The existence of strong complete mappings

Anthony B. Evans

Department of Mathematics and Statistics
Wright State University
Dayton, Ohio 45435

anthony.evans@wright.edu

Submitted: Feb 9, 2011; Accepted: Jan 23, 2012; Published: Feb 7, 2012
Mathematics Subject Classification: 05B15

Abstract

A strong complete mapping of a group G is a bijection $\theta\colon G \to G$ for which both mappings $x \mapsto x\theta(x)$ and $x \mapsto x^{-1}\theta(x)$ are bijections. We characterize finite abelian groups that admit strong complete mappings, thus solving a problem posed by Horton in 1990. We also prove the existence of strong complete mappings for countably infinite groups.

1 Introduction

For a group G, a bijection $\theta\colon G \to G$ is a *complete mapping* of G if the mapping $x \mapsto x\theta(x)$ is a bijection, an *orthomorphism* of G if the mapping $x \mapsto x^{-1}\theta(x)$ is a bijection, and a *strong complete mapping* of G if it is both a complete mapping of G and an orthomorphism of G. The term strong complete mapping was first used by Hsu and Keedwell [10] in 1985: these were called strong orthomorphisms in [1], and strong permutations in [9].

Which groups admit strong complete mappings? To admit strong complete mappings, a group must first admit complete mappings. For infinite groups the existence problem for complete mappings was settled in 1950 by Bateman [2], who proved that any infinite group admits complete mappings. In 1955 Hall and Paige [6] proved that a finite group with a nontrivial, cyclic Sylow 2-subgroup does not admit complete mappings: the converse was proved in 2009 by Wilcox [14], Evans [5], and Bray [3]. Thus the existence problem for complete mappings is settled. For strong complete mappings, however, the existence problem is far from settled.

In 1947 Paige [12] characterized finite abelian groups that admit complete mappings. A finite abelian group admits complete mappings if and only if its Sylow 2-subgroup is either trivial or noncyclic. In determining whether a finite abelian group admits strong complete mappings or not we need to consider, not only the structure of its Sylow 2-subgroup, but also the structure of its Sylow 3-subgroup. For cyclic groups, the existence problem for strong complete mappings was implicitly settled by Hedayat and Federer in the context of Knut Vic designs: a Knut Vic design of order n exists if and only if $\mathbb{Z}_n$ admits strong complete mappings. In 1975 Hedayat and Federer [8] proved that a Knut Vic design of order n exists if neither 2 nor 3 divides n, and Hedayat [7] proved the converse in 1977. For only a few other classes of finite groups has the existence problem for strong complete mappings been settled. As an example: in 1990 Evans [4] and Horton [9] independently proved that noncyclic abelian 2-groups admit strong complete mappings. In 1990 Horton [9] posed the problem of characterizing finite abelian groups that admit strong complete mappings. We will solve this problem by proving that a finite abelian group admits strong complete mappings if and only if neither its Sylow 2-subgroup nor its Sylow 3-subgroup is nontrivial and cyclic.

In Section 2 we will survey known nonexistence results. In Section 3 we will characterize finite abelian groups that admit strong complete mappings, and in Section 4 we will prove that all countably infinite groups, whether abelian or not, admit strong complete mappings.

2 Nonexistence results

For a group to admit strong complete mappings it must first admit complete mappings. In 1955 Hall and Paige [6] proved that a necessary condition for a finite group to admit complete mappings was that its Sylow 2-subgroup be either trivial or noncyclic.

Theorem 1 (Hall, Paige, 1955). *A finite group with a nontrivial, cyclic Sylow 2-subgroup does not admit complete mappings.*

Proof. See Theorem 5 in [6]. □

Hall and Paige conjectured the converse of Theorem 1. This conjecture was proved in 2009: Wilcox [14] proved that any minimal counterexample to the Hall-Paige conjecture must be a nonabelian simple group and further showed that it would have to be the Tits group or a sporadic simple group; Evans [5] reduced the number of possible minimal counterexamples to just one, J_4, Janko's fourth group; and Bray [3] completed the proof by showing that J_4 could not be a minimal counterexample.

Theorem 2 (Wilcox, Evans, Bray, 2009). *A finite group with a trivial or noncyclic Sylow 2-subgroup admits complete mappings.*

In 1990 Evans [4] established a similar necessary condition for a finite group to admit strong complete mappings.

Theorem 3 (Evans, 1990). *If a finite group G has a nontrivial, cyclic Sylow 3-subgroup S and H is a normal subgroup of G for which $G/H \cong S$, then G does not admit strong complete mappings.*

Proof. See Theorem 2 in [4]. □

This result can be strengthened for groups of odd order.

Corollary 4. *A finite group of odd order with a nontrivial, cyclic Sylow* 3*-subgroup does not admit strong complete mappings.*

Proof. If G is of odd order and has a cyclic Sylow 3-subgroup H, then there is a homomorphism from G onto H: see Corollary 1.4.18 in [11] for a proof. The result then follows from Theorem 3. □

Horton [9] proved Corollary 4 for abelian groups. It should be noted that Horton's proof, Evans' proof of Theorem 3, and Hedayat's [7] 1977 proof that the order of a Knut Vic design cannot be divisible by 3, are essentially the same.

Analogous to the Hall-Paige conjecture, we conjecture a partial converse to Theorem 3.

Conjecture 5. A finite group whose Sylow 2-subgroup is trivial or noncyclic and whose Sylow 3-subgroup is also trivial or noncyclic admits strong complete mappings.

If the Sylow 3-subgroup of a finite group is nontrivial and cyclic, but not a homomorphic image, is it still possible for the group to admit strong complete mappings? Of necessity, if such a group admits strong complete mappings, then it must be of even order and its Sylow 2-subgroup must be noncyclic. The smallest case of interest is D_6, the dihedral group of order 12: Shieh, Hsiang, and Hsu [13] have shown that this group admits strong complete mappings.

3 Abelian groups

In 1990 Horton [9] posed the problem of determining which finite abelian groups admit strong complete mappings. Also in 1990, Evans [4] conjectured that these groups would be precisely those with a trivial or noncyclic Sylow 2-subgroup and a trivial or noncyclic Sylow 3-subgroup. In this section we will prove this conjecture true. In Lemmas 6 through 9 we list some known existence results for strong complete mappings.

Lemma 6. *Any group whose order is divisible by neither* 2 *nor* 3 *admits strong complete mappings.*

Proof. Let G be a group of order n. If $\gcd(n, r) = 1$, then the mapping $x \mapsto x^r$ is a bijection. It follows that, if n is divisible by neither 2 nor 3, then $x \mapsto x^2$ is a strong complete mapping. □

Lemma 7. *If $q \geqslant 4$, then the elementary abelian group $GF(q)^+$ admits strong complete mappings.*

Proof. The mapping $x \mapsto ax$ is a strong complete mapping if $a \neq 0, \pm 1$. □

Lemma 8. *If H is a subgroup of an abelian group G and G/H and H both admit strong complete mappings, then G admits strong complete mappings.*

Proof. See Theorem 3 in [4] or Lemma 2.8 in [9]. □

Recently Shieh, Hsiang, and Hsu [13] have proved that Lemma 8 holds for nonabelian groups also.

Lemma 9. *Every noncyclic, abelian 2-group admits strong complete mappings.*

Proof. See Theorem 4 in [4] or Lemma 2.10 in [9]. □

To prove that noncyclic, abelian 3-groups admit strong complete mappings we will need to prove the existence of strong complete map pings for 3-groups of the form $G = \mathbb{Z}_3 \times \mathbb{Z}_{3m}$. For the two smallest cases, $\mathbb{Z}_3 \times \mathbb{Z}_3$ admits strong complete mappings by Lemma 7, and in 1990 Horton [9] found a strong complete mapping of $\mathbb{Z}_3 \times \mathbb{Z}_9$ via a computer search.

Let $G = \mathbb{Z}_3 \times \mathbb{Z}_{3m}$, $m \geqslant 3$ a power of 3, and let K be the subgroup of G generated by $(1, 0)$ and $(0, m)$. Thus K is isomorphic to the vector space of dimension 2 over $GF(3)$. We will use ij to denote the element (i, jm) of K and, if $k = ij \in K$ and $l \in \{0, \ldots, m-1\}$, we will use $[k, l]$ or $[ij, l]$ to denote the element $(i, jm + l)$ of G. For $ij \in K$ we will call i the *first component* and j the *second component* of ij.

For $i = 0, 1, 2$ and $j = 0, 1, \ldots, m-1$, let $\delta_{i,j}, \epsilon_{i,j} \in K$ be defined by

$$\delta_{i,j} = \begin{cases} 01 & \text{if } (3j + i \mod m) < j, \\ 00 & \text{otherwise,} \end{cases}$$

and

$$\epsilon_{i,j} = \begin{cases} 01 & \text{if } (3j + i \mod m) \geqslant m - j, \\ 00 & \text{otherwise.} \end{cases}$$

For $a \in \{1, \ldots, m-1\}$, $\gcd(a, m) = 1$, let a' denote the unique integer in $\{1, \ldots, m-1\}$ satisfying $aa' \equiv 1 \pmod{m}$.

Lemma 10. $\delta_{i,j} = 01$ *if and only if* $j \in \{m/3, \ldots, (m-3)/2\} \cup \{2m/3, \ldots, m-2\}$ *and* $i = 0, 1, 2$, *or* $j = (m-1)/2$ *and* $i = 0$, *or* $j = m-1$ *and* $i = 0, 1$.

Proof. By definition $\delta_{i,j} = 01$ if and only if $(3j + i \mod m) < j$, if and only if $0 \leqslant 3j + i - km < j$ for some integer k. As $0 \leqslant j \leqslant m-1$, this inequality can hold true for $k = 1$ or 2 only. If $k = 1$, then the inequality holds if and only if

$$\frac{m}{3} \leqslant j \leqslant \begin{cases} \frac{m-1}{2} & \text{if } i = 0, \\ \frac{m-3}{2} & \text{if } i = 1, 2. \end{cases}$$

If $k = 2$, then the inequality holds if and only if

$$\frac{2m}{3} \leqslant j \leqslant \begin{cases} m-1 & \text{if } i = 0,1, \\ m-2 & \text{if } i = 2. \end{cases}$$

Hence the result. □

The following lemma gives a sufficient condition for $\mathbb{Z}_3 \times \mathbb{Z}_{3m}$, m a power of 3, to admit strong complete mappings.

Lemma 11. *The group $G = \mathbb{Z}_3 \times \mathbb{Z}_{3m}$, $m \geqslant 3$ a power of 3, admits strong complete mappings if there exist $\alpha_{i,j} \in K$, $i = 0, 1, 2$ and $j = 0, 1, \ldots, m-1$, and $x_j \in \{01, 02\}$, $j = 0, 1, \ldots, m-1$, for which the following three conditions hold.*

1. *For $i = 0, 1, 2$ and $j = 0, 1, \ldots, \frac{m}{3} - 1$, the set*
$$\{\alpha_{i,j}, \alpha_{i,j+m/3}, \alpha_{i,j+2m/3}\}$$
is a system of distinct coset representatives for $\langle 11 \rangle$ in K.

2. *For $j = 0, 1, \ldots, m-1$, the set*
$$\{\alpha_{0,j} - \delta_{0,j}, \alpha_{1,j-2'} - x_{j-2'} - \delta_{1,j-2'}, \alpha_{2,j-1} + x_{j-1} - \delta_{2,j-1}\}$$
is a system of distinct coset representatives for $\langle 10 \rangle$ in K, where the indices are computed modulo m.

3. *For $j = 0, 1, \ldots, m-1$, the set*
$$\{\alpha_{0,j}, \alpha_{1,j-4'}, \alpha_{2,j-2'}\}$$
is a system of distinct coset representatives for $\langle 01 \rangle$ in K, where the second indices are computed modulo m.

Proof. Let H be the subgroup of K generated by 12 and let $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $HA = \langle 11 \rangle$, $H(A - I) = \langle 10 \rangle$, and $H(A + I) = \langle 01 \rangle$. Let $\alpha_{i,j} \in K$, $i = 0, 1, 2$ and $j = 0, 1, \ldots, m-1$, and $x_j \in \{01, 02\}$, $j = 0, 1, \ldots, m-1$, satisfy the conditions of the lemma and let $\theta \colon G \to G$ be define d by

$$\theta([h + ix_j, j]) = [hA + \alpha_{i,j}, (3j + i \mod m)],$$

where $h \in H$, $i \in \{0, 1, 2\}$, and $j \in \{0, 1, \ldots, m-1\}$.

Clearly θ is well-defined.

As
$$3j + i \equiv 3\left(j + \frac{m}{3}\right) + i \equiv 3\left(j + \frac{2m}{3}\right) + i \pmod{m},$$

θ is a bijection if and only if, the set $\{\alpha_{i,j}, \alpha_{i,j+m/3}, \alpha_{i,j+2m/3}\}$, $i = 0, 1, 2$ and $j = 0, 1, \ldots, \frac{m}{3} - 1$, is a system of distinct coset representatives for $\langle 11 \rangle$ in K.

Now $\theta([h + ix_j, j]) - [h + ix_j, j] = [h(A - I) - ix_j + \alpha_{i,j} - \delta_{i,j}, (2j + i \mod m)]$. As $2j \equiv 2(j - 2') + 1 \equiv 2(j - 1) + 2 \pmod{m}$, the mapping $[h + ix_j, j] \mapsto \theta([h + ix_j, j]) - [h + ix_j, j]$ is a bijection if and only if, for $j = 0, 1, \ldots, m-1$, the set $\{\alpha_{0,j} - \delta_{0,j}, \alpha_{1,j-2'} - x_{j-2'} - \delta_{1,j-2'}, \alpha_{2,j-1} + x_{j-1} - \delta_{2,j-1}\}$ is a system of distinct coset representatives for $\langle 10 \rangle$ in K.

Now $\theta([h + ix_j, j]) + [h + ix_j, j] = [h(A + I) + ix_j + \alpha_{i,j} + \epsilon_{i,j}, (4j + i \mod m)]$. As $4j \equiv 4(j-4') + 1 \equiv 4(j-2') + 2 \pmod m$ and $x_j, \epsilon_{i,j} \in H(A+I)$ for all i, j, the mapping $[h+ix_j, j] \mapsto \theta([h+ix_j, j]) + [h+ix_j, j]$ is a bijection if and only if, for $j = 0, 1, \ldots, m-1$, the set $\{\alpha_{0,j}, \alpha_{1,j-4'}, \alpha_{2,j-2'}\}$ is a system of distinct coset representatives for $\langle 01 \rangle$ in K. The result follows. □

We call the condition in Lemma 11(1) the Θ *condition*, the condition in Lemma 11(2) the Δ *condition*, and the condition in Lemma 11(3) the Σ *condition*. We call the sets $\{\alpha_{i,j}, \alpha_{i,j+m/3}, \alpha_{i,j+2m/3}\}$, Θ *sets*; the sets $\{\alpha_{0,j} - \delta_{0,j}, \alpha_{1,j-2'} - x_{j-2'} - \delta_{1,j-2'}, \alpha_{2,j-1} + x_{j-1} - \delta_{2,j-1}\}$, Δ *sets*; and the sets $\{\alpha_{0,j}, \alpha_{1,j-4'}, \alpha_{2,j-2'}\}$, Σ *sets*. The differences, first components minus second components modulo m, will be called the *component differences*. The Θ condition is satisfied if and only if, for each Θ set, the component differences are all distinct. The Δ condition is satisfied if and only if, for each Δ set, the second components are all distinct, and the Σ condition is satisfied if and only if, for each Σ set, the first components are all distinct.

As a corollary to Lemma 11, we can show that $\mathbb{Z}_3 \times \mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_{27}$ admit strong complete mappings.

Corollary 12. *The groups* $\mathbb{Z}_3 \times \mathbb{Z}_9$ *and* $\mathbb{Z}_3 \times \mathbb{Z}_{27}$ *admit strong complete mappings.*

Proof. For $m = 3$, let $x_0 = 01$, $x_1 = 02$, $x_2 = 01$, and, for $i, j = 0, 1, 2$, let $\alpha_{i,j}$ be the ijth entry in the following table.

	0	1	2
0	00	01	02
1	11	20	10
2	11	20	21

It is routine to show that the conditions of Lemma 11 are satisfied and, hence, that $\mathbb{Z}_3 \times \mathbb{Z}_9$ admits strong complete mappings. The corresponding strong complete mapping is described in the table below.

x	$\theta(x)$	x	$\theta(x)$	x	$\theta(x)$	x	$\theta(x)$	x	$\theta(x)$	x	$\theta(x)$
00	00	26	27	17	20	04	22	25	10	15	12
16	26	06	15	24	16	11	18	05	11	22	08
23	13	13	02	07	21	27	05	12	07		
03	14	20	28	14	17	02	06	28	24		
10	01	01	03	21	04	18	23	08	25		

For $m = 9$, let

$$x_j = \begin{cases} 01 & \text{if } j = 3, 4, 5 \\ 02 & \text{otherwise,} \end{cases}$$

and, for $i = 0, 1, 2$ and $j = 0, 1, \ldots, 8$, let $\alpha_{i,j}$ be the ijth entry in the following table.

	0	1	2	3	4	5	6	7	8
0	00	12	00	10	00	12	20	10	10
1	02	22	12	20	12	21	11	21	11
2	02	00	21	00	20	00	20	02	20

It is routine to show that the conditions of Lemma 11 are satisfied and, hence, that $\mathbb{Z}_3 \times \mathbb{Z}_{27}$ admits strong complete mappings. $\square$

More generally, $\mathbb{Z}_3 \times \mathbb{Z}_{3m}$ admits strong complete mappings.

Lemma 13. *If $m = 3^n$, $n \geqslant 2$, then $\mathbb{Z}_3 \times \mathbb{Z}_{3m}$ admits strong complete mappings.*

Proof. We will choose $\alpha_{i,j}$ and x_j, $i = 0, 1, 2$ and $j = 0, \ldots, m-1$, so that the conditions of Lemma 11 are satisfied.

We set

$$x_j = \begin{cases} 01 & \text{if } j = 2m/3, \ldots, (5m-9)/6, \\ 02 & \text{otherwise,} \end{cases}$$

$\alpha_{0,j} = 0k$, where

$$k = \begin{cases} 0 & \text{if } 0 < j < m/3, \text{ or } j = 2m/3, \\ 1 & \text{if } m/3 \leqslant j < 2m/3, \\ 2 & \text{if } 2m/3 < j \leqslant m-1, \text{ or } j = 0, \end{cases}$$

$\alpha_{1,j} = 1k$, where

$$k = \begin{cases} 0 & \text{if } 0 \leqslant (j + 2' \mod m) < m/3, \\ 1 & \text{if } m/3 \leqslant (j + 2' \mod m) < 2m/3, \\ 2 & \text{if } 2m/3 \leqslant (j + 2' \mod m) \leqslant m-1. \end{cases}$$

and $\alpha_{2,j} = 2k$, where

$$k = \begin{cases} 0 & \text{if } 0 < (j + 1 \mod m) \leqslant m/3, \\ 1 & \text{if } m/3 < (j + 1 \mod m) < 2m/3, \text{ or } (j+1 \mod m) = 0, \\ 2 & \text{if } 2m/3 \leqslant (j + 1 \mod m) \leqslant m-1. \end{cases}$$

For each $j = 0, \ldots, m-1$, the first components of the Σ set $\{\alpha_{0,j}, \alpha_{1,j-4'}, \alpha_{2,j-2'}\}$ are $\{0, 1, 2\}$. Hence, for each $j = 0, \ldots, m-1$, the Σ set $\{\alpha_{0,j}, \alpha_{1,j-4'}, \alpha_{2,j-2'}\}$ is a system of distinct coset representatives for $\langle 01 \rangle$, and so the Σ condition is satisfied.

Simple computation shows that the set $\{-\delta_{0,j}, -x_{j-2'} - \delta_{1,j-2'}, x_{j-1} - \delta_{2,j-1}\}$ is equal to $\{00, 01, 02\}$ if $j \in \{1, \ldots, (m-3)/3\} \cup \{(m+1)/2, \ldots, (2m-3)/3\}$, $\{02, 00, 01\}$ if $j \in \{(m+3)/3, \ldots, (m-1)/2\} \cup \{(5m+3)/6, \ldots, m-1\}$, and $\{02, 01, 00\}$ if $j \in \{(2m+3)/3, \ldots, (5m-3)/6\}$. For each of these sets the second components form a permutation of $\{0, 1, 2\}$, and hence each is a system of distinct coset representatives for $\langle 10 \rangle$. As $\alpha_{0,j}$, $\alpha_{0,j-2'}$, and $\alpha_{0,j-1}$ are the same if $j \neq 0, m/3$ or $2m/3$, it follows that the Δ set $\{\alpha_{0,j} - \delta_{0,j}, \alpha_{1,j-2\prime} - x_{j-2'} - \delta_{1,j-2'}, \alpha_{2,j-1} + x_{j-1} - \delta_{2,j-1}\}$ is a system of distinct coset representatives for $\langle 10 \rangle$, if $j \neq 0, m/3$ or $2m/3$. For $j = 0, m/3$ or $2m/3$, the Δ set $\{\alpha_{0,j} - \delta_{0,j}, \alpha_{1,j-2'} - x_{j-2'} - \delta_{1,j-2'}, \alpha_{2,j-1} + x_{j-1} - \delta_{2,j-1}\} = \{02, 11, 20\}$ if $j = 0$, $\{00, 11, 22\}$ if $j = m/3$, and $\{02, 10, 21\}$ if $j = 2m/3$: as, for each of these Δ sets the second components form a permutation of $\{0, 1, 2\}$, each of these Δ sets is a system of distinct coset representatives for $\langle 10 \rangle$. Hence the Δ condition is satisfied.

By our construction the Θ set $\{\alpha_{0,0}, \alpha_{0,m/3}, \alpha_{0,2m/3}\} = \{02, 01, 00\}$ and the Θ set $\{\alpha_{2,m-1}, \alpha_{2,(m-3)/3}, \alpha_{2,(2m-3)/3}\} = \{21, 20, 22\}$, the component differences being $\{1, 2, 0\}$ in both cases. For all other Θ sets $\{\alpha_{i,j}, \alpha_{i,j+m/3}, \alpha_{i,j+2m/3}\} = \{i0, i1, i2\}$ in some order; the component differences a re then clearly $\{0, 1, 2\}$. It follows that each of these Θ sets is a system of distinct coset representatives for $\langle 11 \rangle$. Hence the Θ condition is satisfied, and $\mathbb{Z}_3 \times \mathbb{Z}_{3m}$ admits strong complete mappings by Lemma 11. $\square$

The proof of Lemma 13 yields another construction of a strong complete mapping of $\mathbb{Z}_3 \times \mathbb{Z}_{27}$.

We can now prove the existence of strong complete mappings for noncyclic, abelian 3-groups.

Lemma 14. *Every noncyclic, abelian* 3*-group admits strong complete mappings.*

Proof. If G is a noncyclic, abelian 3-group then it is an exercise to show that G admits a subnormal series $\{e\} = K_1 < \cdots < K_s = G$, in which $H_i = K_i/K_{i-1} \cong GF(q_i)^+$, $q_i > 3$, if $i < s$, and $H_s = K_s/K_{s-1} \cong GF(q)^+$, $q > 3$, or $\mathbb{Z}_{3m} \times \mathbb{Z}_3$, $m = 3^n$. By Lemma 7, H_i admits strong complete mappings for $i < s$. If $H_s \cong GF(q)^+$, $q > 3$, then H_s admits strong complete mappings by Lemma 7. If $H_s \cong \mathbb{Z}_{3m} \times \mathbb{Z}_3$, then H_s admits strong complete mappings by Lemma 7 if $m = 1$, by Corollary 12 if $m = 3$, or by Lemma 13 if $m > 3$. Hence, by repeated applications of Lemma 8, G admits strong complete mappings. $\square$

We are now in a position to characterize finite abelian groups that admit strong complete mappings.

Theorem 15. *A finite abelian group admits strong complete mappings if and only if neither its Sylow* 2*-subgroup nor its Sylow* 3*-subgroup is nontrivial and cyclic.*

Proof. Let G be a finite abelian group and let $p_1, \ldots, p_r$ be the distinct prime divisors of $|G|$. Let H_i be a Sylow p_i-subgroup of G for $i = 1, \ldots, r$. If $p_i = 2$ for some i and H_i is cyclic, then, by Theorem 1, G does not admit complete mappings, and hence does not admit strong complete mappings. Similarly If $p_i = 3$ for some i and H_i is cyclic, then, by Theorem 3, G does not admit strong complete mappings.

Next let us assume that the Sylow 2-subgroup of G is trivial or noncyclic and that the Sylow 3-subgroup of G is also trivial or noncyclic. If $p_i = 2$, then H_i admits strong complete mappings by Lemma 9, if $p_i = 3$, then H_i admits strong complete mappings by Lemma 14, and if $p_i \neq 2$, then H_i admits strong complete mappings by Lemma 6. As $G \cong H_1 \times \cdots \times H_r$, the result follows from repeated applications of Lemma 8. $\square$

4 Countably Infinite Groups

In 1950 Bateman [2] proved, using transfinite induction, that any infinite group admits complete mappings. We will adapt Bateman's proof to prove the existence of strong complete mappings for countably infinite groups.

Theorem 16. *Any countably infinite group admits strong complete mappings.*

Proof. Let $G = \{g_1, g_2, g_3, \ldots\}$ be a countably infinite group. Pick $a, b \in G$ and set $x_1 = a$, $y_1 = b$, $w_1 = a^{-1}b$, and $z_1 = ab$. Set $X_1 = \{x_1\}$, $Y_1 = \{y_1\}$, $W_1 = \{w_1\}$, and $Z_1 = \{z_1\}$. We will recursively define sets X_n, Y_n, W_n, and Z_n, $n = 1, 2, 3, \ldots$.

Having defined X_n, Y_n, W_n, and Z_n, let u_n be the element of G of least index that is not an element of $X_n \cap Y_n \cap W_n \cap Z_n$.

If $u_n \notin X_n$, then set $x_{n+1} = u_n$, set y_{n+1} equal to the element of G of least index which is not in Y_n and for which $x_{n+1}y_{n+1} \notin Z_n$ and $x_{n+1}^{-1}y_{n+1} \notin W_n$. Set $z_{n+1} = x_{n+1}y_{n+1}$ and $w_{n+1} = x_{n+1}^{-1}y_{n+1}$, $X_{n+1} = X_n \cup \{x_{n+1}\}$, $Y_{n+1} = Y_n \cup \{y_{n+1}\}$, $W_{n+1} = W_n \cup \{w_{n+1}\}$, and $Z_{n+1} = Z_n \cup \{z_{n+1}\}$.

If $u_n \in X_n$ but $u_n \notin Y_n$, then set $y_{n+1} = u_n$, set x_{n+1} equal to the element of G of least index which is not in X_n and for which $x_{n+1}y_{n+1} \notin Z_n$ and $x_{n+1}^{-1}y_{n+1} \notin W_n$. Set $z_{n+1} = x_{n+1}y_{n+1}$ and $w_{n+1} = x_{n+1}^{-1}y_{n+1}$, $X_{n+1} = X_n \cup \{x_{n+1}\}$, $Y_{n+1} = Y_n \cup \{y_{n+1}\}$, $W_{n+1} = W_n \cup \{w_{n+1}\}$, and $Z_{n+1} = Z_n \cup \{z_{n+1}\}$.

If $u_n \in X_n \cap Y_n$ but $u_n \notin W_n$, then set $w_{n+1} = u_n$, set x_{n+1} equal to the element of G of least index which is not in X_n and for which $x_{n+1}w_{n+1} \notin Y_n$ and $x_{n+1}^2 w_{n+1} \notin Z_n$. Set $y_{n+1} = x_{n+1}w_{n+1}$ and $z_{n+1} = x_{n+1}^2 w_{n+1}$, $X_{n+1} = X_n \cup \{x_{n+1}\}$, $Y_{n+1} = Y_n \cup \{y_{n+1}\}$, $W_{n+1} = W_n \cup \{w_{n+1}\}$, and $Z_{n+1} = Z_n \cup \{z_{n+1}\}$.

If $u_n \in X_n \cap Y_n \cap W_n$ but $u_n \notin Z_n$, then set $z_{n+1} = u_n$, set x_{n+1} equal to the element of G of least index which is not in X_n and for which $x_{n+1}^{-1}z_{n+1} \notin Y_n$ and $x_{n+1}^{-2}z_{n+1} \notin W_n$. Set $y_{n+1} = x_{n+1}^{-1}z_{n+1}$ and $w_{n+1} = x_{n+1}^{-2}z_{n+1}$, $X_{n+1} = X_n \cup \{x_{n+1}\}$, $Y_{n+1} = Y_n \cup \{y_{n+1}\}$, $W_{n+1} = W_n \cup \{w_{n+1}\}$, and $Z_{n+1} = Z_n \cup \{z_{n+1}\}$.

If we set $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$, $W = \{w_1, w_2, \ldots\}$, and $Z = \{z_1, z_2, \ldots\}$, then each element of G appears exactly once in each of the sequences X, Y, W, and Z. hence, the mapping $x_n \mapsto y_n$ is a strong complete mapp ing of G. □

References

[1] B. A. Anderson. Sequencings and houses. *Congr. Numer.* 43: 23–43, 1984.

[2] P. T. Bateman. A remark on infinite groups. *Amer. Math. Monthly* 57: 623–624, 1950.

[3] J. N. Bray. *Personal communication.*

[4] A. B. Evans. On strong complete mappings. *Congr. Numer.* 70: 241–248, 1990.

[5] A. B. Evans. The admissibility of sporadic simple groups. *J. Algebra* 321: 105–116, 2009.

[6] M. Hall, L. J. Paige. Complete mappings of finite groups. *Pacific J. Math.* 5: 541–549, 1955.

[7] A. Hedayat. A complete solution to the existence and non-existence of Knut Vic designs and orthogonal Knut Vic designs. *J. Combin. Theory Ser. A* 22(3): 331–337, 1977.

[8] A. Hedayat, W. T. Federer. On the non-existence of Knut Vic designs for all even orders. *Ann. Statist.* 3: 445–447, 1975.

[9] J. D. Horton. Orthogonal starters in finite abelian groups. *Discrete Math.* 79(3): 265–278, 1990.

[10] D. F. Hsu, A. D. Keedwell. Generalized complete mappings, neofields, sequenceable groups and block designs. II. *Pacific J. Math.* 117(2): 291–312, 1985.

[11] G. O. Michler. Theory of finite simple groups. *Cambridge University Press*, Cambridge, 2006.

[12] L. J. Paige. A note on finite abelian groups. *Bull. Amer. Math. Soc.* 53: 590–593, 1947.

[13] Y.-P. Shieh, J. Hsiang, D. F. Hsu. On the existence problems of complete mappings. *Preprint.*

[14] S. Wilcox. Reduction of the Hall-Paige conjecture to sporadic simple groups. *J. Algebra* 321: 1407–1428, 20 09.

Corrigendum added 2 October 2018.

In Theorem 16 it is claimed that any countably infinite group G admits strong complete mappings. Matt Ollis pointed out that this proof fails when $\{g^2 \mid g \in G\}$ is finite. The corrected statement of the theorem is:

Theorem 16. *If G is a countably infinite group and $\{g^2 \mid g \in G\}$ is countably infinite, then G admits strong complete mappings.*