On Ryser's conjecture

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Abstract

Motivated by an old problem known as Ryser's Conjecture, we prove that for $r = 4$ and $r = 5$, there exists $\epsilon > 0$ such that every r-partite r-uniform hypergraph $\mathcal{H}$ has a cover of size at most $(r - \epsilon)\nu(\mathcal{H})$, where $\nu(\mathcal{H})$ denotes the size of a largest matching in $\mathcal{H}$.

1 Introduction

In this paper we are concerned with a packing and covering problem in hypergraphs. A *hypergraph* consists of a vertex set V and a set $\mathcal{H}$ of *edges*, where each edge is a nonempty subset of $V = V(\mathcal{H})$. We say $\mathcal{H}$ has *rank* r if the largest size of an edge is r, and that $\mathcal{H}$ is r-*uniform* if every edge has size r. The *packing number* (also called *matching number*) $\nu(\mathcal{H})$ of $\mathcal{H}$ is the size of a largest matching in $\mathcal{H}$, where a *matching* is a set of pairwise disjoint edges in $\mathcal{H}$. The *covering number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the size of a smallest cover of $\mathcal{H}$, where a *cover* is a subset $W \subset V$ such that every edge of $\mathcal{H}$ contains a vertex of W. It is clear that if $\mathcal{H}$ has rank r then $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$, and this is attained for example by the complete r-uniform hypergraph $\mathcal{K}^r_{2r-1}$ with $2r - 1$ vertices, which has $\nu(\mathcal{K}^r_{2r-1}) = 1$ and $\tau(\mathcal{K}^r_{2r-1}) = r$.

Our focus here is on a long-standing open problem known as Ryser's Conjecture, which states that if $\mathcal{H}$ is an r-partite r-uniform hypergraph then $\tau(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H})$ (see e.g. [4, 9]; a stronger version of the conjecture was proposed by Lovász [6]). Here $\mathcal{H}$ being r-*partite* means that its vertex set has a partition $V_1 \cup \cdots \cup V_r$ and every edge contains exactly one vertex of each V_i. When $r = 2$ this is the classical theorem of König, and for $r = 3$, after a number of partial results [8, 10, 5], the conjecture was proved by Aharoni

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[1]. Apart from these two cases, very little is known about the problem. If true, the statement is best possible whenever $r-1$ is a prime power (see e.g. [9]). Until now no nontrivial bound of the form $\tau(\mathcal{H}) \leq (r-\epsilon)\nu(\mathcal{H})$ for $\epsilon > 0$ and any $r \geq 4$ was known.

A hypergraph $\mathcal{H}$ is said to be *intersecting* if $\nu(\mathcal{H}) = 1$. Even for intersecting hypergraphs, Ryser's Conjecture is open for all $r \geq 6$. There are many examples showing the result would be best possible in this case, and they can be quite sparse (see [7]). For $r \leq 5$, however, the conjecture has been proved in the special case of intersecting hypergraphs.

Theorem 1.1. *(Tuza [9]) If $\mathcal{H}$ is an intersecting r-partite hypergraph of rank r and $r \leq 5$ then $\tau(\mathcal{H}) \leq r-1$.*

Our aim in this paper is to prove the following theorem, the proof of which depends on Theorem 1.1, and thus give a nontrivial upper bound for Ryser's problem in the cases $r = 4$ and $r = 5$.

Theorem 1.2. *For each of $r = 4$ and $r = 5$, there exists a positive constant ϵ such that $\tau(\mathcal{H}) \leq (r-\epsilon)\nu(\mathcal{H})$ for every r-partite r-uniform hypergraph $\mathcal{H}$.*

2 General r

We begin the proof of Theorem 1.2 in this section, arguing in terms of general r. We then complete the proof for $r = 4$ and $r = 5$ respectively in the next two sections.

Let $\mathcal{J}$ be an r-partite r-uniform hypergraph, with a fixed partition $V_1 \cup \ldots \cup V_r$. Let $\mathcal{B}$ be a matching of size $\nu(\mathcal{J})$ in $\mathcal{J}$. It is clear that $V(\mathcal{B})$ is a cover of $\mathcal{J}$ of size $r\nu(\mathcal{J})$. For $B_j \in \mathcal{B}$ we let $\mathcal{H}_j$ denote the set of edges of $\mathcal{J}$ that intersect $V(\mathcal{B})$ only in vertices of B_j. Note then that $\mathcal{H}_j$ is intersecting and $B_j \in \mathcal{H}_j$.

We call an edge $A \in \mathcal{J}$ *bad* if $A \cap V(\mathcal{B}) = \{v\}$ for some v. The vertex v is also called *bad*, and we say A is *i-bad* where v is in the ith colour class V_i of the r-partition of $\mathcal{J}$. Note that each bad edge is in $\mathcal{H}_j$ for some j. Let $\mathcal{B}_1 = \{B_j \in \mathcal{B} : B_j \text{ has } r \text{ bad vertices}\}$.

Lemma 2.1. *If $\tau(\mathcal{J}) > (r - 1/2r)|\mathcal{B}|$ then $|\mathcal{B}_1| > |\mathcal{B}|/2$.*

Proof. Suppose that $|\mathcal{B}_1| \leq |\mathcal{B}|/2$. Then there is a colour class i such that at least $|\mathcal{B}|/2r$ of the $B_j \notin \mathcal{B}_1$ have no i-bad vertex. Let $\mathcal{B}^*$ denote the set of these B_j. But then $\bigcup_{B_j \notin \mathcal{B}^*} B_j \cup \bigcup_{B_j \in \mathcal{B}^*} B_j \setminus V_i$ is a cover of $\mathcal{J}$ of size at most $r(|\mathcal{B}| - |\mathcal{B}^*|) + (r-1)|\mathcal{B}^*| \leq (r - 1/2r)|\mathcal{B}|$. □

Lemma 2.1 indicates how our proof of Theorem 1.2 will proceed. Either $\mathcal{J}$ has a suitably small cover, or we can find a special subset of $\mathcal{B}$ whose size is a positive proportion of $|\mathcal{B}|$ (in this case $\mathcal{B}_1$ which is at least half of $\mathcal{B}$) about which we can make a further assumption. We may then cover all edges of $\mathcal{J}$ that intersect any edge of $\mathcal{B}$ that is not in the special subset by taking every vertex of every edge of $\mathcal{B}$ not in the special subset. This will not change the hypergraphs $\mathcal{H}_j$, or the notion of bad, for the edges of $\mathcal{J}$ that remain. We then focus on showing that the remaining edges have a suitably small cover (in this case of size at most $(r-\alpha)|\mathcal{B}_1|$ for some fixed positive α). In our proof of Theorem 1.2 we will apply this procedure $r+2$ times for $r=4$, and $r+3$ times for $r=5$.

By Lemma 2.1 we may assume that $|\mathcal{B}_1| > |\mathcal{B}|/2$. As outlined in the previous paragraph, we let $\mathcal{J}_1 = \{A \in \mathcal{J} : A \cap B_j = \emptyset \text{ for all } B_j \in \mathcal{B} \setminus \mathcal{B}_1\}$. Then $\nu(\mathcal{J}_1) = |\mathcal{B}_1|$, and $\tau(\mathcal{J}) \leq r(|\mathcal{B}| - |\mathcal{B}_1|) + \tau(\mathcal{J}_1)$.

Lemma 2.2. *If $\tau(\mathcal{J}_1) > (r-1/2)|\mathcal{B}_1|$ then there is a matching of 1-bad edges in $\mathcal{J}_1$ of size at least $|\mathcal{B}_1|/2r$.*

Proof. Let $\mathcal{M} = \{M_1, \ldots, M_t\}$ be a maximum matching of 1-bad edges in $\mathcal{J}_1$. Note that since each $\mathcal{H}_j$ is intersecting, all edges of $\mathcal{M}$ are in distinct $\mathcal{H}_j$, say $\mathcal{H}_1, \ldots, \mathcal{H}_t$. Then

$$\bigcup_{j=1}^{t}(M_j \cup B_j) \cup \bigcup_{j>t} B_j \setminus V_1$$

is a cover of $\mathcal{J}_1$ of size at most $(2r-1)|\mathcal{M}| + (r-1)(|\mathcal{B}_1| - |\mathcal{M}| = (r-1)|\mathcal{B}_1| + r|\mathcal{M}|)$. If $|\mathcal{M}| < |\mathcal{B}_1|/2r$ then this is at most $(r-1/2)|\mathcal{B}_1|$. □

By Lemma 2.2 we may assume that there is a matching $\mathcal{M}$ of 1-bad edges in $\mathcal{J}_1$ of size at least $|\mathcal{B}_1|/2r$. Let $\mathcal{B}_2 = \{B_j \in \mathcal{B}_1 : B_j \cap M_k \neq \emptyset \text{ for some } M_k \in \mathcal{M}\}$. Then $|\mathcal{B}_2| = |\mathcal{M}| \geq |\mathcal{B}_1|/2r$. Let $\mathcal{J}_2 = \{A \in \mathcal{J}_1 : A \cap B_j = \emptyset \text{ for all } B_j \in \mathcal{B}_1 \setminus \mathcal{B}_2\}$. Then $\nu(\mathcal{J}_2) = |\mathcal{B}_2|$, and $\tau(\mathcal{J}_1) \leq r(|\mathcal{B}_1| - |\mathcal{B}_2|) + \tau(\mathcal{J}_2)$. We may repeat this argument another $r-1$ times for colour classes $V_2, \ldots, V_r$ until we reach a hypergraph $\mathcal{J}_{r+1}$ and a matching $\mathcal{B}_{r+1}$ in $\mathcal{J}_{r+1}$, in which there exists a matching $\mathcal{M}_i$ of i-bad edges with $|\mathcal{M}_i| = |\mathcal{B}_{r+1}|$ for each i. Each edge of $\mathcal{M}_i$ is in a distinct $\mathcal{H}_j$, and $\nu(\mathcal{J}_{r+1}) = |\mathcal{B}_{r+1}|$. To prove Theorem 1.2 it will suffice to show that $\mathcal{J}_{r+1}$ has a cover of size at most $(r-\alpha)|\mathcal{B}_{r+1}|$ for some fixed positive α.

We denote by $\mathcal{C}_j$ the hypergraph consisting of the r edges of $\bigcup_{i=1}^{r} \mathcal{M}_i$ in $\mathcal{J}_{r+1}$ that intersect B_j, together with the edge B_j itself. Then $\mathcal{C}_j \subset \mathcal{H}_j$.

Lemma 2.3. *For each $\mathcal{C}_j$ we have $\tau(\mathcal{C}_j) \geq 2$, and no cover of $\mathcal{C}_j$ of size two consists of vertices from distinct colour classes.*

Proof. If on the contrary $\tau(\mathcal{C}_j) = 1$ then without loss of generality we may assume that the vertex of B_j of colour 1 covers $\mathcal{C}_j$. But then the $\mathcal{M}_2$-edge in $\mathcal{C}_j$ is not covered. Thus $\tau(\mathcal{C}_j) \geq 2$.

Suppose now that vertices $v \in V_1$ and $w \in V_2$ form a cover of $\mathcal{C}_j$. We may assume without loss of generality that v is in B_j. Then the $\mathcal{M}_3$ edge in $\mathcal{C}_j$ is not covered by v, hence w must not be in B_j. But then the M_2 edge in $\mathcal{C}_j$ is not covered by $\{v, w\}$. □

Next we would like to restrict to a hypergraph in which $V(\mathcal{H}_j) \cap V(\mathcal{C}_k) \neq \emptyset$ if and only if $j = k$. To do this we will need to consider a more general setting in which our r-uniform hypergraph is replaced with a hypergraph of rank r.

A *sunflower* with *centre* C in a hypergraph is a set $\mathcal{S}$ of edges such that $S \cap S' = C$ for all $S \neq S'$ in $\mathcal{S}$. Each edge of $\mathcal{S}$ is called a *petal*. A classical theorem of Erdős and

Rado [3] tells us that every hypergraph of rank r with more than $(t-1)^r r!$ edges contains a sunflower of size t.

Let $\mathcal{H}$ be a hypergraph of rank r. We call a set $\mathcal{S}$ of t edges in $\mathcal{H}$ a *giant sunflower* if it forms a sunflower and $t \geq r(2r-4)+1$. Note that since $t > r$, if an intersecting hypergraph $\mathcal{H}$ contains a giant sunflower $\mathcal{S}$ with centre C, then $\mathcal{H}' = \mathcal{H} \setminus \mathcal{S} \cup \{C\}$ is also intersecting. We refer to the hypergraph $\mathcal{H}'$ as the hypergraph obtained by *picking* the sunflower $\mathcal{S}$.

We apply the following procedure to each $\mathcal{H}_j$ where $B_j \in \mathcal{B}_{r+1}$. If $\mathcal{H}_j = \mathcal{H}_j^0$ contains a giant sunflower $\mathcal{S}_0$, we pick it to obtain $\mathcal{H}_j^1$. We repeat this process with the current hypergraph $\mathcal{H}_j^k$ to get $\mathcal{H}_j^{k+1}$, until for some u we obtain a hypergraph $\mathcal{D}_j = \mathcal{H}_j^u$ that is free of giant sunflowers. Then in particular each $\mathcal{D}_j$ is intersecting. Let $\mathcal{J}' = (\mathcal{J}_{r+1} \setminus \bigcup_j \mathcal{H}_j) \cup \bigcup_j \mathcal{D}_j$. For every edge $A \in \mathcal{H}_j$ there exists a unique edge $\hat{A} \in \mathcal{J}'$ and a sequence of edges $A = A^0, \dots, A^u = \hat{A}$ with $A^k \in \mathcal{H}_j^k$ such that for $i = 1, \dots, u$, either $A^i = A^{i-1}$ or A^{i-1} is a petal of $\mathcal{S}_{i-1}$ and A^i is its centre. We extend this definition to every $A \in \mathcal{J}_{r+1}$ by setting $\hat{A} = A$ for each $A \in \mathcal{J}_{r+1}$ that is not in any $\mathcal{H}_j$.

Note that $\mathcal{J}'$ has rank at most r but may not be r-uniform. Also, we do not know that $\nu(\mathcal{J}') \leq \nu(\mathcal{J}_{r+1})$.

Lemma 2.4. *Any cover of $\mathcal{J}'$ is also a cover of $\mathcal{J}_{r+1}$.*

Proof. Every edge A of $\mathcal{J}_{r+1}$ has a subset $\hat{A}$ that is an edge of $\mathcal{J}'$. □

Thus to prove Theorem 1.2 it will suffice to find a cover of $\mathcal{J}'$ of size $(r-\alpha)|\mathcal{B}_{r+1}|$ for some $\alpha > 0$.

Lemma 2.5. *Let $\{A'_1, \dots, A'_s\}$ be a matching of size $s \leq 2r-3$ in $\mathcal{J}'$. Then there exists a matching $\{A_1, \dots, A_s \in \mathcal{J}_{r+1}\}$ such that*

- *$A'_i \subseteq A_i$ for each i,*
- *if $A'_i \in \mathcal{D}_j$ then $A_i \in \mathcal{H}_j$.*

Proof. If every $A'_i \in \mathcal{J}_{r+1}$ then we set $A_i = A'_i$ for each i. Otherwise, since each $\mathcal{D}_j$ is intersecting, we may assume that $A'_1, \dots, A'_{c-1} \in \mathcal{J}_{r+1}$, and that there are distinct $\mathcal{D}_i$ for $c \leq i \leq s$ such that $A'_i \in \mathcal{D}_i$. Set $A_i = A'_i$ for each $1 \leq i \leq c-1$.

Let A_i for $c \leq i \leq s$ be such that the following hold.

- $A'_i \subseteq A_i$ for each i,
- $A_i \in \mathcal{H}_i^{k_i}$ for some k_i,
- $A_1, \dots, A_s$ are all disjoint,
- $\sum_{i=c}^{s} k_i$ is as small as possible.

Such a choice of A_i exists because $A'_c, \ldots, A'_s$ satisfy the conditions. We claim that $k_i = 0$ for each i, which implies the lemma.

Suppose on the contrary that $A_i \in \mathcal{H}_i^{k_i}$ for some i, where $k_i \geq 1$. Since $\sum_{i=c}^{s} k_i$ is as small as possible we know that $A_i \notin \mathcal{H}_i^{k_i-1}$, which implies that it is the centre of a giant sunflower $\mathcal{S}$ in $\mathcal{H}_i^{k_i-1}$. Let $A_i^* \in \mathcal{H}_i^{k_i-1}$ be a petal of $\mathcal{S}$ that is disjoint from all of $A_1, \ldots, A_{i-1}$ and all of $A_{i+1}, \ldots, A_s$. This is possible because the union of these edges has size at most $r(s-1) \leq r(2r-4)$, and $\mathcal{S}$ has at least $r(2r-4)+1$ petals. But then replacing A_i by A_i^* gives a new family satisfying the conditions, contradicting the fact that $\sum_{i=c}^{s} k_i$ was as small as possible. Thus $k_i = 0$ for each i, completing the proof. □

In fact it follows from the proof of Lemma 2.5 that $A'_i = \hat{A}_i$ for each i.

Lemma 2.6. *Each $\mathcal{D}_j$ has at most $r^{r+1}(2r-4)^r r!$ vertices.*

Proof. In particular there is no sunflower of size $r(2r-4)+1$ in $\mathcal{D}_j$, so by the Erdős-Rado theorem $\mathcal{D}_j$ has at most $(r(2r-4))^r r!$ edges, and hence at most $r^{r+1}(2r-4)^r r!$ vertices. □

Lemma 2.7. *For each $B_j \in \mathcal{B}_{r+1}$ we have $\hat{B}_j = B_j$.*

Proof. Suppose the contrary. Then for some k we have that B_j is a petal of a sunflower $\mathcal{S}_k$ in $\mathcal{H}_j^k$. We may assume without loss of generality that the centre C of $\mathcal{S}_k$ does not contain a vertex of colour 1. Let M be the $\mathcal{M}_1$-edge in $\mathcal{C}_j$. Then $\hat{M} \cap C = \emptyset$, contradicting the fact that $\mathcal{D}_j$ is intersecting. □

Lemma 2.7 implies that if an edge $A \in \mathcal{J}'$ intersects exactly one $B_j \in \mathcal{B}_{r+1}$ then $A \in \mathcal{D}_j$.

Lemma 2.8. *$V(\mathcal{B}_{r+1})$ is a cover of $\mathcal{J}'$.*

Proof. Suppose on the contrary that an edge $A \in \mathcal{J}'$ is disjoint from $V(\mathcal{B}_{r+1})$. Since each $\mathcal{D}_j$ is intersecting and $B_j \in \mathcal{D}_j$, we know that $A \notin \mathcal{D}_j$ for any j, so $A \in \mathcal{J}_{r+1}$. But then since $V(\mathcal{B}_{r+1})$ is a cover of $\mathcal{J}_{r+1}$ we find a contradiction. □

For each j let $\mathcal{C}'_j = \{\hat{A} : A \in \mathcal{C}_j\}$, so $\mathcal{C}'_j \subseteq \mathcal{D}_j$ for each j. To restrict to our hypergraph in which $\mathcal{C}'_j$ shares a vertex with $\mathcal{D}_k$ if and only if $j = k$, for convenience we define an auxiliary directed graph G as follows. The vertex set of G is $\mathcal{B}_{r+1}$. We put an arc from B_k to B_j if and only if $\mathcal{D}_k$ and $\mathcal{C}'_j$ share a vertex.

Lemma 2.9. *The graph G has an independent set $\mathcal{B}''$ of vertices that has size at least $|\mathcal{B}_{r+1}|/(2r^{r+3}(2r-4)^r r! + 1)$. Thus for any $B_j, B_k \in \mathcal{B}''$, if $\mathcal{C}'_j$ shares a vertex with $\mathcal{D}_k$ then $j = k$.*

Proof. Since each $\mathcal{M}_i$ is a matching, no vertex can be in more than $r+1$ edges of $\bigcup_j \mathcal{C}'_j = \bigcup_j \{B_j\} \cup \{\hat{M} : M \in \mathcal{M}_i \text{ for some } 1 \leq i \leq r\}$. By Lemma 2.6 each $\mathcal{D}_k$ has fewer than $r^{r+1}(2r-4)^r r!$ vertices, and so can share a vertex with at most $r^{r+3}(2r-4)^r r!$ C_j's. Thus the outdegree of G is at most $r^{r+3}(2r-4)^r r!$, which implies that it has an independent set of size at most $|V(G)|/(2r^{r+3}(2r-4)^r r! + 1)$. □

Let $\mathcal{J}'' = \{A \in \mathcal{J}' : A \cap B_j = \emptyset \text{ for all } B_j \in \mathcal{B}_{r+1} \setminus \mathcal{B}''\}$. Then $\mathcal{B}''$ is a matching in $\mathcal{J}''$ such that $V(\mathcal{B}'')$ covers $\mathcal{J}''$, and to prove Theorem 1.2 it suffices to prove that $\tau(\mathcal{J}'') < (r-\alpha)|\mathcal{B}''|$ for some fixed positive α. One important consequence of the definition of $\mathcal{B}''$ is the fact that if $B_j, B_k \in \mathcal{B}''$ then $V(\mathcal{C}'_j) \cap V(\mathcal{C}'_k) = \emptyset$.

Lemma 2.10. *Every edge of $\mathcal{J}''$ contains a cover of $\mathcal{C}'_j$ for some j.*

Proof. Suppose not. Then since the $\mathcal{C}'_j$ are all vertex-disjoint, some edge A together with an edge A_j in $\mathcal{C}'_j$ for each j forms a matching of size $|\mathcal{B}''|+1$ in $\mathcal{J}''$. Except for the set I of at most r indices j for which $A \cap V(\mathcal{C}'_j) \neq \emptyset$, we may assume $A_j = B_j$. Then Lemma 2.5 applied to A together with $\{A_j : j \in I\}$ gives a matching in $\mathcal{J}_{r+1}$ of size $|I| + 1$, which by our construction of $\mathcal{J}''$ consists of edges that do not intersect any edge of $\mathcal{B}_{r+1}$ except $\{B_j : j \in I\}$. But then together with $\{B_j : j \notin I\}$ this forms a matching in $\mathcal{J}_{r+1}$ of size $|\mathcal{B}_{r+1}| + 1$, a contradiction. ☐

Lemma 2.10 tells us that for every edge $A \in \mathcal{J}''$ there exists j such that A contains a cover of $\mathcal{C}'_j$. Since every cover of $\mathcal{C}'_j$ is a cover of $\mathcal{C}_j$, Lemma 2.3 tells us that this cover is of size at least 3. Thus j is unique for $r = 4$ and $r = 5$. Let $\mathcal{C}^*_j = \{A \in \mathcal{J}'' : A \text{ contains a cover of } \mathcal{C}'_j\}$, so since $\mathcal{C}'_j$ is intersecting we have $\mathcal{C}'_j \subseteq \mathcal{C}^*_j$. Then $\mathcal{J}'' = \bigcup_j \mathcal{C}^*_j$, where the union is a disjoint union.

Lemma 2.11. *Suppose that $A \cap A' = \emptyset$ for $A, A' \in \mathcal{C}^*_j$. Then there exists $k \neq j$ such that $A \cup A'$ contains a cover of $\mathcal{C}'_k$.*

Proof. Suppose the contrary. Let I denote the set of at most $2(r-3)+1$ indices such that $(A \cup A') \cap V(\mathcal{C}'_j) \neq \emptyset$. Then A and A' together with an edge of $\mathcal{C}'_k$ for all $k \in I \setminus \{j\}$ forms a matching of size $|I| + 1$, consisting of edges that are disjoint from each B_j with $j \notin I$. Then as in the proof of Lemma 2.10 this leads to a matching in $\mathcal{J}_{r+1}$ that is larger than $\mathcal{B}_{r+1}$. This contradiction completes the proof. ☐

3 $r = 4$

We have now done essentially all the required work to prove Theorem 1.2 for $r = 4$.

Lemma 3.1. *Suppose $r = 4$. Then each $\mathcal{C}^*_j$ is intersecting.*

Proof. Suppose on the contrary that $A \cap A' = \emptyset$ where $A, A' \in \mathcal{C}^*_j$. By Lemma 2.3, each of A and A' must have three vertices in $V(\mathcal{C}'_j)$. By Lemma 2.11 we know $A \cup A'$ covers $\mathcal{C}'_k$ for some $k \neq j$. Since every cover of $\mathcal{C}'_k$ is a cover of $\mathcal{C}_k$, and $V(\mathcal{C}'_j) \cap V(\mathcal{C}'_k) = \emptyset$, we may assume that the vertices of colour 1 in A and A' form a cover of $\mathcal{C}'_k$. But then one of these vertices is not in B_k, so one of the edges, say A, contains 3 vertices of $\mathcal{C}'_j$ and one vertex of $\mathcal{C}'_k$ that is not in B_k. Thus $A \in \mathcal{H}_j$, which implies $A \in \mathcal{D}_j$. But then A cannot intersect $\mathcal{C}'_k$ by Lemma 2.9. ☐

We close this section with the $r = 4$ case of Theorem 1.2.

Theorem 3.2. *Suppose $r = 4$. Then there exists $\epsilon > 0$ such that $\tau(\mathcal{J}) \leq (4 - \epsilon)\nu(\mathcal{J})$.*

Proof. Since $\mathcal{J}'' = \bigcup_j \mathcal{C}_j^*$, by Lemma 3.1 we may apply Theorem 1.1 to conclude that each $\mathcal{C}_j^*$ has a cover of size 3. Therefore $\tau(\mathcal{J}'') \leq 3|\mathcal{B}''|$, completing the proof. □

4 $r = 5$

Our approach for the case $r = 5$ will be to start with the hypergraph $\mathcal{J}''$ and the matching $\mathcal{B}''$ as defined in Section 2, and restrict once more to a portion of $\mathcal{J}''$ in which all the hypergraphs $\mathcal{C}_j^*$ are intersecting.

We begin by fixing $B_j \in \mathcal{B}''$, and considering how the edges in $\mathcal{C}_j^*$ can intersect other sets $\mathcal{C}_k'$. In particular, we will need some technical information on pairs of disjoint edges in $\mathcal{C}_j^*$. We will make use of the following classical theorem of Bollobás [2].

Theorem 4.1. *(Bollobás [2]) Suppose sets $F_1, \ldots, F_m$ and $F_1', \ldots, F_m'$ satisfy $F_i \cap F_h' = \emptyset$ if and only if $i = h$. Then*

$$\sum_{i=1}^{m} \binom{|F_i| + |F_i'|}{|F_i|}^{-1} \leq 1.$$

We say that a set of vertices is *multicoloured* if no two of its elements come from the same partition class V_i. For $B_j \in \mathcal{B}''$, suppose (S, S') is a pair of disjoint multicoloured covers of $\mathcal{C}_j'$. Since every cover of $\mathcal{C}_j'$ is a cover of $\mathcal{C}_j$, by Lemma 2.3 we know each of S and S' has size at least three. Let

$$\mathcal{A}(S, S') = \{(A, A') : A, A' \in \mathcal{C}_j^*, A \cap A' = \emptyset, A \cap V(\mathcal{C}_j') = S, A' \cap V(\mathcal{C}_j') = S'\}.$$

Our key lemma in this section is the following.

Lemma 4.2. *Let $B_j \in \mathcal{B}''$, and suppose (S, S') is a fixed pair of disjoint multicoloured covers of $\mathcal{C}_j'$. Let*

$$U = \{B_k \in \mathcal{B}'' \setminus \{B_j\} : A \cup A' \text{ covers } \mathcal{C}_k' \text{ for some } (A, A') \in \mathcal{A}(S, S')\}.$$

Then there exist $B, B' \in \mathcal{B}'' \setminus \{B_j\}$ such that for all but at most 42 elements $B_k \in U$, if $A \cup A'$ covers $\mathcal{C}_k'$ where $(A, A') \in \mathcal{A}(S, S')$ then $(A \cup A') \cap (B \cup B') \neq \emptyset$.

Proof. Note that since $|S|, |S'| \geq 3$, for any $(A, A') \in \mathcal{A}(S, S')$ we know that each of A and A' has at most two vertices outside $V(\mathcal{C}_j')$.

Let U_0 be the set of B_k in U for which there is some $(A, A') \in \mathcal{A}(S, S')$ with $A \cup A'$ covering $\mathcal{C}_k'$, such that $A \cup A'$ has at least 3 vertices in $\mathcal{C}_k'$. Let $U_1 = U \setminus U_0$.

Suppose that $|U_0| \geq 3$. For each $B_k \in U_0$ pick $(A_k, A_k') \in \mathcal{A}(S, S')$ with $|(A_k \cup A_k') \cap V(\mathcal{C}_k')| \geq 3$. Then one of A_k, A_k' must have 2 vertices in $\mathcal{C}_k'$ and the other must have at least 1. Without loss of generality, we may assume that there are at least two sets A_k, say A_1, A_2, such that A_k has 2 vertices in $\mathcal{C}_k'$. In particular, for $i = 1, 2$, A_i is contained in $S \cup V(\mathcal{C}_i')$. Now consider A_3': if it has no vertex in $\mathcal{C}_i'$ then A_3' and A_i are disjoint and contradict Lemma 2.11. On the other hand, A'_3 has at most one vertex outside $B_j \cup V(\mathcal{C}'_3)$. So we must have $|U_0| \leq 2$.

Now we consider U_1. For each $B_k \in U_1$ and $(A_k, A'_k) \in \mathcal{A}(S, S')$ that covers $\mathcal{C}'_k$, by Lemma 2.3 we know that the vertices y_k and y'_k are of the same colour, where $A_k \cap V(\mathcal{C}'_k) = \{y_k\}$ and $A'_k \cap V(\mathcal{C}'_k) = \{y'_k\}$.

Case 1. Suppose that there exist $B_k \in U_1$ and associated (A_k, A'_k) such that for some $B_l \in \mathcal{B}'' \setminus \{B_j, B_k\}$, the vertices x_k and x'_k exist and are both in $\mathcal{C}'_l$, where $\{x_k\} = A_k \setminus (V(\mathcal{C}'_k) \cup V(\mathcal{C}'_j))$ and $\{x'_k\} = A'_k \setminus (V(\mathcal{C}'_k) \cup V(\mathcal{C}'_j))$. We claim that $B = B_k$ and $B' = B_l$ satisfy the lemma in this case. To verify this, we first observe that by Lemma 2.3, one of A_k and A'_k (say A_k) does not contain a vertex of B_k. If $x_k \in A_k$ is not a vertex of B_l, then since its other three vertices are in $\mathcal{C}'_j$, and the $\mathcal{C}'_h$ are all vertex-disjoint, we find $A_k \in \mathcal{D}_j$. But this contradicts Lemma 2.9. Therefore $x_k \in A_k \cap B_l$, so $\{x_k, x'_k\} \cap B_l \neq \emptyset$. We know $\{y_k, y'_k\} \cap B_k \neq \emptyset$ since $\{y_k, y'_k\}$ covers $\mathcal{C}'_k$. Then to prove our claim we show that for every $B_t \in U_1$ and every associated (A_t, A'_t), if the colour of $\{y_t, y'_t\}$ is the same as the colour of $\{y_k, y'_k\}$ then $\{x_k, x'_k\} \subset A_t \cup A'_t$, and if the colour of $\{y_t, y'_t\}$ is not the same as the colour of $\{y_k, y'_k\}$ then either $\{y_k, y'_k\} \subset A_t \cup A'_t$ or $\{x_k, x'_k\} \cap B_l \subset A_t \cup A'_t$.

Let $B_t \neq B_k$ in U_1 be given, and first assume that the colour of $\{y_t, y'_t\}$ (say 2) is the same as the colour of $\{y_k, y'_k\}$. Then A_k and A'_t are both in $\mathcal{C}^*_j$. If they are not disjoint then A'_t must contain x_k. Suppose they are disjoint. Then by Lemma 2.11 the vertex x'_t where $A'_t = S' \cup \{y'_t\} \cup \{x'_t\}$ must exist and $\{x_k, x'_t\}$ must cover $\mathcal{C}'_l$, and hence x_k and x'_t are the same colour (say 1). (Note that $\{y_k, x'_t\}$ cannot cover $\mathcal{C}'_k$ because they are different colours, contradicting Lemma 2.3.) But then since $A'_k = S' \cup \{y'_k\} \cup \{x'_k\}$ and y'_k has colour 2, we see that x'_k has colour 1. Therefore $x'_k = x'_t$, since otherwise there is an edge of $\mathcal{C}'_l$ containing $x'_k \in V(\mathcal{C}'_l)$ that is not covered by $\{x_k, x'_t\}$. Thus $x'_k \in A'_t$. Now the same argument applies to the pair A'_k and A_t. Therefore since $A_t \cap A'_t = \emptyset$ we find that $\{x_k, x'_k\} \subset A_t \cup A'_t$.

If the colour of $\{y_t, y'_t\}$ (say 2) is not the same as the colour of $\{y_k, y'_k\}$ (say 1) then both elements of $\{x_k, x'_k\}$ also have colour 2. If $\mathcal{C}'_t \neq \mathcal{C}'_l$ then consider A_k and A'_t. If they are disjoint then, since $A_k \cap V(\mathcal{C}'_t) = \emptyset$, by Lemma 2.11 they must cover $\mathcal{C}'_k$. Thus $y'_k \in A'_t$. If they are not disjoint then $y_k \in A'_t$. The same argument applies to A'_k and A_t, then since $A_t \cap A'_t = \emptyset$ we conclude $\{y_k, y'_k\} \subset A_t \cup A'_t$. If $\mathcal{C}'_t = \mathcal{C}'_l$, recall that one of x_k and x'_k is the vertex of colour 2 in B_l. But then since $\{y_t, y'_t\}$ covers $\mathcal{C}'_l$ it must contain the vertex of colour 2 in B_l. Therefore $\{x_k, x'_k\} \cap B_l \subset \{y_t, y'_t\} \subset A_t \cup A'_t$. This finishes the proof for Case 1.

Case 2. Suppose that for each $B_k \in U_1$ and associated (A_k, A'_k), the vertices x_k and x'_k (if they exist) do not lie in a common $\mathcal{C}'_l$. To finish the proof we will show that $|U_1| \leq 40$. Suppose not, then there is a subset U_2 of U_1 of size at least 21 in which all $\{y_k, y'_k\}$ are the same colour. For each x_k that exists and lies in a cover of size two of the $\mathcal{C}'_l$ it is in, set z_k to be the other vertex of the cover. Note that z_k is unique by Lemma 2.3. Define z'_k similarly for each x'_k. Define $F_k = (A_k \setminus S) \cup \{z_k\}$ and $F'_k = (A'_k \setminus S') \cup \{z'_k\}$ for each k (if z_k or z'_k do not exist then simply set $F_k = (A_k \setminus S)$, $F'_k = (A'_k \setminus S')$). We claim that these pairs of sets satisfy the conditions for Theorem 4.1. Since x_k and x'_k do not lie in a common B_l, we have that $F_k \cap F'_k = \emptyset$ for each k. Suppose that $F_k \cap F'_l = \emptyset$. Then

A_k and A'_l are disjoint edges in $\mathcal{C}'_j$ that do not cover any $\mathcal{C}'_t$, contradicting Lemma 2.11. Therefore by Theorem 4.1 we find that $|U_2| \leq \binom{6}{3} = 20$. This contradiction completes the proof. $\square$

We define an auxiliary directed graph G on the vertex set $\mathcal{B}''$ as follows. Consider a vertex B_j and a pair (S, S') of disjoint multicoloured covers of $\mathcal{C}'_j$ of size at least three (and at most four), and let U be the set defined in Lemma 4.2 for this choice of B_j and (S, S'). If $|U| \leq 42$ then we put an arc (B_j, B_k) for each $B_k \in U$. If $|U| \geq 43$ then, for B, B' guaranteed by Lemma 4.2, we put arcs (B_j, B) and (B_j, B'), and an arc (B_j, B_k) for each $B_k \in U$ that fails to satisfy the conclusion of Lemma 4.2. We do this for each B_j and each pair (S, S') of disjoint multicoloured covers of $\mathcal{C}'_j$.

Lemma 4.3. *The directed graph G has outdegree less than $44(5)^{16}$, and hence has an independent set $\mathcal{B}^\dagger$ of size at least $|\mathcal{B}''|/100(5)^{16}$.*

Proof. Since $|V(\mathcal{C}'_j)| \leq |V(\mathcal{C}_j)| < r^2$, the number of distinct choices of (S, S') in $\mathcal{C}'_j$ is less than $(|V(\mathcal{C}'_j)|^4)^2 < \binom{r^2}{4}^2 < r^{16} = 5^{16}$. Thus the outdegree of G is less than $49(5)^{16}$. Therefore G has an independent set of size at least $|V(G)|/(98(5)^{16} + 1) < |\mathcal{B}''|/100(5)^{16}$. $\square$

Let $\mathcal{J}^\dagger = \{A \in \mathcal{J}'' : A \cap B_j = \emptyset \text{ for all } B_j \in \mathcal{B}'' \setminus \mathcal{B}^\dagger\}$. Then $\mathcal{B}^\dagger$ is a matching in $\mathcal{J}^\dagger$ such that $V(\mathcal{B}^\dagger)$ covers $\mathcal{J}^\dagger$, and to prove Theorem 1.2 for $r = 5$ it suffices to prove that $\tau(\mathcal{J}^\dagger) < (r - \alpha)|\mathcal{B}^\dagger|$ for some fixed positive α.

Lemma 4.4. *Each $\mathcal{C}^*_j \cap \mathcal{J}^\dagger$ is intersecting.*

Proof. Suppose on the contrary that A and $A' \in \mathcal{C}^*_j$ are edges of $\mathcal{J}^\dagger$ that do not intersect. We know by Lemma 2.11 that $A \cup A'$ covers some $\mathcal{C}'_k$, $k \neq j$. Since then $(A \cup A') \cap V(\mathcal{C}'_k) \neq \emptyset$, it must be true that $B_k \in \mathcal{B}^\dagger$. Let $S = A \cap V(\mathcal{C}'_j)$ and $S' = A' \cap V(\mathcal{C}'_j)$. Since $B_j, B_k \in \mathcal{B}^\dagger$, there cannot be an arc (B_j, B_k) in G. The construction of G implies then that for this choice of B_j and (S, S'), the set U satisfies $|U| \geq 47$ and that B and B' exist satisfying the conclusion of Lemma 4.2. Since $\mathcal{B}^\dagger$ is an independent set in G and $B_j \in \mathcal{B}^\dagger$ we know that $B, B' \notin \mathcal{B}^\dagger$. But then by Lemma 4.2 one of A and A' intersects B or B', and hence it is not an edge of $\mathcal{J}^\dagger$ by definition. This contradiction completes the proof. $\square$

The $r = 5$ case of Theorem 1.2 follows.

Theorem 4.5. *Suppose $r = 5$. Then there exists a fixed $\epsilon > 0$ such that $\tau(\mathcal{H}) \leq (5 - \epsilon)\nu(\mathcal{H})$.*

Proof. Since $\mathcal{J}^\dagger = \bigcup_j \mathcal{C}^*_j \cap \mathcal{J}^\dagger$, by Theorem 1.1 we conclude that each $\mathcal{C}^*_j \cap \mathcal{J}^\dagger$ has a cover of size 4. Therefore $\tau(\mathcal{J}^\dagger) \leq 4|\mathcal{B}^\dagger|$, completing the proof. $\square$

We end with the remark that for each of $r = 4$ and $r = 5$, an explicit lower bound for ϵ could be computed by following the steps of our proof. However, as this value is probably very far from the truth we make no attempt to do this here.

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