively. The third term of w ensures that there are enough vertices of color 3 available. Thus, Γ_1 is well-defined. Because the extra vertices in Γ_1 are able to use at least as many edges of each color set as those of Γ_0 and have at most as many such edges available to use, $\Gamma_1 \in \mathcal{F}(\Delta)$.

The first extra vertex of Γ_1 uses all edges of color 12, so the second extra vertex does not add any additional facets. Meanwhile, the first extra vertex of Γ_1 adds $j_2(\Gamma_1)j_3(\Gamma_1) = j_3(\Gamma_0)(j_2(\Gamma_0) - wg_1(\Gamma_0))$ facets. This yields

$$\begin{aligned}
f_{123}(\Gamma_1) &= g_1(\Gamma_0)g_2(\Gamma_0)(g_3(\Gamma_0) + w) + j_3(\Gamma_0)(j_2(\Gamma_0) - wg_1(\Gamma_0)) \\
&= g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_3(\Gamma_0)j_2(\Gamma_0) + wg_1(\Gamma_0)(g_2(\Gamma_0) - j_3(\Gamma_0)) \\
&\geqslant g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_3(\Gamma_0)j_2(\Gamma_0) \\
&= f_{123}(\Gamma_0).
\end{aligned}$$

Case I A 2 b i: $j_3(\Gamma_0) < g_2(\Gamma_0)$

This ensures that the inequality above is strict, so we can take $\Gamma = \Gamma_1$ and have the second option of the lemma.

Case I A 2 b ii: $w = f_3(\Delta) - g_3(\Gamma_0)$

This ensures that $g_3(\Gamma_1) = f_3(\Delta)$, so we can take $\Gamma = \Gamma_1$ and have the third option of the lemma.

Case I A 2 b iii: $j_3(\Gamma_0) = g_2(\Gamma_0)$ and $w < f_3(\Delta) - g_3(\Gamma_0)$

Case I A 2 b iii (a): $j_1(\Gamma_1) \leqslant g_2(\Gamma_1)$

Define Γ by $g_i(\Gamma) = g_i(\Gamma_1)$ for all $i \in [3]$, $p(\Gamma) = 1$ and $q(\Gamma) = 3$. We have that Γ is well-defined because Γ_1 is. The first extra vertex of Γ uses all available edges of colors 12 and 13 because the first extra vertex of Γ_1 does also. Because $j_1(\Gamma_1) \leqslant g_2(\Gamma_1)$, the second extra vertex of Γ uses all remaining edges of color 23. Hence, $\Gamma \in \mathcal{F}(\Delta)$. Furthermore, since the only differing edges between Γ and Γ_1 are the ones containing the second extra vertex, none of which are contained in a facet, we have $f_{123}(\Gamma) = f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma_0)$, so we satisfy the first option of the lemma.

Case I A 2 b iii (b): $j_1(\Gamma_1) > g_2(\Gamma_1)$

This means that $w \neq \lfloor \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)} \rfloor$. Since $w < f_3(\Delta) - g_3(\Gamma_0)$, we must have $w = \lfloor \frac{j_2(\Gamma_0)}{g_1(\Gamma_0)} \rfloor$. Hence, $j_2(\Gamma_1) = j_2(\Gamma_0) - wg_1(\Gamma_0) < g_1(\Gamma_0)$. That $w \neq \lfloor \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)} \rfloor$ means that $w + 1 \leqslant \lfloor \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)} \rfloor \leqslant \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)}$. This yields $\frac{j_2(\Gamma_0)}{g_1(\Gamma_0)} < w+1 \leqslant \frac{j_1(\Gamma_0)}{g_2(\Gamma_0)}$, so $j_2(\Gamma_0)g_2(\Gamma_0) < j_1(\Gamma_0)g_1(\Gamma_0)$. Since $j_2(\Gamma_0) \geqslant j_1(\Gamma_0)$, we must have $g_2(\Gamma_0) < g_1(\Gamma_0)$.

Case I A 2 b iii (b) (i): $j_1(\Gamma_1) > j_2(\Gamma_1)$

Define Γ by $g_i(\Gamma) = g_i(\Gamma_1)$ for all $i \in [3]$, $p(\Gamma) = 2$ and $q(\Gamma) = 1$. Then Γ is well-defined because Γ_1 is. We can compute that Γ has $g_1(\Gamma)g_2(\Gamma)g_3(\Gamma)$ facets before adding extra vertices. The first extra vertex of Γ uses up all remaining edges of color 12 because $j_3(\Gamma) = j_3(\Gamma_0) = g_2(\Gamma_0) < g_1(\Gamma_0)$. Furthermore, Γ uses all remaining edges of color 23 because $j_1(\Gamma) < j_1(\Gamma_0) \leqslant g_3(\Gamma_0) < g_3(\Gamma)$. Thus, we can compute

$$\begin{aligned} f_{123}(\Gamma) &= g_1(\Gamma)g_2(\Gamma)g_3(\Gamma) + j_3(\Gamma)j_1(\Gamma) \\ &= g_1(\Gamma)g_2(\Gamma)g_3(\Gamma) + j_3(\Gamma_1)j_1(\Gamma_1) \\ &> g_1(\Gamma)g_2(\Gamma)g_3(\Gamma) + j_3(\Gamma_1)j_2(\Gamma_1) \\ &= f_{123}(\Gamma_1) \\ &\geqslant f_{123}(\Gamma_0), \end{aligned}$$

which yields the second option of the lemma because $\Gamma \in \mathcal{A}(\Delta)$ by construction.

Case I A 2 b iii (b) (ii): $j_1(\Gamma_1) \leqslant j_2(\Gamma_1)$

Define Γ_2 by $g_1(\Gamma_2) = g_1(\Gamma_0)$, $g_2(\Gamma_2) = g_2(\Gamma_0)$, $g_3(\Gamma_2) = \lceil \frac{f_{13}(\Delta)}{g_1(\Gamma_0)+1} \rceil$, $p(\Gamma_2) = 1$, and $q(\Gamma_2) = 2$. Let $y = g_3(\Gamma_0) - g_3(\Gamma_2)$. Since

$$f_{13}(\Delta) = g_1(\Gamma_0)g_3(\Gamma_0) + j_2(\Gamma_0) \leqslant g_1(\Gamma_0)g_3(\Gamma_0) + g_3(\Gamma_0) = (g_1(\Gamma_0) + 1)g_3(\Gamma_0),$$

we have $g_3(\Gamma_2) = \lceil \frac{f_{13}(\Delta)}{g_1(\Gamma_0)+1} \rceil \leqslant \lceil g_3(\Gamma_0) \rceil = g_3(\Gamma_0)$, and so $y \geqslant 0$. Hence, Γ_2 is well-defined because Γ_0 is. It uses all edges of color 12 because Γ_0 does. It uses all edges of color 13 because $g_1(\Gamma_0) + 1$ vertices of color 1 and $g_3(\Gamma_2)$ of color 3 can use up to $(g_1(\Gamma_0)+1)g_3(\Gamma_2) = (g_1(\Gamma_0)+1)\lceil \frac{f_{13}(\Delta)}{g_1(\Gamma_0)+1} \rceil \geqslant f_{13}(\Delta)$ edges of color 13. Finally, Γ_2 uses all edges of color 23 because

$$j_1(\Gamma_2) = j_1(\Gamma_0) + yg_2(\Gamma_0) \leqslant j_2(\Gamma_0) + yg_1(\Gamma_0) = j_2(\Gamma_2) \leqslant g_3(\Gamma_2),$$

as Γ_2 also uses all edges of color 13. Therefore, $\Gamma_2 \in \mathcal{F}(\Delta)$.

Before adding any extra vertices, Γ_2 has $g_1(\Gamma_0)g_2(\Gamma_0)(g_3(\Gamma_0) - y)$ facets. The first extra vertex adds $g_2(\Gamma_0)(j_2(\Gamma_0) + yg_1(\Gamma_0))$ facets. This uses all edges of color 12, so the second extra vertex does not add any additional facets. Thus, we compute

$$\begin{aligned}
f_{123}(\Gamma_2) &= g_1(\Gamma_0)g_2(\Gamma_0)(g_3(\Gamma_0) - y) + g_2(\Gamma_0)(j_2(\Gamma_0) + yg_1(\Gamma_0)) \\
&= g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + g_2(\Gamma_0)j_2(\Gamma_0) \\
&= g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + j_3(\Gamma_0)j_2(\Gamma_0) \\
&= f_{123}(\Gamma_0).
\end{aligned}$$

Define Γ by $g_1(\Gamma) = g_1(\Gamma_2) + 1$, $g_2(\Gamma) = g_2(\Gamma_2)$, $g_3(\Gamma) = g_3(\Gamma_2) - 1$, $p(\Gamma) = 3$, and $q(\Gamma) = 2$. There are enough edges of color 23 to do this because Γ_2 is well-defined. There are enough edges of color 12 because

$$f_{12}(\Delta) = g_1(\Gamma_0)g_2(\Gamma_0) + j_3(\Gamma_0) = g_1(\Gamma_0)g_2(\Gamma_0) + g_2(\Gamma_0) = g_1(\Gamma)g_2(\Gamma).$$

There are enough edges of color 13 because

$$\begin{aligned}
g_1(\Gamma)g_3(\Gamma) &= (g_1(\Gamma_0) + 1)\Big(\Big\lceil \frac{f_{13}(\Delta)}{g_1(\Gamma_0) + 1}\Big\rceil - 1\Big) \\
&< (g_1(\Gamma_0) + 1)\Big(\frac{f_{13}(\Delta)}{g_1(\Gamma_0) + 1}\Big) \\
&= f_{13}(\Delta).
\end{aligned}$$

Therefore, Γ is well-defined.

It is easy to check that Γ and Γ_2 use exactly the same edges of colors 12 and 23. They have exactly the same vertices of all colors, so Γ uses all available edges because Γ_2 does. Hence, $\Gamma \in \mathcal{F}(\Delta)$. The only possible difference is that Γ could include some edges containing $v^1_{g_1(\Gamma_2)+1}$ but not $v^3_{g_3(\Gamma_2)}$, while Γ_2 contains exactly the same number of edges containing $v^3_{g_3(\Gamma_2)}$ but not $v^1_{g_1(\Gamma_2)+1}$. Both of these vertices are adjacent to exactly $g_2(\Gamma_0)$ vertices of color 2, so every edge that is in either Γ or Γ_2 but not both is contained in exactly $g_2(\Gamma_0)$ facets. Therefore, $f_{123}(\Gamma) = f_{123}(\Gamma_2) \geqslant f_{123}(\Gamma_0)$, which gives us the first option of the lemma.

Case I B: $j_3(\Gamma_0) > g_2(\Gamma_0)$

The first extra vertex of Γ_0 adds $g_2(\Gamma_0)j_2(\Gamma_0)$ facets, while the second extra vertex adds $j_1(\Gamma_0)(j_3(\Gamma_0) - g_2(\Gamma_0))$. Thus,

$$f_{123}(\Gamma_0) = g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0) + g_2(\Gamma_0)j_2(\Gamma_0) + j_1(\Gamma_0)(j_3(\Gamma_0) - g_2(\Gamma_0)).$$

Case I B 1: $j_2(\Gamma_0) + g_1(\Gamma_0) < g_3(\Gamma_0)$

Define Γ by $p(\Gamma) = 1$, $q(\Gamma) = 2$, $g_1(\Gamma) = g_1(\Gamma_0)$, $g_2(\Gamma) = g_2(\Gamma_0)$, and $g_3(\Gamma) = g_3(\Gamma_0) - 1$. We have already seen that this is well-defined, and it is immediate from the definition that $\Gamma \in \mathcal{A}(\Delta)$. The first extra vertex of Γ adds $g_2(\Gamma_0)(j_2(\Gamma_0) + g_1(\Gamma_0))$ facets.

Case I B 1 a: $j_1(\Gamma_0) + g_2(\Gamma_0) < g_3(\Gamma_0)$

This allows the second extra vertex of Γ to use all remaining edges of color 23, so it adds $(j_1(\Gamma_0)+g_2(\Gamma_0))(j_3(\Gamma_0)-g_2(\Gamma_0))$ facets. Thus, we have

$$\begin{aligned} f_{123}(\Gamma) &= g_1(\Gamma_0)g_2(\Gamma_0)(g_3(\Gamma_0)-1)+g_2(\Gamma_0)(j_2(\Gamma_0)+g_1(\Gamma_0)) \\ &\qquad +(j_1(\Gamma_0)+g_2(\Gamma_0))(j_3(\Gamma_0)-g_2(\Gamma_0)) \\ &> g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0)+g_2(\Gamma_0)j_2(\Gamma_0)+j_1(\Gamma_0)(j_3(\Gamma_0)-g_2(\Gamma_0)) \\ &= f_{123}(\Gamma_0), \end{aligned}$$

giving us the second option of the lemma.

Case I B 1 b: $j_1(\Gamma_0)+g_2(\Gamma_0)\geqslant g_3(\Gamma_0)$

The second extra vertex of Γ can only use $g_3(\Gamma)=g_3(\Gamma_0)-1$ edges of color 23, so it only adds $(g_3(\Gamma_0)-1)(j_3(\Gamma_0)-g_2(\Gamma_0))$ facets. Still, we have

$$g_3(\Gamma_0)-1\geqslant j_2(\Gamma_0)+g_1(\Gamma_0)>j_2(\Gamma_0)\geqslant j_1(\Gamma_0),$$

from which we can compute

$$\begin{aligned} f_{123}(\Gamma) &= g_1(\Gamma_0)g_2(\Gamma_0)(g_3(\Gamma_0)-1)+g_2(\Gamma_0)(j_2(\Gamma_0)+g_1(\Gamma_0)) \\ &\qquad +(g_3(\Gamma_0)-1)(j_3(\Gamma_0)-g_2(\Gamma_0)) \\ &> g_1(\Gamma_0)g_2(\Gamma_0)g_3(\Gamma_0)+g_2(\Gamma_0)j_2(\Gamma_0)+j_1(\Gamma_0)(j_3(\Gamma_0)-g_2(\Gamma_0)) \\ &= f_{123}(\Gamma_0), \end{aligned}$$

which is the second option in the lemma.

Case I B 2: $j_2(\Gamma_0)+g_1(\Gamma_0)\geqslant g_3(\Gamma_0)$

Case I B 2 a: $j_2(\Gamma_0)=g_3(\Gamma_0)$

Define Γ by $g_1(\Gamma)=g_1(\Gamma_0)+1$, $g_2(\Gamma)=g_2(\Gamma_0)$, $g_3(\Gamma)=g_3(\Gamma_0)$, $p(\Gamma)=2$, and $q(\Gamma)=3$. The first extra vertex of Γ_0 is adjacent to all vertices of color 3 because $j_2(\Gamma_0)=g_3(\Gamma_0)$ and the first $g_2(\Gamma_0)$ vertices of color 2 because $j_3(\Gamma_0)>g_2(\Gamma_0)$. The difference between Γ_0 and Γ is that the extra vertex of Γ_0 of color 1 is still in Γ but no longer an extra vertex, and Γ tacks on an extra isolated vertex of color 3 that Γ_0 lacks. Hence, $\Gamma_0\subseteq\Gamma$, so Γ must use at least as many edges and facets as Γ_0. This means that $\Gamma\in\mathcal{F}(\Delta)$ and satisfies the first option of the lemma.

Case I B 2 b: $j_2(\Gamma_0)<g_3(\Gamma_0)$

Define Γ by $g_1(\Gamma)=g_1(\Gamma_0)+1$, $g_2(\Gamma)=g_2(\Gamma_0)$, $g_3(\Gamma)=g_3(\Gamma_0)-1$, $p(\Gamma)=3$, and $q(\Gamma)=2$. We know that there are enough edges of color 23 for Γ because there are enough for Γ_0, which needs more. There are enough edges of color 12 because Γ only needs $g_2(\Gamma_0)$ more edges than Γ_0 and $j_3(\Gamma_0)>g_2(\Gamma_0)$. There are enough edges of color 13 because

$$\begin{aligned} f_{13}(\Delta) &= g_1(\Gamma_0)g_3(\Gamma_0)+j_2(\Gamma_0) \\ &\geqslant g_1(\Gamma_0)g_3(\Gamma_0)+g_3(\Gamma_0)-g_1(\Gamma_0) \\ &> (g_1(\Gamma_0)+1)(g_3(\Gamma_0)-1) \\ &= g_1(\Gamma)g_3(\Gamma). \end{aligned}$$

The vertices of Γ are precisely the vertices of Γ_0 (with the last vertex of color 1 coming because $g_1(\Gamma_0) < f_1(\Delta)$), so Γ is well-defined. Furthermore, because Γ and Γ_0 have the same vertices, they can each use the same number of edges of each color set. Therefore, $\Gamma \in \mathcal{F}(\Delta)$ because Γ_0 is also.

One can easily check that Γ and Γ_0 have exactly the same edges of colors 12 and 23. The edges of color 13 can only differ in that Γ may have some edges that Γ_0 lacks containing $v^1_{g_1(\Gamma_0)+1}$ but not $v^3_{g_3(\Gamma_0)}$, while Γ_0 could have some edges that are missing from Γ and contain $v^3_{g_3(\Gamma_0)}$ but not $v^1_{g_1(\Gamma_0)+1}$. Any edge that is contained only in Γ has at least $g_2(\Gamma_0)$ facets, as both of its vertices are present before adding any extra vertices. Any edge that is contained only in Γ_0 is contained in at most $g_2(\Gamma_0)$ facets, as $v^3_{g_3(\Gamma_0)}$ is not adjacent to $v^2_{g_2(\Gamma_0)+1}$ because $j_2(\Gamma_0) < g_3(\Gamma)$. Therefore, each differing edge of Γ has at least as many facets as each one of Γ_0, so $f_{123}(\Gamma) \geqslant f_{123}(\Gamma_0)$, which gives us the first option in the lemma.

Case II: $j_1(\Gamma_0) > j_2(\Gamma_0)$

Define Γ_1 by $p(\Gamma_1) = 2$, $q(\Gamma_1) = 1$, and $g_i(\Gamma_1) = g_i(\Gamma_0)$ for all $i \in [3]$. We have that $f_{123}(\Gamma_1) \geqslant f_{123}(\Gamma_0)$ by Lemma 5.1. Because Γ_0 and Γ_1 have the same vertices, including the same extra vertices, $\Gamma_1 \in \mathcal{F}(\Delta)$. Applying Case I to Γ_1 gives that the lemma holds for Γ_0. □

Proof of Lemma 3.10: Because $\Gamma \in \mathcal{E}(\Delta) \subseteq \mathcal{D}(\Delta)$, it must use all available edges. Let $k = 6 - r(\Gamma) - i$. Since there is not an extra vertex of color $r(\Gamma)$, we have $j_k(\Gamma) \leqslant g_{r(\Gamma)}(\Gamma)$. We trivially must have $j_k(\Gamma) \geqslant 0$, so we have $g_{r(\Gamma)}(\Gamma)b_i(\Delta) \leqslant g_{r(\Gamma)}(\Gamma)b_i(\Delta) + j_k(\Gamma) \leqslant g_{r(\Gamma)}(\Gamma)b_i(\Delta) + g_{r(\Gamma)}(\Gamma)$. The middle term is $f_{ir(\Gamma)}(\Delta)$, so we have $g_{r(\Gamma)}(\Gamma)b_i(\Delta) \leqslant f_{ir(\Gamma)}(\Delta) \leqslant g_{r(\Gamma)}(\Gamma)(b_i(\Delta)+1)$. Dividing the two inequalities as appropriate, we get $g_{r(\Gamma)}(\Gamma) \leqslant \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)}$ and $g_{r(\Gamma)}(\Gamma) \geqslant \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1}$, respectively. Chain these together to get $\frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} \leqslant g_{r(\Gamma)}(\Gamma) \leqslant \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)}$. Since $g_{r(\Gamma)}(\Gamma)$ is an integer, we have $\left\lceil \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} \right\rceil \leqslant g_{r(\Gamma)}(\Gamma) \leqslant \left\lfloor \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)} \right\rfloor$, which gives one inequality of the lemma.

Next, we compute

$$\begin{aligned}
\frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)} - \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} &= f_{ir(\Gamma)}(\Delta)\Big(\frac{1}{b_i(\Delta)} - \frac{1}{b_i(\Delta)+1}\Big) \\
&= \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)(b_i(\Delta)+1)} \\
&< \frac{(b_i(\Delta)+1)(b_{r(\Gamma)}(\Delta)+1)}{b_i(\Delta)(b_i(\Delta)+1)} \\
&= \frac{b_{r(\Gamma)}(\Delta)+1}{b_i(\Delta)} \\
&\leqslant \frac{b_i(\Delta)+1}{b_i(\Delta)} \\
&\leqslant 2.
\end{aligned}$$

Thus, $g_{r(\Gamma)}(\Gamma)$ is an integer contained in an interval of length less than two. There can be at most two such integers. We have seen that $\left\lceil \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)+1} \right\rceil$ is the smallest possible such integer and $\left\lfloor \frac{f_{ir(\Gamma)}(\Delta)}{b_i(\Delta)} \right\rfloor$ is the largest, so if $g_{r(\Gamma)}(\Gamma)$ has two possible values, these must be both of them. If $g_{r(\Gamma)}(\Gamma)$ has only one possible value, then these expressions both give that one value. $\square$

Proof of Lemma 3.15: The lemma is stated as it is to make it clear that the upper bound depends only on the choice of $g_2(\Gamma)$, but it is easier to prove an alternate form. We can compute

$$\begin{aligned} v(\Delta, g_2(\Gamma)) &= b_1(\Delta)f_{23}(\Delta) + (f_{12}(\Delta) - b_1(\Delta)g_2(\Gamma))\Big(f_{13}(\Delta) - \frac{b_1(\Delta)f_{23}(\Delta)}{g_2(\Gamma)}\Big) \\ &= b_1(\Delta)f_{23}(\Delta) + j_3(\Gamma)\Big(f_{13}(\Delta) - \frac{b_1(\Delta)(g_2(\Gamma)g_3(\Gamma) + j_1(\Gamma))}{g_2(\Gamma)}\Big) \\ &= b_1(\Delta)f_{23}(\Delta) + j_3(\Gamma)\Big(f_{13}(\Delta) - b_1(\Delta)g_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)}{g_2(\Gamma)}\Big) \\ &= b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

Case I: $p(\Gamma) = 3$

Case I A: $j_2(\Gamma) \leqslant b_1(\Delta)$

The first extra vertex uses all remaining edges of both of its color sets. This does not leave any remaining edges of color 13 for use by the second extra vertex, so the second extra vertex does not add any additional facets. Therefore,

$$f_{123}(\Gamma) = b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + j_1(\Gamma)j_2(\Gamma).$$

Case I A 1: $j_1(\Gamma) \leqslant j_3(\Gamma)$

$$\begin{aligned} f_{123}(\Gamma) &= b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + j_1(\Gamma)j_2(\Gamma) \\ &= b_1(\Delta)(f_{23}(\Delta) - j_1(\Gamma)) + j_2(\Gamma)j_3(\Gamma) + j_2(\Gamma)(j_1(\Gamma) - j_3(\Gamma)) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

The last line comes because $j_3(\Gamma) \geqslant j_1(\Gamma)$ and $j_3(\Gamma) \leqslant g_2(\Gamma)$, as there is not an extra vertex of color 2.

Case I A 2: $j_3(\Gamma) < j_1(\Gamma)$

$$\begin{aligned} f_{123}(\Gamma) &= b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + j_1(\Gamma)j_2(\Gamma) \\ &= b_1(\Delta)(f_{23}(\Delta) - j_1(\Gamma)) + j_2(\Gamma)j_3(\Gamma) + j_2(\Gamma)(j_1(\Gamma) - j_3(\Gamma)) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - b_1(\Delta)j_1(\Gamma) + b_1(\Delta)(j_1(\Gamma) - j_3(\Gamma)) \\ &= b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - b_1(\Delta)j_3(\Gamma) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

As in the previous case, the last line comes because $j_1(\Gamma) \leqslant g_2(\Gamma)$.

Case I B: $j_2(\Gamma) \geqslant b_1(\Delta)$

The first extra vertex is adjacent to all previous vertices of color 1, so it adds $j_1(\Gamma)b_1(\Delta)$ facets. The second extra vertex adds $j_3(\Gamma)(j_2(\Gamma) - b_1(\Delta))$ facets. Thus, we have

$$\begin{aligned} f_{123}(\Gamma) &= b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + j_1(\Gamma)b_1(\Delta) + j_3(\Gamma)(j_2(\Gamma) - b_1(\Delta)) \\ &= b_1(\Delta)(f_{23}(\Delta) - j_1(\Gamma)) + j_1(\Gamma)b_1(\Delta) + j_3(\Gamma)(j_2(\Gamma) - b_1(\Delta)) \\ &= b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - b_1(\Delta)j_3(\Gamma) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

Case II: $p(\Gamma) = 1$

Case II A: $j_2(\Gamma) \leqslant g_3(\Gamma)$

The first extra vertex uses all available edges of color 13 and adds $j_2(\Gamma)j_3(\Gamma)$ facets. This does not leave any edges of this color set to be used by the second available vertex, so the other vertex does not add any more facets. This gives us

$$\begin{aligned} f_{123}(\Gamma) &= b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + j_2(\Gamma)j_3(\Gamma) \\ &= b_1(\Delta)(f_{23}(\Delta) - j_1(\Gamma)) + j_2(\Gamma)j_3(\Gamma) \\ &= b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - b_1(\Delta)j_1(\Gamma) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

Case II B: $j_2(\Gamma) > g_3(\Gamma)$

Case II B 1: $j_3(\Gamma) \geqslant j_1(\Gamma)$

There are enough spare edges of color 13 to make the first extra vertex adjacent to all previous vertices of color 3. Thus, the first extra vertex adds $g_3(\Gamma)j_3(\Gamma)$ facets. The second extra vertex brings an additional $j_1(\Gamma)(j_2(\Gamma) - g_3(\Gamma))$ facets. We can use these to compute

$$\begin{aligned} f_{123}(\Gamma) &= b_1(\Delta)g_2(\Gamma)g_3(\Gamma) + g_3(\Gamma)j_3(\Gamma) + j_1(\Gamma)(j_2(\Gamma) - g_3(\Gamma)) \\ &\leqslant b_1(\Delta)(f_{23}(\Delta) - j_1(\Gamma)) + g_3(\Gamma)j_3(\Gamma) + j_3(\Gamma)(j_2(\Gamma) - g_3(\Gamma)) \\ &= b_1(\Delta)f_{23}(\Delta) - b_1(\Delta)j_1(\Gamma) + j_2(\Gamma)j_3(\Gamma) \\ &\leqslant b_1(\Delta)f_{23}(\Delta) + j_2(\Gamma)j_3(\Gamma) - \frac{b_1(\Delta)j_1(\Gamma)j_3(\Gamma)}{g_2(\Gamma)}. \end{aligned}$$

Case II B 2: $j_3(\Gamma) < j_1(\Gamma)$

Define Γ_1 by $g_i(\Gamma_1) = g_i(\Gamma)$ for all $i \in [3]$, $p(\Gamma_1) = 3$, and $q(\Gamma_1) = 1$. By Lemma 5.1, $f_{123}(\Gamma) \leqslant f_{123}(\Gamma_1)$. Furthermore, Γ_1 satisfies the bound of this lemma by Case I. □

The next lemma says that we can get close to the bound of Lemma 3.15. While it is not a useful lemma for finding the maximum number of facets in practice, it does give us a worst case approximation that will allow a bound on the number of values of $g_2(\Gamma)$ we could potentially need to check.

Lemma 5.2 *Let Δ be a 3-colored simplicial complex with $f_{12}(\Delta) \leqslant f_{13}(\Delta)$ and $f_{12}(\Delta) \leqslant f_{23}(\Delta)$. Suppose that there is a $\Gamma_0 \in \mathcal{F}(\Delta)$ with $g_1(\Gamma_0) = b_1(\Delta)$ and $r(\Gamma_0) = 2$. Then there is a $\Gamma \in \mathcal{F}(\Delta)$ such that $f_{123}(\Gamma) \geqslant v(\Delta, g_2(\Gamma_0)) - f_{12}(\Delta)$.*

Proof: Define Γ by $g_i(\Gamma) = g_i(\Gamma_0)$ for all $i \in [3]$. If $j_2(\Gamma_0) \geqslant b_1(\Delta)$, then let $p(\Gamma) = 3$ and $q(\Gamma) = 1$. Otherwise, let $p(\Gamma) = 1$ and $q(\Gamma) = 3$. Either Γ is the same complex as Γ_0 or else it swaps $p(\Gamma_0)$ with $q(\Gamma_0)$. This does not affect any of the criteria for $\mathcal{F}(\Delta)$, so $\Gamma \in \mathcal{F}(\Delta)$.

We can compute $b_1(\Delta) = \left\lfloor \frac{f_{12}(\Delta) f_{13}(\Delta)}{f_{23}(\Delta)} \right\rfloor \leqslant \left\lfloor \frac{f_{13}(\Delta) f_{23}(\Delta)}{f_{12}(\Delta)} \right\rfloor = b_3(\Delta)$ because $f_{12}(\Delta) \leqslant f_{23}(\Delta)$. Suppose first that $g_3(\Gamma) \leqslant b_1(\Delta) - 2$. In this case, we have

$$\begin{aligned} f_{13}(\Delta) &\leqslant (g_1(\Gamma)+1)(g_3(\Gamma)+1) \\ &\leqslant (b_1(\Delta)+1)(b_1(\Delta)-1) \\ &= b_1(\Delta)^2 - 1 \\ &< b_1(\Delta)^2 \\ &\leqslant b_1(\Delta) b_3(\Delta) \\ &\leqslant f_{13}(\Delta), \end{aligned}$$

a contradiction. Therefore, $g_3(\Gamma) \geqslant b_1(\Delta) - 1$. Thus, if $j_2(\Gamma) < b_1(\Delta)$, then $j_2(\Gamma) \leqslant g_3(\Gamma)$. As such, in the proof of Lemma 3.15, we are either in case I B or case II A.

As with the previous lemma, it is more convenient to prove the alternative form

$$f_{123}(\Gamma) \geqslant b_1(\Delta) f_{23}(\Delta) + j_2(\Gamma) j_3(\Gamma) - \frac{b_1(\Delta) j_1(\Gamma) j_3(\Gamma)}{g_2(\Gamma)} - f_{12}(\Delta).$$

For simplicity, let

$$z(\Gamma) = b_1(\Delta) f_{23}(\Delta) + j_2(\Gamma) j_3(\Gamma) - \frac{b_1(\Delta) j_1(\Gamma) j_3(\Gamma)}{g_2(\Gamma)} - f_{12}(\Delta),$$

so that we are trying to prove that $f_{123}(\Gamma) \geqslant z(\Gamma)$. We break this into the same cases as before and do not repeat the computations, but only check how far from inequality we are.

If $j_2(\Gamma) \geqslant b_1(\Delta)$, then from the arithmetic of Case I B of Lemma 3.15, we have

$$\begin{aligned} f_{123}(\Gamma) - z(\Gamma) &= b_1(\Delta) j_3(\Gamma) \Big(\frac{j_1(\Gamma)}{g_2(\Gamma)} - 1 \Big) + f_{12}(\Delta) \\ &\geqslant -b_1(\Delta) j_3(\Gamma) + f_{12}(\Delta) \\ &\geqslant f_{12}(\Delta) - g_1(\Gamma) g_2(\Gamma) \geqslant 0. \end{aligned}$$

Similarly, if $j_2(\Gamma) < b_1(\Delta)$, then the arithmetic of Case II A yields

$$\begin{aligned} f_{123}(\Gamma) - z(\Gamma) &= b_1(\Delta) j_1(\Gamma) \Big(\frac{j_3(\Gamma)}{g_2(\Gamma)} - 1 \Big) + f_{12}(\Delta) \\ &\geqslant -b_1(\Delta) j_1(\Gamma) + f_{12}(\Delta) \\ &\geqslant f_{12}(\Delta) - g_1(\Gamma) g_2(\Gamma) \geqslant 0. \qquad \square \end{aligned}$$

Proof of Lemma 3.6: As $b_1(\Delta) = 0$, we have $\sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}} < 1$, or equivalently, $f_{12}(\Delta)f_{13}(\Delta) < f_{23}(\Delta)$. If $f_2(\Delta) < f_{12}(\Delta)$, then we have $\frac{f_{12}(\Delta)}{f_2(\Delta)} > 1$, so $\left\lfloor \frac{f_{12}(\Delta)}{f_2(\Delta)} \right\rfloor \geqslant 1$. Furthermore, $\frac{f_{23}(\Delta)}{f_2(\Delta)} > \frac{f_{23}(\Delta)}{f_{12}(\Delta)} > f_{13}(\Delta)$, so $\left\lfloor \frac{f_{23}(\Delta)}{f_2(\Delta)} \right\rfloor \geqslant f_{13}(\Delta)$. Thus, $\left\lfloor \frac{f_{12}(\Delta)}{f_2(\Delta)} \right\rfloor \left\lfloor \frac{f_{23}(\Delta)}{f_2(\Delta)} \right\rfloor \geqslant f_{13}(\Delta)$, so by Lemma 3.2, $\mathcal{D}(\Delta) = \emptyset$, a contradiction. By the same argument, if $f_3(\Delta) < f_{13}(\Delta)$, then $\mathcal{D}(\Delta) = \emptyset$.

Otherwise, define Γ by $g_1(\Gamma) = 1$, $g_2(\Gamma) = f_{12}(\Delta)$, $g_3(\Gamma) = f_{13}(\Delta)$. Do not give Γ any extra vertices. We have seen that there are enough vertices of each color to do this. There are clearly enough edges of colors 12 and 13. Since $f_{23}(\Gamma) = f_{12}(\Delta)f_{13}(\Delta) < f_{23}(\Delta)$, there are also enough edges of color 23. Hence, Γ is well-defined. Since $f_{123}(\Gamma) = f_{12}(\Delta)f_{13}(\Delta)$, we get $m(\Delta) \geqslant f_{12}(\Delta)f_{13}(\Delta)$.

Conversely, each choice of an edge of color 12 and edge of color 13 specifies at least one vertex of each color, so there can be at most one facet containing these two edges. Each facet must use an edge of each color set, so any $\Gamma_1 \in \mathcal{A}(\Delta)$ can have at most $f_{12}(\Delta)f_{13}(\Delta)$ facets. Therefore, $m(\Delta) \leqslant f_{12}(\Delta)f_{13}(\Delta)$, and so the statement of the lemma follows. $\square$

This next lemma is useful in a worst case approximation of the number of values of $g_2(\Gamma)$ we may need to check.

Lemma 5.3 *Let Δ be a 3-colored simplicial complex with $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$ and let $\Gamma_1, \Gamma_2 \in \mathcal{F}(\Delta)$ such that $g_1(\Gamma_1) = g_1(\Gamma_2) = b_1(\Delta)$ and $r(\Gamma_1) = r(\Gamma_2) = 2$. If $x \in \mathbb{Z}$ such that $g_2(\Gamma_1) < x < g_2(\Gamma_2)$, then there is a complex $\Gamma_3 \in \mathcal{F}(\Delta)$ such that $g_2(\Gamma_3) = x$, $g_1(\Gamma_3) = b_1(\Delta)$ and $r(\Gamma_3) = 2$.*

Proof: Try to define Γ_3 by $g_2(\Gamma_3) = x$, $g_1(\Gamma_3) = \lfloor \frac{f_{12}(\Delta)}{x} \rfloor$, $g_3(\Gamma_3) = \lfloor \frac{f_{23}(\Delta)}{x} \rfloor$, $p(\Gamma_3) = 1$, and $q(\Gamma_3) = 3$. It follows from the definition that there are enough edges of colors 12 and 23 for Γ_3 to be well-defined, and that Γ_3 uses all edges of these two color sets if $f_1(\Delta) > g_1(\Gamma_3)$ and $f_3(\Delta) > g_3(\Gamma_3)$. Since $\Gamma_1 \in \mathcal{F}(\Delta)$, we must have $\mathcal{D}(\Delta) \neq \emptyset$, so Γ_3 satisfies this condition for $\mathcal{F}(\Delta)$.

Because $\Gamma_1, \Gamma_2 \in \mathcal{F}(\Delta)$, we must have $b_1(\Delta)g_2(\Gamma_1) < b_1(\Delta)g_2(\Gamma_2) \leqslant f_{12}(\Delta)$ and $f_{12}(\Delta) \leqslant (b_1(\Delta)+1)g_2(\Gamma_1)$. If $f_1(\Delta) \leqslant b_1(\Delta)$, we would have $f_{12}(\Gamma_1) \leqslant b_1(\Delta)g_2(\Gamma_1) < f_{12}(\Delta)$, a contradiction. Hence, $f_1(\Delta) > b_1(\Delta)$. Furthermore,

$$b_1(\Delta)x < b_1(\Delta)g_2(\Gamma_2) \leqslant f_{12}(\Delta) \leqslant (b_1(\Delta)+1)g_2(\Gamma_1) < (b_1(\Delta)+1)x.$$

That $b_1(\Delta) < \frac{f_{12}(\Delta)}{x} < b_1(\Delta) + 1$ guarantees that $g_1(\Gamma_3) = b_1(\Delta) < f_1(\Delta)$. Similarly, we have

$$g_3(\Gamma_2)x < g_3(\Gamma_2)g_2(\Gamma_2) \leqslant f_{23}(\Delta) \leqslant (g_3(\Gamma_1)+1)g_2(\Gamma_1) < (g_3(\Gamma_1)+1)x.$$

This means $g_3(\Gamma_2) < \frac{f_{23}(\Delta)}{x} < g_3(\Gamma_1) + 1$, from which $g_3(\Gamma_2) \leqslant g_3(\Gamma_3) < g_3(\Gamma_1)$. Since Γ_1 is well-defined, $f_3(\Delta) \geqslant g_3(\Gamma_1) > g_3(\Gamma_3)$.

Next, we can compute

$$g_1(\Gamma_3)g_3(\Gamma_3) < g_1(\Gamma_1)g_3(\Gamma_1) \leqslant f_{13}(\Delta) \leqslant (g_1(\Gamma_2)+1)(g_3(\Gamma_2)+1)$$

$$\leqslant (g_1(\Gamma_3)+1)(g_3(\Gamma_3)+1).$$

This ensures that Γ_3 has enough edges of color 13 to be well-defined because Γ_1 is and enough vertices to use all of the edges because Γ_2 does.

Because Γ_3 uses at most as many vertices of color 2 as Γ_2 and at most as many of colors 1 and 3 as Γ_1, Γ_3 is well-defined. We immediately have $\Gamma_3 \in \mathcal{A}(\Delta)$ by construction. We have seen that Γ_3 uses all available edges of each color set, so $\Gamma_3 \in \mathcal{F}(\Delta)$. □

Proof of Lemma 3.16: We start with some preliminary computations. We wish to find the value of $t > 0$ that maximizes $v(\Delta, t)$. One can readily compute $\frac{\partial}{\partial t} v(\Delta,t) = -b_1(\Delta)f_{13}(\Delta) + \frac{1}{t^2} b_1(\Delta)f_{12}(\Delta)f_{23}(\Delta)$. Setting the derivative equal to zero and solving for t gives $t = s(\Delta)$. Furthermore, $\frac{\partial^2}{\partial t^2} v(\Delta,t) = -\frac{1}{t^3} b_1(\Delta)f_{12}(\Delta)f_{23}(\Delta)$, which is negative for all $t > 0$, so this is a maximum.

Next, we compute how far from maximizing $v(\Delta,t)$ a given value of t is. For the former, if we define z by $t = s(\Delta)+z$, we compute

$$\begin{aligned}
& v(\Delta, s(\Delta)) - v(\Delta, s(\Delta)+z) \\
= \;& b_1(\Delta)f_{23}(\Delta) + f_{12}(\Delta)f_{13}(\Delta) + b_1(\Delta)^2 f_{23}(\Delta) - b_1(\Delta)f_{13}(\Delta)s(\Delta) \\
& -\frac{b_1(\Delta)f_{12}(\Delta)f_{23}(\Delta)}{s(\Delta)} - b_1(\Delta)f_{23}(\Delta) - f_{12}(\Delta)f_{13}(\Delta) - b_1(\Delta)^2 f_{23}(\Delta) \\
& + b_1(\Delta)f_{13}(\Delta)(s(\Delta)+z) + \frac{b_1(\Delta)f_{12}(\Delta)f_{23}(\Delta)}{s(\Delta)+z} \\
= \;& b_1(\Delta)\Big(f_{13}(\Delta)z - \frac{f_{12}(\Delta)f_{23}(\Delta)z}{s(\Delta)(s(\Delta)+z)}\Big) \\
= \;& b_1(\Delta)\Big(f_{13}(\Delta)z - \frac{f_{13}(\Delta)s(\Delta)z}{(s(\Delta)+z)}\Big) \\
= \;& b_1(\Delta)f_{13}(\Delta)z\Big(1 - \frac{s(\Delta)}{(s(\Delta)+z)}\Big) \\
= \;& \frac{b_1(\Delta)f_{13}(\Delta)z^2}{(s(\Delta)+z)}.
\end{aligned}$$

Suppose that there is a complex $\Gamma_0 \in \mathcal{F}(\Delta)$ with $g_1(\Gamma_0) = b_1(\Delta)$, $g_2(\Gamma_0) = b_2(\Delta)$, and $r(\Gamma_0) = 2$. For this complex, we get $s(\Delta)+z = b_2(\Delta)$. Since $b_2(\Delta) = \lfloor s(\Delta) \rfloor$, we get $|z| < 1$. As such, we get $v(\Delta, s(\Delta)+z) = v(\Delta, s(\Delta)) - \frac{b_1(\Delta)f_{13}(\Delta)z^2}{b_2(\Delta)}$. Applying Lemma 5.2 yields

$$\begin{aligned}
f_{123}(\Gamma_0) &\geqslant v(\Delta, s(\Delta)) - \frac{b_1(\Delta)f_{13}(\Delta)z^2}{b_2(\Delta)} - f_{12}(\Delta) \\
&> v(\Delta, s(\Delta)) - \frac{b_1(\Delta)f_{13}(\Delta)}{b_2(\Delta)} - f_{12}(\Delta).
\end{aligned}$$

For simplicity, let $x(\Delta) = \frac{b_1(\Delta)f_{12}(\Delta)}{b_2(\Delta)} + f_{12}(\Delta)$. It thus suffices to check the values of $g_2(\Gamma)$ where $v(\Delta, g_2(\Gamma)) > f_{123}(\Gamma_0)$. We set $g_2(\Gamma) = s(\Delta)+z$ and use Lemma 3.15

compute

$$\begin{aligned}
0 &< v(\Delta, g_2(\Gamma)) - f_{123}(\Gamma_0)\\
&\leqslant v(\Delta, s(\Delta)+z) - v(\Delta, s(\Delta)) + x(\Delta)\\
&= v(\Delta, s(\Delta)) - \frac{b_1(\Delta)f_{13}(\Delta)z^2}{(s(\Delta)+z)} - v(\Delta, s(\Delta)) + x(\Delta)\\
&= x(\Delta) - \frac{b_1(\Delta)f_{13}(\Delta)z^2}{(s(\Delta)+z)}.
\end{aligned}$$

The inequality $x(\Delta) - \frac{b_1(\Delta)f_{13}(\Delta)z^2}{(s(\Delta)+z)} > 0$ clearly holds if $z = 0$ and fails if z gets far enough away from zero. Thus, to find the values of z that make it true, it suffices to find the values that give equality and take the interval between them. We compute $b_1(\Delta)f_{13}(\Delta)z^2 - zx(\Delta) - s(\Delta)x(\Delta) = 0$. The quadratic formula gives

$$z = \frac{x(\Delta) \pm \sqrt{x(\Delta)^2 + 4b_1(\Delta)f_{13}(\Delta)s(\Delta)x(\Delta)}}{2b_1(\Delta)f_{13}(\Delta)}.$$

The difference between the two roots is

$$\begin{aligned}
&\frac{\sqrt{x(\Delta)^2 + 4b_1(\Delta)f_{13}(\Delta)s(\Delta)x(\Delta)}}{b_1(\Delta)f_{13}(\Delta)}\\
= \ &\sqrt{\Big(\frac{x(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)^2 + \frac{4b_1(\Delta)f_{13}(\Delta)s(\Delta)x(\Delta)}{b_1(\Delta)^2 f_{13}(\Delta)^2}}\\
= \ &\sqrt{\Big(\frac{x(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)^2 + \frac{4s(\Delta)x(\Delta)}{b_1(\Delta)f_{13}(\Delta)}}\\
= \ &\sqrt{\Big(\frac{x(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)\Big(4s(\Delta) + \frac{x(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)}\\
= \ &\sqrt{\Big(\frac{1}{b_2(\Delta)} + \frac{f_{12}(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)\Big(4s(\Delta) + \frac{1}{b_2(\Delta)} + \frac{f_{12}(\Delta)}{b_1(\Delta)f_{13}(\Delta)}\Big)}\\
< \ &\sqrt{\frac{2\sqrt{f_{13}(\Delta)}}{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}} + \frac{2f_{12}(\Delta)\sqrt{f_{23}(\Delta)}}{f_{13}(\Delta)\sqrt{f_{12}(\Delta)f_{13}(\Delta)}}}\\
&*\sqrt{\frac{4\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{\sqrt{f_{13}(\Delta)}} + \frac{2\sqrt{f_{13}(\Delta)}}{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}} + \frac{2f_{12}(\Delta)\sqrt{f_{23}(\Delta)}}{f_{13}(\Delta)\sqrt{f_{12}(\Delta)f_{13}(\Delta)}}}\\
= \ &\sqrt{8 + 8\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)^2} + \frac{4f_{13}(\Delta)}{f_{12}(\Delta)f_{23}(\Delta)} + \frac{8}{f_{13}(\Delta)} + \frac{4f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)^3}}\\
< \ &\sqrt{24 + 8\frac{f_{12}(\Delta)f_{23}(\Delta)}{f_{13}(\Delta)^2}}\\
< \ &5 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}
\end{aligned}$$

Above, we used that $f_{23}(\Delta) \leqslant f_{12}(\Delta)f_{13}(\Delta)$ (because $b_1(\Delta) \geqslant 1$) and $b_2(\Delta) \geqslant \frac{1}{2}s_2(\Delta)$ (because $b_2(\Delta) \geqslant b_1(\Delta) \geqslant 1$ and $s_2(\Delta) - b_2(\Delta) < 1$). Hence, there are fewer than $6 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ integers in the interval, and so fewer than $6 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ possible values of $g_2(\Gamma)$ to check.

Above, we assumed that one could take $g_2(\Gamma_0) = b_2(\Delta)$ and $g_1(\Gamma_0) = b_1(\Delta)$. If this is not the case, then by Lemma 5.3, either all $\Gamma \in \mathcal{F}(\Delta)$ with $g_1(\Gamma) = b_1(\Delta)$ have $g_2(\Gamma) > b_2(\Delta)$ or else all have $g_2(\Gamma) < b_2(\Delta)$. Since $g_2(\Gamma)$ is an integer, all such Γ have $g_2(\Gamma)$ on the same side of $s(\Delta)$.

We have seen that $v(\Delta, t)$ attains its maximum at $t = s(\Delta)$ and that $\frac{\partial^2}{\partial t^2}v(\Delta, t) < 0$ for all $t > 0$. Let

$$\begin{aligned} m_2(\Delta) &= \max\{v(\Delta, g_2(\Gamma)) - f_{12}(\Delta) \mid \Gamma \in \mathcal{F}(\Delta), r(\Gamma) = 2, g_1(\Gamma) = b_1(\Delta)\}, \\ c_1(\Delta) &= \min\{g_2(\Gamma) \mid \Gamma \in \mathcal{F}(\Delta), r(\Gamma) = 2, g_1(\Gamma) = b_1(\Delta), \\ &\qquad v(\Delta, g_2(\Gamma)) > m_2(\Delta)\}, \qquad \text{and} \\ c_2(\Delta) &= \max\{g_2(\Gamma) \mid \Gamma \in \mathcal{F}(\Delta), r(\Gamma) = 2, g_1(\Gamma) = b_1(\Delta), \\ &\qquad v(\Delta, g_2(\Gamma)) > m_2(\Delta)\}. \end{aligned}$$

Suppose that $c_1(\Delta) > b_2(\Delta)$. If $c_2(\Delta) - c_1(\Delta) \geqslant 5 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$, then

$$\begin{aligned} & v(\Delta, c_1(\Delta)) - v(\Delta, c_2(\Delta)) \\ > \; & v(\Delta, s(\Delta)) - v(\Delta, s(\Delta) + c_2(\Delta) - c_1(\Delta)) \qquad (\text{because } \tfrac{\partial^2}{\partial t^2}v(\Delta, t) < 0) \\ > \; & v(\Delta, s(\Delta)) - v\left(\Delta, s(\Delta) + 5 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}\right) \\ > \; & v(\Delta, s(\Delta)) - v\left(\Delta, s(\Delta) + \frac{x(\Delta) + \sqrt{x(\Delta)^2 + 4b_1(\Delta)f_{13}(\Delta)s(\Delta)x(\Delta)}}{2b_1(\Delta)f_{13}(\Delta)}\right) \\ = \; & f_{12}(\Delta). \end{aligned}$$

This gives that $m_2(\Delta) \geqslant v(\Delta, c_1(\Delta)) - f_{12}(\Delta) > v(\Delta, c_2(\Delta))$, a contradiction. Similarly, if $c_2(\Delta) < b_2(\Delta)$ and $c_2(\Delta) - c_1(\Delta) \geqslant 5 + 2\sqrt{2}\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$, we get $v(\Delta, c_2(\Delta)) - v(\Delta, c_1(\Delta)) > f_{12}(\Delta)$, which gives that $m_2(\Delta) \geqslant v(\Delta, c_2(\Delta)) - f_{12}(\Delta) > v(\Delta, c_1(\Delta))$, a contradiction. Therefore, if either $c_1(\Delta) > b_2(\Delta)$ or $c_2(\Delta) < b_2(\Delta)$, the lemma holds. It is clear from the definitions that $c_1(\Delta) \leqslant c_2(\Delta)$, so the only other possibility is that $c_1(\Delta) \leqslant b_2(\Delta) \leqslant c_2(\Delta)$. In this case, by Lemma 5.3, there is such a complex Γ_0, and so we have already seen that the lemma holds. □

Proof of Lemma 3.13: We must have $f_1(\Delta) \geqslant g_1(\Gamma)$, and so if there is a Γ with $b_1(\Delta) = g_1(\Gamma)$, then we must have $f_1(\Delta) \geqslant b_1(\Delta)$.

Suppose that $f_1(\Delta) = b_1(\Delta)$. Note that this means that Γ cannot have an extra vertex of color 1. Additionally, since $r(\Gamma) = 2$, Γ does not have an extra vertex of color 2. There are few enough edges of color 12 for Γ to use them all if and only if $f_1(\Delta)g_2(\Gamma) \geqslant f_{12}(\Delta)$. There are enough edges of color 12 for Γ to be well-defined if and only if $b_1(\Delta)g_2(\Gamma) \leqslant f_{12}(\Delta)$. Hence, equality must hold, and so we get $g_2(\Gamma) = \frac{f_{12}(\Delta)}{f_1(\Delta)}$.

Otherwise, we have $f_1(\Delta) > b_1(\Delta)$. There are enough vertices of color 2 to define the complex if and only if $g_2(\Gamma) \leqslant f_2(\Delta)$, which is condition (8). There are enough vertices of color 1 to handle the edges of color 12 if and only if $f_1(\Delta)g_2(\Gamma) \geqslant f_{12}(\Delta)$, or equivalently, $g_2(\Gamma) \geqslant \frac{f_{12}(\Delta)}{f_1(\Delta)}$, which is condition (2). There are enough vertices of color 3 to deal with the edges of color 23 if and only if $f_3(\Delta)g_2(\Gamma) \geqslant f_{23}(\Delta)$, or equivalently, $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{f_3(\Delta)}$, which is condition (5).

There are enough edges of color 12 for the complex to be well-defined if and only if $b_1(\Delta)g_2(\Gamma) \leqslant f_{12}(\Delta)$, or equivalently, $g_2(\Gamma) \leqslant \frac{f_{12}(\Delta)}{b_1(\Delta)}$, which is condition (6). There are enough edges of color 23 for the complex to be well-defined if and only if $g_2(\Gamma)g_3(\Gamma) \leqslant f_{23}(\Delta)$, or equivalently, $g_3(\Gamma) \leqslant \frac{f_{23}(\Delta)}{g_2(\Gamma)}$. There are enough edges of color 13 for the complex to be well-defined if and only if $b_1(\Delta)g_3(\Gamma) \leqslant f_{13}(\Delta)$, or equivalently, $g_3(\Gamma) \leqslant \frac{f_{13}(\Delta)}{b_1(\Delta)}$. Since $g_3(\Gamma)$ is an integer, we can take floors of its upper bounds to get $g_3(\Gamma) \leqslant \left\lfloor \frac{f_{23}(\Delta)}{g_2(\Gamma)} \right\rfloor$ and $g_3(\Gamma) \leqslant \left\lfloor \frac{f_{13}(\Delta)}{b_1(\Delta)} \right\rfloor$.

There are few enough edges of color 12 for Γ to use them all if and only if $f_{12}(\Delta) \leqslant (b_1(\Delta)+1)g_2(\Gamma)$, or equivalently, $g_2(\Gamma) \geqslant \frac{f_{12}(\Delta)}{b_1(\Delta)+1}$, which is condition (3). There are few enough edges of color 23 for the complex to use them all if and only if $g_2(\Delta)(g_3(\Gamma)+1) \geqslant f_{23}(\Delta)$, or equivalently, $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{g_3(\Delta)+1}$.

There are few enough edges of color 13 for Γ to use them all if and only if both $f_{13}(\Delta) \leqslant f_3(\Delta)(b_1(\Delta)+1)$ and $(b_1(\Delta)+1)(g_3(\Gamma)+1) \geqslant f_{13}(\Delta)$. The former is condition (1), and the latter is equivalent to $g_3(\Gamma) \geqslant \frac{f_{13}(\Delta)}{b_1(\Delta)+1} - 1$. Since $g_3(\Gamma)$ is an integer, we can take the ceiling and get $g_3(\Gamma) \geqslant \left\lceil \frac{f_{13}(\Delta)}{b_1(\Delta)+1} \right\rceil - 1$.

Thus, we have that in order to make $g_3(\Gamma)$ compatible with the choice of $g_1(\Gamma) = b_1(\Delta)$ and not use more vertices of color 3 than are available, our bounds are $g_3(\Gamma) \geqslant \left\lceil \frac{f_{13}(\Delta)}{b_1(\Delta)+1} \right\rceil - 1$, $g_3(\Gamma) \leqslant \left\lfloor \frac{f_{13}(\Delta)}{b_1(\Delta)} \right\rfloor$, and $g_3(\Gamma) \leqslant f_3(\Delta)$. In order to then make $g_2(\Gamma)$ compatible with $g_3(\Gamma)$, our bounds are $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{g_3(\Gamma)+1}$ and $g_2(\Gamma)g_3(\Gamma) \leqslant f_{23}(\Delta)$, from which $g_2(\Gamma) \leqslant \frac{f_{23}(\Delta)}{g_3(\Gamma)}$. We plug in our bounds on $g_3(\Gamma)$ to get $g_2(\Gamma) \leqslant \frac{f_{23}(\Delta)}{\left\lceil \frac{f_{13}(\Delta)}{b_1(\Delta)+1} \right\rceil - 1}$ and $g_2(\Gamma) \geqslant \frac{f_{23}(\Delta)}{\left\lfloor \frac{f_{13}(\Delta)}{b_1(\Delta)} \right\rfloor + 1}$, which are conditions (7) and (4), respectively. □

6 Some examples

Theorem 3.17 explained how to compute $m(\Delta)$, and thereby characterize the flag f-vectors of three-colored complexes. In this section, we give some examples of how the procedure works, with both some typical cases and some extremal ones to argue that it would likely be impractical to greatly improve upon Theorem 3.17, so it a satisfactory solution to the problem. We start with a few trivial examples.

Example 6.1 Let $f_1(\Delta) = 3$, $f_2(\Delta) = 5$, $f_3(\Delta) = 7$, $f_{12}(\Delta) = 23$, $f_{13}(\Delta) = 14$, and

$f_{23}(\Delta) = 18$. We compute $f_{12}(\Delta) = 23 > 15 = f_1(\Delta)f_2(\Delta)$, so there is no 3-colored complex Δ having the given face numbers, and we stop.

Example 6.2 Let $f_1(\Delta) = 3$, $f_2(\Delta) = 5$, $f_3(\Delta) = 7$, $f_{12}(\Delta) = 13$, $f_{13}(\Delta) = 16$, and $f_{23}(\Delta) = 18$. The inequalities of part (1) of Theorem 3.17 hold, so we move on. In part (2), we compute

$$\left\lfloor \frac{f_{12}(\Delta)}{f_1(\Delta)} \right\rfloor \left\lfloor \frac{f_{13}(\Delta)}{f_1(\Delta)} \right\rfloor = \left\lfloor \frac{13}{3} \right\rfloor \left\lfloor \frac{16}{3} \right\rfloor = 20 \geqslant 18 = f_{23}(\Delta).$$

Thus, Lemma 3.2 asserts that $m(\Delta) = f_1(\Delta)f_{23}(\Delta) = (3)(18) = 54$.

The proof of the Lemma 3.2 also explains how to find a complex Γ with the desired flag f-numbers and 54 facets. We start with 3 vertices of color 1, 4 vertices of color 2, and 5 vertices of color 3, and fill in all edges connecting two of these vertices of distinct colors, except that there are two edges that would connect a vertex of color 2 to one of color 3 that are missing. This gives us exactly 54 facets. We then add the remaining vertices and edges to have the desired complex.

In this case, we are able to attain Walker's bound of $f_{123}(\Delta) \leqslant f_1(\Delta)f_{23}(\Delta) = 54$.

Example 6.3 Let $f_1(\Delta) = 17$, $f_2(\Delta) = 31$, $f_3(\Delta) = 25$, $f_{12}(\Delta) = 15$, $f_{13}(\Delta) = 12$, and $f_{23}(\Delta) = 279$. The inequalities of point (1) hold and those of point (2) fail, so neither settles the problem and we move on. Step (3) advises us to ensure that $f_{12}(\Delta) \leqslant f_{13}(\Delta) \leqslant f_{23}(\Delta)$. This does not hold with the numbers as given, as $15 > 12$. We want to rearrange the colors such that $f_{12}(\Delta) = 12$, $f_{13}(\Delta) = 15$, and $f_{23}(\Delta) = 279$. This can be done by swapping colors 2 and 3, which also gives us $f_2(\Delta) = 25$ and $f_3(\Delta) = 31$.

Step (4) starts by computing

$$b_1(\Delta) = \left\lfloor \sqrt{\frac{f_{12}(\Delta)f_{13}(\Delta)}{f_{23}(\Delta)}} \right\rfloor = \left\lfloor \sqrt{\frac{(12)(15)}{279}} \right\rfloor \approx \lfloor .803 \rfloor = 0.$$

Since $b_1(\Delta) = 0$, Lemma 3.6 tells us that

$$m(\Delta) = f_{12}(\Delta)f_{13}(\Delta) = (12)(15) = 180.$$

We can build a complex with exactly 180 facets by starting with a complete tripartite graph on one vertex of color 1, 12 vertices of color 2, and 15 vertices of color 3, and then adding the remaining vertices and edges any way you like. For comparison, the smallest of Walker's bounds is $f_{123}(\Delta) \leqslant \sqrt{f_{12}(\Delta)f_{13}(\Delta)f_{23}(\Delta)} \approx 224$.

Example 6.4 Let $f_{12}(\Delta) = f_{13}(\Delta) = f_{23}(\Delta) = 3$ and $f_1(\Delta) = f_2(\Delta) = f_3(\Delta) = 2$. Steps (1) and (2) do not solve the problem. The number of edges are sorted as in step (3). For part (4), we compute $b_1(\Delta) = b_2(\Delta) = b_3(\Delta) = 1$. Next, we construct complexes as in the other steps.

step	$g_1(\Gamma)$	$g_2(\Gamma)$	$g_3(\Gamma)$	$r(\Gamma)$	$f_{123}(\Gamma)$
5	1	2	2	1	undefined
5	2	1	2	2	undefined
6	1	2	1	2	4
6	1	3	1	2	undefined
7	2	1	1	1	4
7	3	1	1	1	undefined
8	2	1	1	1	previous
8	3	1	1	1	undefined
9	1	1	2	3	4
12	1	2	1	2	previous

In this example, we found a valid simplicial complex at five steps. In two of the five cases, we had exactly the same constants as at a previous step, so there was no need to construct the complex again. In the three cases where we did construct the complex, we ended up getting exactly the same complex all three times, but with the vertices and edges merely added in a different order. This happened because in this example, there is only one color-shifted complex with the given flag f-numbers. This phenomenon of constructing the same complex in multiple ways can easily happen if $g_i(\Gamma) \approx b_i(\Delta)$ for all $i \in [3]$. We clearly have $m(\Delta) = 4$. For comparison, the smallest of Walker's bounds is $f_{123}(\Delta) \leqslant 3\sqrt{3} \approx 5.2$.

The next example is a typical use of the full Theorem 3.17. It has few enough complexes in $\mathcal{F}(\Delta)$ that it is easy to compute them all, so that Lemma 3.16 doesn't particularly matter.

Example 6.5 Let $f_1(\Delta) = 533$, $f_2(\Delta) = 471$, $f_3(\Delta) = 818$, $f_{12}(\Delta) = 4972$, $f_{13}(\Delta) = 5311$, and $f_{23}(\Delta) = 5630$. We can quickly compute that steps (1) and (2) do not solve the problem, and the numbers of edges are already sorted as step (3) dictates. Step (4) asks us to compute $b_1(\Delta) = 68$, $b_2(\Delta) = 72$, and $b_3(\Delta) = 77$.

The remaining steps essentially ask us to brute force the various complexes in $\mathcal{F}(\Delta)$. We list the step at which we construct each complex, the parameters of the complex, and the number of facets. When we hit on parameters used earlier, we note it and do not reconstruct a complex that we have already used.

step	$g_1(\Gamma)$	$g_2(\Gamma)$	$g_3(\Gamma)$	$r(\Gamma)$	$f_{123}(\Gamma)$
5	68	73	78	1	undefined
5	69	72	78	2	undefined
6	68	73	77	2	382896
8	69	72	76	1	382736
9	6	6	818	3	not in $\mathcal{F}(\Delta)$
11	68	73	77	2	previous

There is no complex for step (7) because the condition of Lemma 3.10 is violated. We could have quickly discarded the two undefined complexes of step (5) on the basis that it has $g_i(\Gamma) > b_i(\Delta)$ for two values of i. We do not bother to invoke Lemma 3.16 for steps (11)-(13), as there are few enough complexes that we can find them all by brute force. By inspection, $m(\Delta) = 382896$. For comparison, the tightest of Walker's bounds is approximately $f_{123}(\Delta) \leqslant 385574$.

The next example gives a typical demonstration of the power of Lemma 3.16. The class $\mathcal{F}(\Delta)$ is huge, but this lemma lets us compute few enough complexes that we can list them all here.

Example 6.6 Let $f_1(\Delta) = 13$, $f_2(\Delta) = 5471$, $f_3(\Delta) = 3818$, $f_{12}(\Delta) = 1843$, $f_{13}(\Delta) = 2157$, and $f_{23}(\Delta) = 3150248$. We can quickly compute that steps (1) and (2) do not solve the problem, and the numbers of edges are already sorted as step (3) dictates. Step (4) asks us to compute $b_1(\Delta) = 1$, $b_2(\Delta) = 1640$, and $b_3(\Delta) = 1920$.

This time, there aren't very many possible complexes outside of steps (11)-(13), but in these final steps, we get complexes in $\mathcal{F}(\Delta)$ with $g_2(\Gamma)$ ranging from 1460 to 1843. A direct brute force approach would require checking several hundred complexes. Fortunately, Lemma 3.15 immediately allows us to limit the computations to values of $g_2(\Delta)$ ranging from 1637 to 1644.

step	$g_1(\Gamma)$	$g_2(\Gamma)$	$g_3(\Gamma)$	$r(\Gamma)$	$f_{123}(\Gamma)$
5	1	1842	2156	1	undefined
5	1	1640	1920	2	3198156
6	1	1640	1920	2	previous
9	0	825	3818	3	not in $\mathcal{F}(\Delta)$
11	1	1640	1920	2	previous
11	1	1641	1919	2	3198122
11	1	1642	1918	2	3198086
11	1	1643	1917	2	3198048
11	1	1644	1916	2	3198008
11	1	1639	1922	2	3198098
11	1	1638	1923	2	3198013
11	1	1637	1924	2	3198040

By inspection, $m(\Delta) = 3198156$. For comparison, the tightest of Walker's bounds is approximately $f_{123}(\Delta) \leqslant 3538833$.

If one of the complexes computed later had more facets than the ones we computed before reaching step (11), that could have further restricted how many complexes we would have to compute in step (11). Regardless, this is still far more efficient than having to compute the number of facets of every single complex in $\mathcal{F}(\Delta)$. Note that it was sufficient to try 8 complexes. For comparison, Theorem 3.17 said that we would need to do the computations for at most 114 complexes.

Finally, we wish to note that finding the complex with the maximal number of vertices can force $g_2(\Delta)$ to be arbitrarily far away from $b_2(\Delta)$. More precisely, the difference can be on the order of $\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ even as this quantity becomes arbitrarily large. Thus, the bound of Theorem 3.17 on how many complexes we need to check is off by at worst a constant factor.

Example 6.7 Pick any real number t and let $f_1(\Delta) = 2$, $f_2(\Delta) = \lfloor 100^t \rfloor$, $f_3(\Delta) = \lfloor 100^t \rfloor$, $f_{12}(\Delta) = \lfloor 100^t \rfloor$, $f_{13}(\Delta) = \lfloor 100^t + 2(10)^t \rfloor$, and $f_{23}(\Delta) = \left\lfloor \lfloor \frac{2}{3} f_{12}(\Delta) \rfloor (\lfloor \frac{2}{3} f_{13}(\Delta) \rfloor + .45) \right\rfloor$. We can compute $b_1(\Delta) = 1$, $b_2(\Delta) \approx \frac{2}{3} f_{12}(\Delta)$, and $b_3(\Delta) \approx \frac{2}{3} f_{13}(\Delta)$. If we apply Theorem 3.17 compute the value of $g_2(\Delta)$ that maximizes the number of facets, we usually get $b_2(\Delta) - g_2(\Delta) \approx .35 \frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$.

In the previous example, $f_{12}(\Delta)$ was close to $f_{13}(\Delta)$, which means that $\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)} \approx \sqrt{\frac{f_{23}(\Delta)}{f_{13}(\Delta)}}$. The next example sets generalizes the previous example and shows that the number of complexes required can still be on the order of $\frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}$ even as $\frac{f_{13}(\Delta)}{f_{12}(\Delta)}$ is arbitrarily large.

Example 6.8 Pick any positive real number t and any integer w and let $f_1(\Delta) = 2$, $f_2(\Delta) = \lfloor 100^t \rfloor$, $f_3(\Delta) = \lfloor w100^t \rfloor$, $f_{12}(\Delta) = \lfloor 100^t \rfloor$, $f_{13}(\Delta) = \lfloor w100^t + 2\sqrt{w}(10)^t \rfloor$, and $f_{23}(\Delta) = \left\lfloor \lfloor \frac{2}{3} f_{12}(\Delta) \rfloor (\lfloor \frac{2}{3} f_{13}(\Delta) \rfloor + .45) \right\rfloor$. We can compute $b_1(\Delta) = 1$, $b_2(\Delta) \approx \frac{2}{3} f_{12}(\Delta)$, and $b_3(\Delta) \approx \frac{2}{3} f_{13}(\Delta)$. Furthermore, if t is large enough that $\frac{f_{13}(\Delta)}{f_{12}(\Delta)} \approx w$, the complex that maximizes $f_{123}(\Gamma)$ has

$$g_2(\Gamma) \approx b_2(\Gamma) - .23 \frac{10^t}{\sqrt{w}} \approx b_2(\Gamma) - .35 \frac{\sqrt{f_{12}(\Delta)f_{23}(\Delta)}}{f_{13}(\Delta)}.$$

In these examples, in order for the complex Γ that maximizes the number of facets to have $g_2(\Gamma)$ far away from $b_2(\Delta)$, it is necessary that many consecutive possible values of $g_2(\Gamma)$ have $j_1(\Gamma)$ much larger than 0 and much smaller than $g_2(\Gamma)$. If this happens, then decreasing $g_2(\Gamma)$ by 1 increases $g_3(\Gamma)$ by the same amount (w in the above example) many consecutive times. This additional structure makes it easy to get a formula for $f_{123}(\Gamma)$ as a function of $g_2(\Gamma)$ that holds for many consecutive values of $g_2(\Gamma)$, which can greatly reduce the computations needed to find $m(\Delta)$ in the particularly bad cases where Theorem 3.17 calls for constructing a large number of simplicial complexes. Thus, even the worst cases are not nearly so bad as they seem.

Of course, one could still hope for a quick and clever solution to this problem as has happened with some previous characterizations of f-vectors of various classes of complexes. The next example explains why an easy characterization is improbable, as adding one extra vertex or edge can dramatically change the complex that maximizes the number of facets.

Example 6.9 Let $f_1(\Delta) = 2$, $f_2(\Delta) = 6683$, $f_3(\Delta) = 7000$, $f_{12}(\Delta) = 10000$, $f_{13}(\Delta) = 10200$, and $f_{23}(\Delta) = 45331745$. We can compute that $m(\Delta) = 56664978$. Furthermore, there is only one complex $\Gamma \in \mathcal{F}(\Delta)$ such that $f_{123}(\Gamma) = 56664978$, and it has $p(\Gamma) = 1$, $q(\Gamma) = 3$, $g_1(\Gamma) = 1$, $g_2(\Gamma) = 6683$, and $g_3(\Gamma) = 6783$.

If we set $f_2(\Delta) = 6682$ and leave the rest of the flag f-numbers unchanged, this obviously excludes the previously optimal complex. This time, we get $m(\Delta) = 56664977$, which corresponds to two complexes $\Gamma_1, \Gamma_2 \in \mathcal{F}(\Delta)$. The two complexes are defined by $p(\Gamma_1) = 3$, $q(\Gamma_1) = 1$, $g_1(\Gamma_1) = 1$, $g_2(\Gamma_1) = 6643$, $g_3(\Gamma_1) = 6823$, $p(\Gamma_2) = 2$, $q(\Gamma_2) = 1$, $g_1(\Gamma_2) = 1$, $g_2(\Gamma_2) = 6642$, and $g_3(\Gamma_2) = 6824$. What happened in this example is that $g_2(\Gamma)$ for the unique $\Gamma \in \mathcal{D}(\Delta)$ was quite far to one side of $b_2(\Delta) = 6666$, and changing the number of allowed vertices of one color by 1 made it so that there were two complexes $\Gamma_1, \Gamma_2 \in \mathcal{D}(\Delta)$, both of which $g_2(\Gamma_1)$ and $g_2(\Gamma_2)$ quite far on the other side of $b_2(\Delta)$.

Furthermore, we can get similar results by adding one edge. Let $f_{13}(\Delta) = 10201$ and leave the rest of the flag f-numbers the same as in the original example. This time, we get $m(\Delta) = 56668334$, and there are again two complexes $\Gamma_1, \Gamma_2 \in \mathcal{F}(\Delta)$ such that $f_{123}(\Gamma_1) = f_{123}(\Gamma_2) = 56668334$. These two complexes are defined by exactly the same parameters as Γ_1 and Γ_2 had in the previous paragraph; the extra edge merely adds some extra facets. This time, the big change in the structure of the complex is not due to a cap on the number of vertices; the same complexes would still be the only ones in $\mathcal{D}(\Delta)$ even if $f_2(\Delta)$ were greatly increased. One can still define Γ by the same parameters as before, but this time, $f_{123}(\Gamma) = 56668295 < m(\Delta)$.

This same behavior also occurs with smaller numbers, but if $g_2(\Gamma)$ differs from $b_2(\Delta)$ by only 1 or 2, it is much less clear what happened.

7 More colors

Having characterized the flag f-vectors of 3-colored complexes, it is natural to ask whether the characterization carries over to more colors. Unfortunately, even the case of four colors is dramatically more complicated than that of three.

The basic approach of the three color case does carry over, however. Recall that we started by ignoring the discreteness of faces and allowing non-integer numbers of vertices. The same scheme can be done with more colors, and is along the lines of what Walker did in [7].

If given a proposed flag f-vector on n colors $\{f_S\}_{S\subseteq[n]}$, one can propose that the faces of color set S be a complete $|S|$-partite complex on some vertices of each color of S. That is, if $S = \{i_1, i_2, \ldots, i_n\}$, we can suppose that the faces of color set S consist of all ways to choose one vertex out of $f_S^{i_1}$ of color set i_1, one vertex out of $f_S^{i_2}$ of color set i_2, and so forth, with the restriction that $f_S = f_S^{i_1} f_S^{i_2} \ldots f_S^{i_n}$. The simplicial complex restriction that any subface of a face must itself be a face corresponds to the requirement $f_T^i \geqslant f_S^i$ for every $i \in T$ and $T \subset S$.

As Walker did, we can take the logarithms of both sides and get $\log(f_S) = \log(f_S^{i_1}) + \log(f_S^{i_2}) + \ldots + \log(f_S^{i_n})$. This turns the problem into a linear programming problem of maximizing $\log(f_{[n]})$ subject to the known values of $\log(f_S)$ and the inequalities $f_T^i \geqslant f_S^i$.

If one can find the optimal solution in the continuous case, one could hope that the optimal solution in the discrete case would be nearby.

Unfortunately, not only is it unclear how to find an efficient solution in the discrete case, but with four or more colors, having a solution in the continuous case doesn't even guarantee that there is a solution in the discrete case. As we saw earlier, if we set $f_{12}(\Delta) = f_{13}(\Delta) = f_{23}(\Delta) = 3$, the optimal solution in the continuous case is $f_{123} = 3\sqrt{3} > 5$, but the discrete case only allows 4 facets. If we use these same numbers as part of a flag f-vector for a four-colored complex and try to require $f_{123}(\Delta) = 5$, we may well find solutions in the continuous case, but there will be no solution in the discrete case. Unlike the case of three colors, faces of dimension two are no longer facets, and cannot be ignored simply by posing the problem as one of maximizing the number of facets.

Regardless of whether this method can be extended to higher dimensions, it does provide a non-trivial class of examples where the exact characterization is known. Any proposed theorem toward characterizing the flag f-vectors of colored complexes or the flag h-vectors of balanced Cohen-Macaulay complexes or balanced shellable complexes can now be checked against the known, exact result in the case of three colors.

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