A Note on Random k-SAT for Moderately Growing k

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Submitted: Aug 12, 2011; Accepted: Jan 15, 2012; Published: Jan 21, 2012
Mathematics Subject Classifications: 05D40, 68Q25

Abstract

Consider a random instance I of k-SAT with n variables and m clauses. Suppose that θ, $c > 0$ are any fixed real numbers. Let $k = k(n) \geq \left(\frac{1}{2} + \theta\right) \log_2 n$. We prove that

$$\lim_{n\to\infty} Pr(I \text{ is satifiable}) = \begin{cases} 1 & m \leq \left(1 - \frac{c}{\sqrt{n}}\right) 2^k n \ln 2 \\ 0 & m \geq \left(1 + \frac{c}{\sqrt{n}}\right) 2^k n \ln 2. \end{cases}$$

Keywords: k-SAT, phase transition, the second moment method.

1 Introduction

Let $C_k(V)$ be the set of all possible $2^k n^k$ k-clauses on V, where a k-clause is a disjunction of k boolean variables or their negations and V is a set of n boolean variables. A random instance I of k-SAT is formed by selecting uniformly, independently and with replacement m clauses from $C_k(V)$ and taking their conjunction [1, 3].

A. Frieze and N.C. Wormald [3] proved the following result.

Theorem A. *Suppose $k - \log n \to \infty$. Let $m_0 = -\frac{n \ln 2}{\ln(1-2^{-k})}$ and let $\varepsilon_n > 0$ be such that $n\varepsilon_n \to \infty$. Then*

$$\lim_{n\to\infty} Pr(I \text{ is satisfiable}) = \begin{cases} 1 & m \leq (1-\varepsilon_n)m_0 \\ 0 & m \geq (1+\varepsilon_n)m_0. \end{cases}$$

Not long afterwards, A. Coja-Oghlan and A. Frieze [2] proved the following result.

Theorem B. Suppose $k - \log n \to \infty$ but $k - \log n = o(\ln n)$. Let $m = 2^k(n \ln 2 + c)$ for an absolute constant c. Then

$$\lim_{n\to\infty} Pr(I \text{ is satisfiable}) = 1 - e^{-e^{-c}}.$$

For a lot of random Constraint Satisfaction Problem (CSP for short) models, the second moment method is harnessed to estimate the desired lower bounds on the satisfiability threshold. Ultimately, we often need to bound sums which have common structure of $\sum_{\omega=0}^{n} \binom{n}{\omega} p(n)^\omega (1-p(n))^{n-\omega} Z(\omega, n)^m$. Take the random CSP model proposed in [3] for example, specifically, the sum is (4). Let

$$G(\tau) = 2^{-n}\binom{n}{\omega} g(\tau)^m, \tag{1}$$

and the global maximum is $G_{max} = G(\tau_{max})$.

In [3], Frieze and Wormald estimated (4) by locating the global maximum $G_{max} = G(\tau_{max})$, and then estimating the contribution of the terms close to τ_{max} by G_{max}.

In this paper, by using the properties of the Gamma Function Γ and the inequality [4] $\frac{1}{2(\omega+1)} < \sum_{i=1}^{\omega} \frac{1}{i} - \gamma - \ln \omega < \frac{1}{2\omega}$ (where γ is Euler-Mascheroni Constant), we can analyze the monotonicity of G very close to τ_{max}. Thus, we can divide the infinitely small neighbourhood of τ_{max} into several smaller intervals, then estimate the contribution of each interval, respectively, by using the monotonicity of G.

Theorem 1. *Suppose that $\theta, c > 0$ are any fixed real numbers. Let $k \geq (\frac{1}{2} + \theta) \log n$ and let $m_0 = -\frac{n \ln 2}{\ln(1-2^{-k})}$. Then*

$$\lim_{n\to\infty} Pr(I \text{ is satisfiable}) = \begin{cases} 1 & m \leq \left(1 - \frac{c}{\sqrt{n}}\right)m_0 \\ 0 & m \geq \left(1 + \frac{c}{\sqrt{n}}\right)m_0. \end{cases}$$

In this note $\log x$ means $\log_2 x$, and $\ln x$ means the natural logarithm.

2 Proof of Theorem 1

Let $X = X(I)$ be the number of satisfying assignments for I and let $\tau = \frac{\omega}{n}$. Then [3]

$$E[X] = 2^n(1-2^{-k})^m, \tag{2}$$

$$E[X^2] = 2^n \sum_{\omega=0}^{n} \binom{n}{\omega} \left(1 - 2^{1-k} + 2^{-k}\tau^k\right)^m. \tag{3}$$

Simple calculation yields

$$\frac{E[X^2]}{E[X]^2} = 2^{-n}\sum_{\omega=0}^{n}\binom{n}{\omega} g(\tau)^m, \tag{4}$$

where $g(\tau) = 1 + \frac{1}{2^k}\left(\tau^k - \frac{1}{2^k}\right)\left(1 - \frac{1}{2^k}\right)^{-2}$.

The upper bound: By simple calculation

$$E[X] = 2^n(1-2^{-k})^m \leq 2^n(1-2^{-k})^{-(1+\frac{c}{\sqrt{n}})\frac{n\ln 2}{\ln(1-2^{-k})}} = 2^{-c\sqrt{n}},$$

by the Markov Inequality $Pr(I \text{ is satisfiable}) \leq E(X)$, $\lim_{n\to\infty} Pr(I \text{ is satisfiable}) = 0$ when $m \geq (1+\frac{c}{\sqrt{n}})m_0$.

The lower bound: Since $\theta > 0$ is an arbitrarily small constant, we require that $\theta < \frac{1}{4}$ in the following of this paper.

Let $t = \left(\frac{1}{2}+\theta\right)\log n$, which is the smallest clause length permitted. Let $m_1 = \left(1-\frac{c}{\sqrt{n}}\right)2^t n\ln 2$. Define a partition of the interval $[0,1]$: $\tau_1 = \left(1+\frac{1}{n^\zeta}\right)/2$, where $\zeta = \frac{1-\theta}{2}$; $\tau_2 = 1-\frac{\ln t}{t}$; $\tau_3 = 1-\frac{\alpha}{t}$, where $\alpha \in \left(0, \ln(1+2\theta)\right)$ is a constant; $\tau_4 = 1-\frac{1}{\sqrt{n}}$; $\tau_5 = 1-\frac{1}{n^{1/2+2\theta}}$ and $\tau_6 = 1-\frac{1}{n}$.

We require that $m \leq \left(1-\frac{c}{\sqrt{n}}\right)m_0$ in the following of this paper unless otherwise specified.

2.1. **A rough estimate**. First we will give a rough upper bound for the sum in (4), which is easier to analyse.

Lemma 1. *Let* $\Phi(\omega) = 2^{-n}\binom{n}{\omega}f(\tau)^{m_1}$, *where* $f(\tau) = 1+\frac{1}{2^t}\left[(1+\frac{2}{2^t})\tau^t - \frac{1}{2^t}\right]$. *Then* $\frac{E[X^2]}{E[X]^2} \leq \frac{1}{2} + \left(1+o(1)\right)\sum_{\omega=n/2}^{n}\Phi(\omega)$.

To prove Lemma 1, first we will give the following two claims. Claim 1 is used to prove Claim 2, and Claim 2 is used to prove Lemma 1.

Claim 1. *For any positive real number* x, $(1+x)\ln(1+x) < x + x^2/2$.

Claim 2. *Let* $\varphi_r(x) = x^{-1}\ln(1+x^{1+r}+2x^{2+r}-x^2)$. *Then there exists a constant* ε (i.e., *independent of* r), *such that for any* $r \in [0,1]$, $\varphi_r'(x) > 0$, $x \in (0,\varepsilon)$.

Proof. For any $r \in [0,1]$, define u_r and v_r on $(0,+\infty)$ as

$$\begin{aligned} u_r(x) &= x^{r-1} + 2x^r - 1, \\ v_r(x) &= rx^{r-1} + 2(1+r)x^r - 1. \end{aligned} \tag{5}$$

With Claim 1 in mind, and note that $u_r > 0$, then

$$(1+x^2u_r)\ln(1+x^2u_r) < x^2u_r + x^4u_r^2/2. \tag{6}$$

Simple calculation yields

$$\varphi_r' = \frac{u_r + v_r}{1 + x^2 u_r} - \frac{\ln(1 + x^2 u_r)}{x^2}. \tag{7}$$

By (6) and (7),

$$\varphi_r' > \frac{v_r - x^2 u_r^2/2}{1 + x^2 u_r}. \tag{8}$$

For any $x > 0$, define $\overline{u_x}(r) = u_r(x)$ and $\overline{v_x}(r) = v_r(x)$ on $[0, 1]$. Then

$$\begin{aligned} \overline{u_x}' &= (1 + 2x)x^{r-1} \ln x, \\ \overline{v_x}' &= \big[1 + 2x(1 + \ln x) + r(1 + 2x) \ln x\big] x^{r-1}. \end{aligned} \tag{9}$$

Note that $\lim_{x \to 0^+} x(1 + \ln x) = 0$, there exists a constant $\varepsilon \in \big(0, \frac{1}{4}\big)$ (i.e., *independent of* r) such that $\overline{v_x}'(0) > 0$, $x \in (0, \varepsilon)$. $\overline{u_x}' < 0$, $x \in (0, \varepsilon)$. Then for any $x \in (0, \varepsilon)$

(i) For any $r \in \big[0, \frac{1}{2}\big]$,

$$\begin{aligned} u_r(x) &= \overline{u_x}(r) \le \overline{u_x}(0) = 1 + \frac{1}{x}, \\ v_r(x) &= \overline{v_x}(r) \ge \min\left\{\overline{v_x}(0), \overline{v_x}\Big(\frac{1}{2}\Big)\right\} = 1. \end{aligned} \tag{10}$$

Note that $x < \varepsilon < \frac{1}{4}$, by (8) and (10),

$$\varphi_r' > \frac{1 - x^2\big(1 + \frac{1}{x}\big)^2/2}{1 + x^2 u_r} = \frac{1 - (1 + x)^2/2}{1 + x^2 u_r} > \frac{1 - \big(1 + \frac{1}{4}\big)^2/2}{1 + x^2 u_r} > 0. \tag{11}$$

(ii) Keep $x < \frac{1}{4}$ in mind, then for any $r \in \big(\frac{1}{2}, 1\big]$,

$$\begin{aligned} u_r(x) &= \overline{u_x}(r) \le \overline{u_x}\Big(\frac{1}{2}\Big) = \frac{1}{\sqrt{x}} + 2\sqrt{x} - 1 < \frac{1}{\sqrt{x}}, \\ v_r(x) &= \overline{v_x}(r) \ge \min\left\{\overline{v_x}\Big(\frac{1}{2}\Big), \overline{v_x}(1)\right\} = 4x. \end{aligned} \tag{12}$$

By (8) and (12),

$$\varphi_r' > \frac{4x - x^2\big(\frac{1}{\sqrt{x}}\big)^2/2}{1 + x^2 u_r} > 0. \tag{13}$$

2.2. **Proof of Lemma 1**. For any $\omega \le n/2$, $g(\tau) \le 1$. By (4),

$$\frac{E[X^2]}{E[X]^2} - \frac{1}{2} \le 2^{-n} \sum_{\omega = n/2}^{n} \binom{n}{\omega} g(\tau)^m. \tag{14}$$

Keep Claim 2 in mind, and note that $m \leq \big(1 - \frac{c}{\sqrt{n}}\big) m_0 < \big(1 - \frac{c}{\sqrt{n}}\big) 2^k n \ln 2$, then

$$\frac{E[X^2]}{E[X]^2} - \frac{1}{2} \leq 2^{-n} \sum_{\omega=n/2}^{n} \binom{n}{\omega} g(\tau)^{\left(1-\frac{c}{\sqrt{n}}\right)2^k n \ln 2},$$

where

$$\begin{aligned}
g(\tau)^{\left(1-\frac{c}{\sqrt{n}}\right)2^k n \ln 2} &= \left[1 + \frac{1}{2^k}\left(\tau^k - \frac{1}{2^k}\right)\left(1 + \frac{2}{2^k} + O\left(\frac{1}{2^{2k}}\right)\right)\right]^{\left(1-\frac{c}{\sqrt{n}}\right)2^k n \ln 2} \\
&= \left\{1 + \frac{1}{2^k}\left[\left(1 + \frac{2}{2^k}\right)\tau^k - \frac{1}{2^k}\right] + o\left(\frac{1}{2^k n}\right)\right\}^{\left(1-\frac{c}{\sqrt{n}}\right)2^k n \ln 2} \\
&= \big(1 + o(1)\big)\left\{1 + \frac{1}{2^k}\left[\left(1 + \frac{2}{2^k}\right)\tau^k - \frac{1}{2^k}\right]\right\}^{\left(1-\frac{c}{\sqrt{n}}\right)2^k n \ln 2} \\
&= \big(1 + o(1)\big) \exp\left\{\left(1 - \frac{c}{\sqrt{n}}\right)\left[\varphi_{-\log\tau}\left(\frac{1}{2^k}\right)\right] n \ln 2\right\} \\
&\leq \big(1 + o(1)\big) \exp\left\{\left(1 - \frac{c}{\sqrt{n}}\right)\left[\varphi_{-\log\tau}\left(\frac{1}{2^t}\right)\right] n \ln 2\right\} \\
&= \big(1 + o(1)\big) f(\tau)^{m_1},
\end{aligned}$$

where $o(1)$ is independent of τ (i.e., independent of ω). Then

$$\frac{E[X^2]}{E[X]^2} - \frac{1}{2} \leq \big(1 + o(1)\big) \sum_{\omega=n/2}^{n} \Phi(\omega).$$

2.3. **The monotonicity of** Φ. Generally, the general term of the sum in (4), $G(\tau)$, as defined in (1), has two local maxima, one approaches $\frac{1}{2}$, and the other approaches 1 (see [3]). We can regard $\frac{1}{2}$ and 1 as singularities of G, since the proportion of each term and monotonicity of terms close to the two points change suddenly, also the sum in (4) is mostly contributed by o(n) terms very close to the two points.

In this section, by studying the monotonicity of G, we show some asymptotic structure of the function close to its singularities, and thus yields Theorem 1.

Lemma 2. Define Φ_c on $[0, n]$ as

$$\Phi_c(x) = 2^{-n} \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} f\left(\frac{x}{n}\right)^{m_1}, \tag{15}$$

where Γ is Gamma Function. Then $\Phi_c(\omega) = \Phi(\omega),\ \omega = 1, 2, \ldots, n$ and

$$\begin{aligned}
&\Phi_c' < 0,\ x \in (n\tau_1, n\tau_2); \\
&\Phi_c' > 0,\ x \in (n\tau_3, n\tau_4); \\
&\Phi_c' < 0,\ x \in (n\tau_5, n\tau_6).
\end{aligned} \tag{16}$$

Proof. Taking the logarithm of both sides of (15), and differentiating,

$$[\ln \Phi_c]' = -[\ln \Gamma(x+1)]' - [\ln \Gamma(n-x+1)]' + m_1\Big[\ln f\Big(\frac{x}{n}\Big)\Big]'. \tag{17}$$

We can rewrite (17) as

$$\frac{\Phi_c'}{\Phi_c} = A(x) + B(x), \tag{18}$$

where

$$\begin{aligned} A(x) &= -\frac{\Gamma'(x+1)}{\Gamma(x+1)} + \frac{\Gamma'(n-x+1)}{\Gamma(n-x+1)}, \\ B(x) &= m_1\Big[\ln f\Big(\frac{x}{n}\Big)\Big]'. \end{aligned}$$

For any real positive number x,

$$-\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \gamma + \sum_{i=1}^{\infty}\Big(\frac{1}{i+x} - \frac{1}{i}\Big), \tag{19}$$

where γ is Euler-Mascheroni Constant.

If x is an Integer $\omega = 0, 1, 2, \ldots$, then

$$-\frac{\Gamma'(\omega+1)}{\Gamma(\omega+1)} = \gamma - \sum_{i=1}^{\omega}\frac{1}{i}, \tag{20}$$

where [4]

$$-\ln\omega - \frac{1}{2\omega} < \gamma - \sum_{i=1}^{\omega}\frac{1}{i} < -\ln\omega - \frac{1}{2(\omega+1)}. \tag{21}$$

By (20) and (21),

$$\ln\Big(\frac{1}{\tau} - 1\Big) + R(\omega) < A(\omega) < \ln\Big(\frac{1}{\tau} - 1\Big) + R(\omega+1), \tag{22}$$

where $R(\omega) = \frac{1}{2}\big(\frac{1}{n-\omega+1} - \frac{1}{\omega}\big),\ \omega = 1, 2, \cdots, n$.

Simple calculation yields

$$B(x) = \big(\ln 2 + o(1)\big)t\Big(\frac{x}{n}\Big)^{t-1}. \tag{23}$$

Choose a constant ξ such that $\zeta < -\big(\frac{1}{2} + \theta\big)\log\xi < 1$, then $\xi \in \big(\frac{1}{2}, 1\big)$. Define $\tau_2^- = 1 - \frac{2\ln t}{t}$.

By (19), A(x) is decreasing in $(0,\infty)$, hence we can handle $n\tau_1$, $n\xi$, $n\tau_2^-$, $n\tau_2$, $n\tau_3$, $n\tau_4$, $n\tau_5$, $n\tau_6$ as integers in the following. Note that $R(\omega) < R(\omega+1),\ \omega = 1, 2, \ldots, n$.

(i) If $x \in (n\tau_1, n\tau_2)$, then

(a) If $x \in (n\tau_1, n\xi)$, by (22)

$$\begin{aligned} A(x) &\leq A(n\tau_1) \leq \ln\Big(\frac{1}{\tau_1}-1\Big) + R(n\tau_1+1) \leq \ln\Big(\frac{1}{\tau_1}-1\Big) + R(n\xi+1) \\ &= \ln\Big(\frac{1-n^{-\zeta}}{1+n^{-\zeta}}\Big) + O\Big(\frac{1}{n}\Big) \leq -\frac{1}{n^\zeta} + O\Big(\frac{1}{n}\Big) = -\frac{1}{n^\zeta} + o(\frac{1}{n^\zeta}), \\ B(x) &\leq \big(\ln 2 + o(1)\big)t\xi^t = \big(\ln 2 + o(1)\big)tn^{(\frac{1}{2}+\theta)\log\xi} = o\Big(\frac{1}{n^\zeta}\Big). \end{aligned} \tag{24}$$

Then $\Phi_c' < 0$ by (18) and (24).

(b) If $x \in [n\xi, n\tau_2^-]$, by (22)

$$\begin{aligned} A(x) &\leq A(n\xi) \leq \ln\Big(\frac{1}{\xi}-1\Big) + R(n\xi+1) = \ln\Big(\frac{1}{\xi}-1\Big) + o(1), \\ B(x) &\leq \big(\ln 2 + o(1)\big)t(\tau_2^-)^t = \big(\ln 2 + o(1)\big)t\Big(1 - \frac{2\ln t}{t}\Big)^t \\ &= \big(\ln 2 + o(1)\big)t\exp\Big\{t\ln\Big(1-\frac{2\ln t}{t}\Big)\Big\} \leq \big(\ln 2 + o(1)\big)t\exp\big\{-2\ln t\big\} \\ &= \big(\ln 2 + o(1)\big)\frac{1}{t} = o(1). \end{aligned} \tag{25}$$

Note that $\xi \in \big(\frac{1}{2}, 1\big)$, then $\ln\big(\frac{1}{\xi}-1\big) < 0$. Then $\Phi_c' < 0$ by (18) and (25).

(c) If $x \in (n\tau_2^-, n\tau_2)$, by (22)

$$\begin{aligned} A(x) &\leq A(n\tau_2^-) \leq \ln\Big(\frac{1}{\tau_2^-}-1\Big) + R(n\tau_2^-+1) \\ &\leq \ln\Big(\frac{\frac{2\ln t}{t}}{1-\frac{2\ln t}{t}}\Big) + R(n) = -\big(1+o(1)\big)\ln t, \\ B(x) &\leq \big(\ln 2 + o(1)\big)t\tau_2^t = \big(\ln 2 + o(1)\big)t\Big(1-\frac{\ln t}{t}\Big)^t \\ &= \big(\ln 2 + o(1)\big)t\exp\Big\{t\ln\Big(1-\frac{\ln t}{t}\Big)\Big\} \leq \big(\ln 2 + o(1)\big)t\exp\big\{-\ln t\big\} \\ &= \ln 2 + o(1). \end{aligned} \tag{26}$$

Then $\Phi_c' < 0$ by (18) and (26). $\Phi_c' < 0$, $x \in (n\tau_1, n\tau_2)$ follows from (a), (b) and (c).

(ii) If $x \in (n\tau_3, n\tau_4)$, by (22)

$$\begin{aligned} A(x) &\geq A(n\tau_4) \geq \ln\Big(\frac{1}{\tau_4}-1\Big) \geq -\frac{\ln n}{2}, \\ B(x) &\geq \big(\ln 2 + o(1)\big)t\tau_3^t = \Big[\Big(\frac{1}{2}+\theta\Big)e^{-\alpha} + o(1)\Big]\ln n. \end{aligned} \tag{27}$$

Note that $\alpha < \ln(1+2\theta)$, hence $(\frac{1}{2}+\theta)e^{-\alpha} - \frac{1}{2} > 0$, then $\Phi_c' > 0$ follows from (18) and (27).

(iii) If $x \in (n\tau_5, n\tau_6)$, by (22)

$$\begin{aligned} A(x) &\leq A(n\tau_5) \leq \ln\Big(\frac{1}{\tau_5} - 1\Big) + R(n) = -\Big(\frac{1}{2} + 2\theta + o(1)\Big)\ln n, \\ B(x) &\leq \big(\ln 2 + o(1)\big)t = \Big(\frac{1}{2} + \theta + o(1)\Big)\ln n. \end{aligned} \tag{28}$$

Hence $\Phi_c' < 0$ by (18) and (28).

In order to simplify the proof of the following several Lemmas, we introduce the following three claims.

Claim 3. $\binom{n}{\omega} = o(\Psi(\tau)^n)$ *provided that* $n\tau(1-\tau) \to \infty$ as $n \to \infty$, *where* $\Psi(\tau) = 1/(\tau^\tau(1-\tau)^{1-\tau})$.

Proof. By using Stirling's formula, and note that $n\tau(1-\tau) \to \infty$ is equivalent to $n\tau \to \infty$ and $n(1-\tau) \to \infty$, then

$$\binom{n}{n\tau} = \big(1+o(1)\big)\frac{1}{\sqrt{2\pi n\tau(1-\tau)}}\frac{n^n}{(n\tau)^{n\tau}(n-n\tau)^{n-n\tau}} = o(\Psi(\tau)^n). \tag{29}$$

Claim 4. $f(\tau_1)^{m_1} = 1 + o(1)$.

Proof. Note that $\frac{1}{2^{3t}} = o(\frac{1}{m_1})$ and $\ln(1+x) < x$, $x \in (0, +\infty)$, then

$$\begin{aligned} f(\tau_1) &= 1 + \frac{1}{2^t}\Big[\Big(1+\frac{2}{2^t}\Big)\tau_1^t - \frac{1}{2^t}\Big] \\ &= 1 + \frac{1}{2^{2t}}\Big[\exp\Big\{t\ln\Big(1+\frac{1}{n^\zeta}\Big)\Big\} - 1\Big] + \frac{2}{2^{3t}}\Big(1+\frac{1}{n^\zeta}\Big)^t \\ &\leq 1 + \frac{1}{2^{2t}}\Big[\exp\Big\{\frac{t}{n^\zeta}\Big\} - 1\Big] + o(\frac{1}{m_1}) \\ &= 1 + O\Big(\frac{t}{2^{2t}n^\zeta}\Big) + o\Big(\frac{1}{m_1}\Big) \\ &= 1 + O\Big(\frac{t}{2^t n^{1+\theta/2}}\Big) + o\Big(\frac{1}{m_1}\Big) \\ &= 1 + o\Big(\frac{1}{m_1}\Big). \end{aligned} \tag{30}$$

Note that $f(\tau_1) > 1$, by (30), $f(\tau_1)^{m_1} = 1 + o(1)$.

Claim 4 solves the crucial puzzle of estimating the sum in (4) close to $\frac{1}{2}$ successfully. As the claim shows, τ_1 is a turning point, by it, we divide the neighborhood of $\frac{1}{2}$ into two parts, and then estimate the two parts separately (see Claim 5, Lemmas 3 and 6).

If $k \leq \frac{1}{2}\log n$ ($k \to \infty$ as $n \to \infty$), then the sum in (4) diverges and the second moment method failed to obtain nontrivial result. The reasons are as follows:

Arbitrarily fix two positive numbers $h_2 > h_1$. Let $\tau = \frac{1}{2} + \frac{h}{2\sqrt{n}}$, $h \in (h_1, h_2)$. Then

$$\begin{aligned}
g(\tau) &= 1 + \frac{1}{2^k}\Big(\tau^k - \frac{1}{2^k}\Big)\Big(1 - \frac{1}{2^k}\Big)^{-2} \\
&\geq 1 + \frac{1}{4^k}\left(\Big(1 + \frac{h}{\sqrt{n}}\Big)^k - 1\right) \\
&= 1 + \frac{1}{4^k}\left(\exp\Big\{k \ln\Big(1 + \frac{h}{\sqrt{n}}\Big)\Big\} - 1\right) \\
&\geq 1 + \frac{1}{4^k}\left(\exp\Big\{\frac{hk}{2\sqrt{n}}\Big\} - 1\right) \\
&\geq 1 + \frac{hk}{2}\frac{1}{4^k\sqrt{n}} \geq 1 + \frac{h_1 k}{2}\frac{1}{2^k n}. \qquad (31)
\end{aligned}$$

Hence $g(\tau)^m$ diverges in the interval $\big(\frac{1}{2} + \frac{h_1}{2\sqrt{n}}, \frac{1}{2} + \frac{h_2}{2\sqrt{n}}\big)$ uniformly as $n \to \infty$. On the other hand, by the de Moivre-Laplace theorem

$$\lim_{n\to\infty} 2^{-n} \sum_{\omega=n(\frac{1}{2}+\frac{h_1}{2\sqrt{n}})}^{n(\frac{1}{2}+\frac{h_2}{2\sqrt{n}})} \binom{n}{\omega} = \frac{1}{\sqrt{2\pi}} \int_{h_1}^{h_2} e^{-\frac{x^2}{2}}\, dx. \qquad (32)$$

Hence the sum in (4) diverges.

Claim 5. $\Phi(n\tau_1) = o\big(\frac{1}{n}\big)$.

Proof. Note that $3\zeta > 1$ follows from $\theta < \frac{1}{4}$, then

$$\begin{aligned}
\ln\big[2^{-n}\Psi(\tau_1)^n\big] &= -\frac{n}{2}\Big[\Big(1 + \frac{1}{n^\zeta}\Big)\ln\Big(1 + \frac{1}{n^\zeta}\Big) + \Big(1 - \frac{1}{n^\zeta}\Big)\ln\Big(1 - \frac{1}{n^\zeta}\Big)\Big] \\
&= -\frac{n}{2}\Big[\Big(1 + \frac{1}{n^\zeta}\Big)\Big(\frac{1}{n^\zeta} - \frac{1}{2n^{2\zeta}}\Big) + \Big(1 - \frac{1}{n^\zeta}\Big)\Big(-\frac{1}{n^\zeta} - \frac{1}{2n^{2\zeta}}\Big)\Big] + o(1) \\
&= -\frac{n^\theta}{2} + o(1).
\end{aligned}$$

By Claims 3 and 4, $\Phi(n\tau_1) = o\big(2^{-n}\Psi(\tau_1)^n\big) = o\big(\exp\big\{-\frac{n^\theta}{2}\big\}\big) = o\big(\frac{1}{n}\big)$.

Lemma 3. $\sum_{\omega=n\tau_1}^{n\tau_2} \Phi(\omega) = o(1)$.

Proof. By Lemma 2 and Claim 5,

$$\sum_{\omega=n\tau_1}^{n\tau_2} \Phi(\omega) \leq \sum_{\omega=n\tau_1}^{n\tau_2} \Phi(n\tau_1) \leq n\Phi(n\tau_1) = o(1).$$

Lemma 4. $\sum_{\omega=n\tau_3}^{n\tau_4} \Phi(\omega) = o(1)$.

Proof. Note that $\ln \Psi(\tau_4) = \left(\frac{1}{2} + o(1)\right) \frac{\ln n}{\sqrt{n}}$, then

$$\Psi(\tau_4)^n = 2^{(\frac{1}{2}+o(1))\sqrt{n}\log n}. \tag{33}$$

$\tau_4^t = \exp\left\{t \ln\left(1 - \frac{1}{\sqrt{n}}\right)\right\} \le \exp\left\{-\frac{t}{\sqrt{n}}\right\} = 1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$, then

$$\begin{aligned} f(\tau_4) &\le 1 + \frac{1}{2^t}\left[\left(1 + \frac{2}{2^t}\right)\left(1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) - \frac{1}{2^t}\right] \\ &= 1 + \frac{1}{2^t}\left(1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right). \end{aligned} \tag{34}$$

Then

$$\begin{aligned} f(\tau_4)^{m_1} &\le \left[1 + \frac{1}{2^t}\left(1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right)\right]^{m_1} \\ &= \exp\left\{m_1 \ln\left[1 + \frac{1}{2^t}\left(1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right)\right]\right\} \\ &\le \exp\left\{\frac{m_1}{2^t}\left(1 - \frac{t}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right)\right\} \\ &= \exp\left\{\left(1 - \frac{t}{\sqrt{n}} - \frac{c}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) n \ln 2\right\} \\ &\le \exp\left\{\left(1 - \frac{t}{\sqrt{n}}\right) n \ln 2\right\} = 2^{n - t\sqrt{n}}. \end{aligned} \tag{35}$$

By Lemma 2 and Claim 3, (33) and (35),

$$\sum_{\omega=n\tau_3}^{n\tau_4} \Phi(\omega) \le n\Phi(n\tau_4) \le n2^{-n}\Psi(\tau_4)^n f(\tau_4)^{m_1} \le n2^{-(\theta+o(1))\sqrt{n}\log n} = o(1).$$

Lemma 5. $\sum_{\omega=n\tau_5}^{n-1} \Phi(\omega) = o(1)$.

Proof. Note that $\Phi(n\tau_5) = o(\frac{1}{n})$ (see (47)), by Lemma 2

$$\sum_{\omega=n\tau_5}^{n-1} \Phi(\omega) \le \sum_{\omega=n\tau_5}^{n-1} \Phi(n\tau_5) \le n\Phi(n\tau_5) = o(1). \tag{36}$$

2.4. **Bounds of the sum in (4) in other intervals**. To bound the sum in (4), except for the two infinitely small neighbourhoods of $\frac{1}{2}$ and 1, traditional methods, such as Stirling's formula, the monotonicity of $\binom{n}{\omega}$, etc., are enough to deal with it.

Lemma 6. $\sum_{\omega=n/2}^{n\tau_1} \Phi(\omega) \le \frac{1}{2} + o(1)$.

Proof. Keep Claim 4 in mind, then

$$\begin{aligned}\sum_{\omega=n/2}^{n\tau_1}\Phi(\omega)&=2^{-n}\sum_{\omega=n/2}^{n\tau_1}\binom{n}{\omega}f(\tau)^{m_1}\le 2^{-n}\sum_{\omega=n/2}^{n\tau_1}\binom{n}{\omega}f(\tau_1)^{m_1}\\&=\big(1+o(1)\big)2^{-n}\sum_{\omega=n/2}^{n\tau_1}\binom{n}{\omega}\le\frac{1}{2}+o(1).\end{aligned}$$

Lemma 7. $\sum_{\omega=n\tau_2}^{n\tau_3}\Phi(\omega)=o(1)$.

Proof. $\lim_{\tau\to1^-}\Psi(\tau)=1$, hence $\Psi(\tau_2)=2^{o(1)}$. By Claim 3,

$$\binom{n}{\omega}\le\binom{n}{n\tau_2}\le\Psi(\tau_2)^n=2^{o(1)n}. \tag{37}$$

Choose a constant $\varepsilon>0$ such that $e^{-\alpha}<1-\varepsilon$, then

$$f(\tau_3)=1+\frac{1}{2^t}\Big[\Big(1+\frac{2}{2^t}\Big)\tau_3^t-\frac{1}{2^t}\Big]=1+\big(e^{-\alpha}+o(1)\big)\frac{1}{2^t}\le1+\frac{1-\varepsilon}{2^t}. \tag{38}$$

Then

$$\begin{aligned}f(\tau)^{m_1}&\le f(\tau_3)^{m_1}\le\Big\{1+\frac{1-\varepsilon}{2^t}\Big\}^{2^tn\ln2}\\&=\exp\Big\{2^tn\ln2\ln\Big(1+\frac{1-\varepsilon}{2^t}\Big)\Big\}\le\exp\big\{(1-\varepsilon)n\ln2\big\}\\&=2^{(1-\varepsilon)n}.\end{aligned} \tag{39}$$

By (37) and (39),

$$\sum_{\omega=n\tau_2}^{n\tau_3}\Phi(\omega)\le n2^{-n}2^{o(1)n}2^{(1-\varepsilon)n}=n2^{-(\varepsilon+o(1))n}=o(1). \tag{40}$$

Lemma 8. $\sum_{\omega=n\tau_4}^{n\tau_5}\Phi(\omega)=o(1)$.

Proof. For any $\tau\in[\tau_4,\tau_5]$, there exists a unique $\beta\in[0,2\theta]$ such that $\tau=1-\frac{1}{n^{1/2+\beta}}$. Then $\ln\Psi(\tau)=\big(\frac12+\beta+o(1)\big)\frac{\ln n}{n^{1/2+\beta}}$, hence

$$\Psi(\tau)^n=2^{(\frac12+\beta+o(1))\frac{\log n}{n^\beta}\sqrt{n}}. \tag{41}$$

$$\begin{aligned}\tau^t&=\Big(1-\frac{1}{n^{1/2+\beta}}\Big)^t=\exp\Big\{t\ln\Big(1-\frac{1}{n^{1/2+\beta}}\Big)\Big\}\\&\le\exp\Big\{-\frac{t}{n^{1/2+\beta}}\Big\}=1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big).\end{aligned} \tag{42}$$

Then

$$\begin{aligned} f(\tau) &\le 1 + \frac{1}{2^t}\Big[\Big(1+\frac{2}{2^t}\Big)\Big(1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big)\Big)-\frac{1}{2^t}\Big] \\ &= 1+\frac{1}{2^t}\Big[1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big)\Big]. \end{aligned} \tag{43}$$

Then

$$\begin{aligned} 2^{-n}f(\tau)^{m_1} &\le 2^{-n}\Big[1+\frac{1}{2^t}\Big(1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big)\Big)\Big]^{m_1} \\ &= 2^{-n}\exp\Big\{m_1\ln\Big[1+\frac{1}{2^t}\Big(1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big)\Big)\Big]\Big\} \\ &\le 2^{-n}\exp\Big\{\frac{m_1}{2^t}\Big(1-\frac{t}{n^{1/2+\beta}}+o\Big(\frac{1}{\sqrt{n}}\Big)\Big)\Big\} \\ &= 2^{-(c+\frac{t}{n^\beta}+o(1))\sqrt{n}}. \end{aligned} \tag{44}$$

Define Λ on $[0, 2\theta]$ as

$$\Lambda(\beta)\sqrt{n} = \log[2^{-n}\Psi(\tau)^n f(\tau)^{m_1}]. \tag{45}$$

By (41) and (44),

$$\Lambda(\beta) \le -c + (\beta-\theta+o(1))\frac{\log n}{n^\beta} + o(1) \le -c + o(1). \tag{46}$$

By (29), $\binom{n}{\omega} = o(\Psi(\tau))^n$ is uniformly for all $n\tau_4 \le \omega \le n\tau_5$. By (45) and (46),

$$\Phi(\omega) \le 2^{-(c+o(1))\sqrt{n}}. \tag{47}$$

Hence

$$\sum_{\omega=n\tau_4}^{n\tau_5} \Phi(\omega) \le n2^{-(c+o(1))\sqrt{n}} = o(1). \tag{48}$$

Lemma 9. $\Phi(n) = o(1)$.

Proof.

$$\begin{aligned} \Phi(n) &= 2^{-n}f(1)^{m_1} = 2^{-n}\Big(1+\frac{1}{2^t}+\frac{1}{2^{2t}}\Big)^{m_1} \\ &= 2^{-n}\exp\Big\{m_1\ln\Big(1+\frac{1}{2^t}+\frac{1}{2^{2t}}\Big)\Big\} \le 2^{-n}\exp\Big\{m_1\Big(\frac{1}{2^t}+\frac{1}{2^{2t}}\Big)\Big\} \\ &= 2^{-n}\exp\Big\{(1-\frac{c}{\sqrt{n}})(1+\frac{1}{2^t})n\ln 2\Big\} = 2^{-(c+o(1))\sqrt{n}} = o(1). \end{aligned}$$

The proof of the lower bound now follows from lemmas 1, 3, 4, 5, 6, 7, 8 and 9.

Acknowledgments

This research was supported by National Natural Science Fund of China (Grant No. 11171013, 60973033).

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