# The negative $q$-binomial

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**Abstract**

Interpretations for the $q$-binomial coefficient evaluated at $-q$ are discussed. A $(q,t)$-version is established, including an instance of a cyclic sieving phenomenon involving unitary spaces.

**Keywords**: q-binomial, Gaussian polynomial, (q,t)-binomial, Ennola duality, unitary group, unitary space, characteristic map, invariant theory, cyclic sieving phenomenon

## 1 The $q$-binomial

The $q$-binomial coefficient is defined for integers $k$ and $n$, with $0 \leq k \leq n$, and an indeterminate $q$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}} \tag{1.1}$$

where $(q)_n = (1-q^1)(1-q^2)\cdots(1-q^n)$. It is well-known [1, p. 39] that the $q$-binomial coefficient is a polynomial in $q$ with non-negative integer coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\omega \in \Omega_{n,k}} q^{\mathrm{inv}(\omega)}, \tag{1.2}$$

where $\Omega_{n,k}$ is the set of words $\omega = (\omega_1, \ldots, \omega_n)$ in $\{0,1\}^n$ having $k$ ones and $n-k$ zeroes, and $\mathrm{inv}(\omega)$ is the number of *inversions* in $\omega$, that is, pairs $(i,j)$ with $1 \leq i < j \leq n$ and $\omega_i = 1, \omega_j = 0$; see [1, p. 40]. When $q$ is a prime power, the $q$-binomial coefficient (1.1) is an integer counting the number of $k$-dimensional spaces in the $n$-dimensional vector space $\mathbb{F}_q^n$ over the field $\mathbb{F}_q$.

Section 2 combinatorially interprets the $q$-binomial coefficient when $q$ is a negative integer (Theorem 2.1), while Section 3 establishes a positivity theorem for a $(q,t)$-analogue when $q$ is negative (Theorem 3.1). Section 4 provides a different interpretation for the negative $q$-binomial, counting unitary subspaces, and related to *Ennola duality* for finite unitary groups. Section 5 proves a cyclic sieving phenomenon involving the $(q,t)$-analogue at negative $q$ and unitary subspaces. Section 6 collects some remarks and remaining questions suggested by these results.

# 2 The negative $q$-binomial

Let $q \geq 2$ be an integer and define

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q := (-1)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q}. \tag{2.1}$$

It is not hard to derive from the the product expression in (1.1) that this primed $q$-binomial will be positive, and it follows from (1.2) that it is an integer, with

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q \leq \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{2.2}$$

Our main result Theorem 2.1 is an analogue of (1.2) for the primed $q$-binomial coefficient that clearly demonstrates (2.2). It expands the primed $q$-binomial coefficient as a sum over words $\omega$ in a subset $\Omega'_{n,k} \subset \Omega_{n,k}$, with weights $\mathrm{wt}(\omega)$ satisfying $1 \leq \mathrm{wt}(\omega) \leq q^{\mathrm{inv}(\omega)}$ for $q \geq 2$. This subset $\Omega'_{n,k}$ and weight $\mathrm{wt}(\omega)$ come from a *pairing* algorithm explained next.

**Definition.** Given $\omega = (\omega_1, \ldots, \omega_n)$ in $\Omega_{n,k}$, pair some of its adjacent entries $(\omega_i, \omega_{i+1})$, and leave others unpaired, according to the following recursive rule:

- When $n = 1$, leave the unique letter $\omega_1$ in $\omega$ unpaired.
- When $n \geq 2$ and $k$ is odd, pair the first two entries $\omega_1, \omega_2$, and recursively pair the remaining word $(\omega_3, \ldots, \omega_n)$.

- When $n \geq 2$ and $k$ is even, leave the first entry $\omega_1$ unpaired, and recursively pair the remaining word $(\omega_2, \omega_3, \ldots, \omega_n)$.

Two examples of pairings for words, with pairings indicated by underlining, are

$$\omega^{(1)} = \underline{01}\ \underline{1}\ \underline{00}\ \underline{10}\ \underline{1}\ \underline{01}$$
$$\omega^{(2)} = \underline{1}\ \underline{10}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{00}\ \underline{1}.$$

Define

$$\Omega'_{n,k} := \{\omega \in \Omega_{n,k} : \omega \text{ has no paired } \underline{01}\}.$$

For example, $\omega^{(2)}$ lies in $\Omega'_{n,k}$, but $\omega^{(1)}$ does not. For $\omega \in \Omega'_{n,k}$, define

$$\begin{aligned} p(\omega) &:= \text{number of } \underline{10} \text{ pairs in } \omega \\ a(\omega) &:= \mathrm{inv}(\omega) - p(\omega) \\ \mathrm{wt}(\omega) &:= q^{a(\omega)}(q-1)^{p(\omega)}. \end{aligned}$$

Note that $a(\omega) \geq 0$ since each pair $\underline{10}$ contributes at least 1 to the value of $\mathrm{inv}(\omega)$. In fact, it is helpful to think of $a(\omega)$ as a perturbation of the inversion statistic $\mathrm{inv}(\omega)$ as follows: each occurrence of 1 in $\omega$ would normally contribute to $\mathrm{inv}(\omega)$ the number of zeroes to its right, but when this 1 occurs in a pair $\underline{10}$ it contributes one fewer than usual to $a(\omega)$.

Note also that $1 \leq \mathrm{wt}(\omega) \leq q^{\mathrm{inv}(\omega)}$ for $q \geq 2$. This brings us to the main result.

**Theorem 2.1.** *For $0 \leq k \leq n$, one has*

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q = \sum_{\omega \in \Omega'_{n,k}} \mathrm{wt}(\omega) = \sum_{\omega \in \Omega'_{n,k}} q^{a(\omega)}(q-1)^{p(\omega)}.$$

For example, if $(n,k) = (5,2)$, one has this data and calculation:

| $\omega \in \Omega'_{5,2}$ | $\mathrm{wt}(\omega)$ |
|---|---|
| $\underline{0}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{1}$ | $1$ |
| $\underline{0}\ \underline{0}\ \underline{1}\ \underline{10}$ | $q(q-1)$ |
| $\underline{0}\ \underline{1}\ \underline{00}\ \underline{1}$ | $q^2$ |
| $\underline{0}\ \underline{1}\ \underline{10}\ \underline{0}$ | $q^3(q-1)$ |
| $\underline{1}\ \underline{00}\ \underline{10}$ | $q^3(q-1)$ |
| $\underline{1}\ \underline{10}\ \underline{0}\ \underline{0}$ | $q^5(q-1)$ |

$$\begin{aligned} &1 + q(q-1) + q^2 + q^3(q-1) + q^3(q-1) + q^5(q-1) \\ &= q^6 - q^5 + 2q^4 - 2q^3 + 2q^2 - q + 1 \\ &= (-1)^{2\cdot(5-2)} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{-q} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}'_q. \end{aligned}$$

*Proof of Theorem 2.1.* Induct on $n$, with easily verified base cases $n = 0, 1$. In the inductive step, we use this $q$-Pascal recurrence (2.3), and its iterate (2.4)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{2.3}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-2 \\ k \end{bmatrix}_q + q^{n-k-1}(q+1) \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q + q^{2(n-k)} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q. \tag{2.4}$$

When $k$ is even, replacing $q \mapsto -q$ in (2.3) gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q' + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q'. \tag{2.5}$$

Since $k$ is even, the leading entry $\omega_1$ will be either be an unpaired $\underline{0}$ or $\underline{1}$. In either case, $\omega_1$ contributes 0 to $p(\omega)$. If $\omega_1 = \underline{0}$ it contributes 0 to $a(\omega)$, and corresponds to the first summand on the right of (2.5), while if $\omega_1 = \underline{1}$ it contributes $n-k$ to $a(\omega)$, and corresponds to the second summand on the right of (2.5).

When $k$ is odd, replacing $q \mapsto -q$ in (2.4) gives

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q' &= q^0(q-1)^0 \begin{bmatrix} n-2 \\ k \end{bmatrix}_q' \\ &\quad + q^{n-k-1}(q-1)^1 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q' \\ &\quad + q^{2(n-k)}(q-1)^0 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q'. \end{aligned} \tag{2.6}$$

Since $k$ is odd, $(\omega_1, \omega_2)$ will be paired, and since $\omega$ lies in $\Omega'_{n,k}$, the pair $(\omega_1, \omega_2)$ takes one of the three forms $\underline{00}$, $\underline{10}$, or $\underline{11}$,

- contributing 0, $n-k-1$, or $2(n-k)$, respectively, to $a(\omega)$,
- contributing 0, 1, or 0, respectively, to $p(\omega)$, and
- leaving $k$ ones, $k-1$ ones, or $k-2$ ones, respectively, in $(\omega_3, \ldots, \omega_n)$.

Thus the three forms correspond to the three summands of (2.6) □

We note that one can reformulate Theorem 2.1 as an expansion of the $q$-binomial coefficient, with no negative signs, as follows.

**Corollary 2.2.** *If* $0 \le k \le n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\omega \in \Omega'_{n,k}} q^{a(\omega)}(q+1)^{p(\omega)}.$$

*Proof.* Setting $q \mapsto -q$ in Theorem 2.1 and multiplying by $(-1)^{k(n-k)}$ gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\omega \in \Omega'_{n,k}} (-1)^{a(\omega)+p(\omega)+k(n-k)} q^{a(\omega)} (q+1)^{p(\omega)}.$$

Comparing with the corollary, it suffices to show that for each $\omega$ in $\Omega'_{n,k}$ one has the following parity condition:

$$\mathrm{inv}(\omega) = a(\omega) + p(\omega) \quad \equiv \quad k(n-k) \pmod 2. \tag{2.7}$$

This can be checked via induction on $n$ using the recursive definition of $\Omega'_{n,k}$:

Case 1. $k$ is even.
If $\omega = \underline{0}\omega'$, then $\mathrm{inv}(\omega) = \mathrm{inv}(\omega') \equiv k(n-1-k) \equiv k(n-k) \bmod 2$.
If $\omega = \underline{1}\omega'$, then $\mathrm{inv}(\omega) = n-k+\mathrm{inv}(\omega') \equiv n-k+(k-1)(n-k) \equiv k(n-k) \bmod 2$.

Case 2. $k$ is odd.
If $\omega = \underline{00}\omega'$ or $\omega = \underline{11}\omega'$, then $\mathrm{inv}(\omega) \equiv \mathrm{inv}(\omega') \equiv k(n-k-2) \equiv k(n-k) \bmod 2$.
If $\omega = \underline{10}\omega'$, then

$$\mathrm{inv}(\omega) = n-k+\mathrm{inv}(\omega') \equiv n-k+(k-1)(n-1-k) \equiv k(n-k) \bmod 2.$$

□

# 3 The $(q,t)$-binomial at negative $q$

In [10, p. 43] the authors consider a certain $(q,t)$-analogue of the $q$-binomial: a polynomial in $t$ with positive integer coefficients, depending upon a positive integer $q$, and whose limit as $t$ goes to 1 is the $q$-binomial [9, Corollary 3.2]. Here we first review the definition and interpretation of this $(q,t)$-binomial, and then establishing a positivity result for it (Theorem 3.1) when $q$ is a *negative* integer.

This $(q,t)$-version of the binomial coefficient is defined by (see [9] or [10, p. 43])

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \prod_{i=1}^{k} \frac{1-t^{q^n-q^{i-1}}}{1-t^{q^k-q^{i-1}}}. \tag{3.1}$$

When $q$ a positive integer, one can show[1] that this turns out to be a polynomial in $t$ with nonnegative integer coefficients. One can easily check from the above definition that its degree in $t$ is $k(q^n-q^k)$, and that

$$t^{k(q^n-q^k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \tag{3.2}$$

so that its coefficient sequence will be symmetric about the power $t^{\frac{1}{2}k(q^n-q^k)}$.

[1] It follows, e.g., by iterating the $(q,t)$-Pascal recurrence (3.3) from the proof of Theorem 3.1.

When $q$ is a prime power, and hence the order of a finite field $\mathbb{F}_q$, these properties of the $(q,t)$-binomial follow from its interpretation as the Hilbert series for a certain graded ring, briefly reviewed here; see [10, §9] or [9] for more detail. One starts with a $S := \mathbb{F}_q[x_, \ldots, x_n]$ a polynomial algebra on which $G := GL_n(\mathbb{F}_q)$ acts by linear substitutions of variables. One has nested subalgebras $S^G \subset S^P$ of $G$-invariant polynomials $S^G$, and $P$-invariant polynomials $S^P$, where $P$ is the parabolic subgroup of $G$ that stabilizes a particular $k$-dimensional subspace of $\mathbb{F}_q^n$. Then the $(q,t)$-binomial coefficient in (3.1) is the Hilbert series in the variable $t$ for the quotient ring $S^P/(S^G_+)$, in which $(S^G_+)$ denotes the ideal of $S^P$ generated by the $G$-invariant polynomials of strictly positive degree.

**Theorem 3.1.** *When $q \leq -2$ is a negative integer, the $(q,t)$-binomial defined as a rational function in* (3.1) *lies in $(-1)^{k(n-k)}\mathbb{N}[t,t^{-1}]$, meaning that it is a Laurent polynomial in $t$ whose nonzero coefficients all have sign $(-1)^{k(n-k)}$.*

*Furthermore, its coefficient sequence is symmetric about $t^{\frac{1}{2}k(q^n-q^k)}$, with monic coefficients on its smallest and largest powers of $t$, which are the following powers:*

$$\begin{array}{lll}
\{ \quad t^0 \quad , \quad t^{k(q^n-q^k)} \quad \} & \text{if } n,k \text{ are both even,} \\
\{ \quad t^{k(q^n-q^k)} \quad , \quad t^0 \quad \} & \text{if } n,k \text{ are both odd,} \\
\{ \quad t^{kq^n-\frac{1-q^k}{1-q}} \quad , \quad t^{-kq^k+\frac{1-q^k}{1-q}} \quad \} & \text{if } n \text{ is odd and } k \text{ is even,} \\
\{ \quad t^{-kq^k+\frac{1-q^k}{1-q}} \quad , \quad t^{kq^n-\frac{1-q^k}{1-q}} \quad \} & \text{if } n \text{ is even and } k \text{ is odd.}
\end{array}$$

*Proof.* The main issue is proving that this $(q,t)$-binomial with $q \leq -2$ is a Laurent polynomial of the appropriate sign. Given this, the coefficient symmetry follows from the validity of (3.2) for any integer value of $q$. The last assertion of the theorem follows by symmetry, after examining in each case the beginning of the Laurent expansion of the product on the right side of (3.1). We omit the details.

The first assertion is proven by induction on $n$, as in the proof of Theorem 2.1. The base cases $n = 0, 1$ are again easily verified. In the inductive step, one proceeds in two cases, based on the parity of $n-k$.

Case 1. $n-k$ is even.
We use the analogue [9, Prop. 4.1] of (2.3) with $k$ replaced by $n-k$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q,t^q} + t^{q^k-1} \prod_{i=0}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q,t^q} \tag{3.3}$$

We check that both summands on the right of (3.3) lie in $(-1)^{k(n-k)}\mathbb{N}[t,t^{-1}] = \mathbb{N}[t,t^{-1}]$. By induction on $n$, the first term lies in $(-1)^{(k-1)(n-k)}\mathbb{N}[t,t^{-1}] = \mathbb{N}[t,t^{-1}]$.

For the second term, again by induction on $n$, one knows that its $(q,t)$-binomial factor lies in $(-1)^{k(n-k-1)}\mathbb{N}[t,t^{-1}] = (-1)^k\mathbb{N}[t,t^{-1}]$. It then suffices to verify that the product over $i = 0, 1, \ldots, k-1$ in the second term has each of its $k$ factors lying in $(-1)^1\mathbb{N}[t,t^{-1}]$. To verify this, let $m := q^k - q^i$ and then this factor can be rewritten

$$\begin{aligned} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} &= \frac{1-t^{qm}}{1-t^m} = -t^{qm}\frac{1-t^{-qm}}{1-t^m} \\ &= -t^{qm}\left(1+t^m+t^{2m}+\cdots+t^{(-q-1)m}\right). \end{aligned} \tag{3.4}$$

This lies in $(-1)^1\mathbb{N}[t,t^{-1}]$, hence the second term of (3.3) lies in $\mathbb{N}[t,t^{-1}]$.

Case 1. $n-k$ is odd.
We use the analogue of (2.4) with $k$ replaced by $n-k$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = A + B + C + D \tag{3.5}$$

where
$$A := \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}}$$
$$B := t^{q^k-q} \prod_{i=0}^{k-2} \frac{1-t^{q^{k+1}-q^{i+2}}}{1-t^{q^k-q^{i+1}}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}}$$
$$C := t^{q^k-1} \prod_{i=0}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}}$$
$$D := t^{q^{k+1}+q^k-q-1} \prod_{i=0}^{k-1} \frac{1-t^{q^{k+2}-q^{i+2}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{q,t^{q^2}}$$

The last term $D$ is easy to deal with alone. By induction on $n$, its $(q,t)$-binomial factor lies in $(-1)^{k(n-2-k)}\mathbb{N}[t,t^{-1}] = (-1)^{k(n-k)}\mathbb{N}[t,t^{-1}]$. We claim that each factor for $i = 0,1,\ldots,k-1$ within the product inside $D$ lies in $\mathbb{N}[t,t^{-1}]$, since it can be expressed

$$\frac{1-t^{q^{k+2}-q^{i+2}}}{1-t^{q^k-q^i}} = \frac{1-t^{q^2m}}{1-t^m} = 1 + t^m + t^{2m} + \cdots + t^{(q^2-1)m}.$$

where $m := q^k - q^i$ as before.

The factors in $B$ and $C$ which correspond to $q^{n-k-1}(q-1)^1$ in (2.6), are no longer Laurent polynomials in $t$ with non-negative coefficients. Thus more care must be taken to prove that $A+B+C$ lies in $(-1)^{k(n-k)}\mathbb{N}[t,t^{-1}]$ for $q \le -2$. We start by combining common factors in $B$ and $C$:

$$B + C = \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} \left( t^{q^k-q} + t^{q^k-1} \frac{1-t^{q^{k+1}-q}}{1-t^{q^k-1}} \right)$$

and rewrite this parenthesized factor within $B+C$ as follows:

$$\begin{aligned} & t^{q^k-q} + t^{q^k-1} \frac{1-t^{q(q^k-1)}}{1-t^{q^k-1}} \\ &= t^{q^k-q} - t^{q^k-1} \left( t^{(-1)(q^k-1)} + t^{(-2)(q^k-1)} + \cdots + t^{q(q^k-1)} \right) \\ &= (t^{q^k-q} - 1) - h \end{aligned}$$

where $h := \left( t^{(-1)(q^k-1)} + t^{(-2)(q^k-1)} + \cdots + t^{(q+1)(q^k-1)} \right)$.

One can also rewrite the other two factors appearing in $B + C$:

$$\prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} = \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \prod_{i=0}^{k-2} \frac{1-t^{q^n-q^{i+2}}}{1-t^{q^{k+1}-q^{i+2}}}$$

$$= \prod_{i=0}^{k-2} \frac{1-t^{q^n-q^{i+2}}}{1-t^{q^k-q^{i+1}}} \quad (\text{ via telescoping })$$

$$= \frac{1-t^{q^n-q^k}}{1-t^{q^k-q}} \prod_{i=0}^{k-3} \frac{1-t^{q^n-q^{i+2}}}{1-t^{q^k-q^{i+2}}} = \frac{1-t^{q^n-q^k}}{1-t^{q^k-q}} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}}.$$

Therefore $A + B + C$ equals

$$\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}} + \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} \cdot ((t^{q^k-q}-1)-h)$$

$$= \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}} + \frac{1-t^{q^n-q^k}}{1-t^{q^k-q}} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}} (t^{q^k-q}-1)$$

$$- \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} \cdot h$$

$$= \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}} \left(1 + \frac{1-t^{q^n-q^k}}{1-t^{q^k-q}}(t^{q^k-q}-1)\right)$$

$$- \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} \cdot h$$

$$= t^{q^n-q^k} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}} - \prod_{i=1}^{k-1} \frac{1-t^{q^{k+1}-q^{i+1}}}{1-t^{q^k-q^i}} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q,t^{q^2}} \cdot h$$

In this last expression, the first summand $t^{q^n-q^k} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q,t^{q^2}}$ lies in

$$(-1)^{(k-2)(n-k)}\mathbb{N}[t,t^{-1}] = (-1)^{k(n-k)}\mathbb{N}[t,t^{-1}].$$

by induction on $n$.

The second summand has three factors, of which

- the $(q,t)$-binomial lies in $(-1)^{(k-1)(n-k-1)}\mathbb{N}[t,t^{-1}]$ by induction on $n$,
- the product over $i = 1, 2, \ldots, k-1$ has each factor in $-\mathbb{N}[t,t^{-1}]$ as observed in (3.4) within the proof of Case 1, and
- the factor of $-h$ also lies in $-\mathbb{N}[t,t^{-1}]$.

Thus, using the fact that $n-k$ is odd, the second summand lies in

$$(-1)^{(k-1)(n-k-1)} \cdot (-1)^{k-1} \cdot (-1)^1 \mathbb{N}[t,t^{-1}] = (-1)^{k(n-k)}\mathbb{N}[t,t^{-1}].$$

□

# 4 Nondegenerate unitary subspaces and Ennola duality

The analogy between the binomial coefficients counting subsets and the $q$-binomials counting subspaces is well-developed, as is the analogy between the *symmetric group* $\mathfrak{S}_n$ and the finite *general linear group* $GL_n := GL_n(\mathbb{F}_q)$. The authors thank John Shareshian for pointing out an extension of this analogy to nondegenerate subspaces of $(\mathbb{F}_{q^2})^n$ as a *unitary space*[2], and the *unitary group* $U_n := U_n(\mathbb{F}_{q^2})$:

$$\begin{aligned}
\binom{n}{k} &= \#\{\text{subsets of cardinality } k \text{ in } \{1,2,\ldots,n\}\} \\
&= [\mathfrak{S}_n : \mathfrak{S}_k \times \mathfrak{S}_{n-k}] \\
\\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \#\{\mathbb{F}_q\text{-subspaces of dimension } k \text{ in } (\mathbb{F}_q)^n\} \\
&= q^{-k(n-k)}[GL_n : GL_k \times GL_{n-k}] = [GL_n : P_{k,n-k}] \\
\\
\begin{bmatrix} n \\ k \end{bmatrix}_{-q} &= (-q)^{-k(n-k)} \# \left\{ \begin{array}{c} \text{nondegenerate} \\ \mathbb{F}_{q^2}\text{-subspaces of dimension } k \text{ in } (\mathbb{F}_{q^2})^n \end{array} \right\} \\
&= (-q)^{-k(n-k)}[U_n : U_k \times U_{n-k}].
\end{aligned} \tag{4.1}$$

Here $P_{k,n-k}$ denotes the parabolic subgroup of $GL_n$ that stabilizes one particular choice of a $k$-dimensional $\mathbb{F}_q$-subspace, with $GL_k \times GL_{n-k}$ its Levi subgroup of index $[P_{k,n-k} : GL_k \times GL_{n-k}] = q^{k(n-k)}$. The formulas in (4.1) follow easily from the transitivity of the actions on subsets, subspaces, nondegenerate subspaces of the groups $\mathfrak{S}_n, GL_n, U_n$, along with these well-known cardinalities (see, for example, Grove [4, Chapters 1, 10, 11]):

$$\begin{aligned}
|\mathfrak{S}_n| &= n! \\
|GL_n| &= q^{\binom{n}{2}}(q-1)(q^2-1)(q^3-1)(q^4-1)\cdots(q^n-1) \\
|U_n| &= q^{\binom{n}{2}}(q+1)(q^2-1)(q^3+1)(q^4-1)\cdots(q^n-(-1)^n).
\end{aligned}$$

Instead, we would like to place (4.1) within the context of *Ennola duality*, relating unipotent characters of $GL_n$ to those of the finite *unitary group* $U_n := U_n(\mathbb{F}_{q^2})$. We review a portion of this material here– see Thiem and Vinroot [15] for a more extensive treatment.

We first review the notion of a unipotent character for $GL_n$ or $U_n$. A *torus* $T$ in a finite group of Lie type $G$ is an abelian subgroup containing only semisimple elements. Given any linear character $\theta : T \to \mathbb{C}$ of a *maximal* torus $T$, there is a virtual character $R_T^G(\theta)$ of $G$ called the *Deligne–Lusztig character* of the pair $(T, \theta)$ (see [3, Chapter 11], [2, Chapter 7]). A natural $\mathbb{C}$-subspace of the space of class functions of $G$ is

$$\mathcal{U}(G) = \mathbb{C}\text{-span}\{R_T^G(1) \mid T \text{ a maximal torus}\},$$

[2]Meaning that one equips $(\mathbb{F}_{q^2})^n$ with a nondegenerate Hermitian form $(\cdot,\cdot)$, sesquilinear with respect to the conjugation action $\bar{\alpha} := \alpha^q$ in $\text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$, such as $(x,y) := \sum_{i=1}^n x\overline{y}$. See, e.g., Grove [4, Chapter 10] and Section 5 below.

where 1 is the trivial character of $T$. In the case where $G$ is $GL_n$ or $U_n$, for each partition $\lambda$ of $n$, there is a unique maximal torus $T_\lambda$ up to conjugacy, and $\mathcal{U}(G)$ has a $\mathbb{C}$-basis given by irreducible characters which we will denote $\chi^\lambda_{GL}$ and $\chi^\lambda_U$; one calls characters in this space *unipotent characters*. We conform here to Macdonald's convention [6, Chap. IV] that $\chi^{(1^n)}_{GL}$ is the trivial representation of $GL_n$; this differs from some conventions by the *conjugation* operation $\lambda \leftrightarrow \lambda'$ on partitions, that is, transposing their Ferrers diagrams.

With this convention, if one defines

$$n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i,$$

then the degree of the $GL_n$-character $\chi^\lambda_{GL}$ is a polynomial in $q$, of degree $\binom{n}{2} - n(\lambda)$, having the following explicit product expression [6, Chap. IV, (6.7)]:

$$\chi^\lambda_{GL}(1) = f^\lambda(q) := q^{n(\lambda')}(q)_n \prod_{(i,j)} \left(1 - q^{h_{ij}}\right)^{-1}. \tag{4.2}$$

Here the product runs over $(i,j)$ with $i \geq 1$ and $j \leq \lambda_i$, that is, over the cells in the Ferrers diagram for $\lambda$, and $h_{ij} := \lambda_i - i + \lambda'_j - j + 1$ is the *hooklength* at cell $(i,j)$, where $\lambda'_j$ is the length of the $j^{th}$ column in the diagram. The degrees of $\chi^\lambda_{\mathfrak{S}}$ and $\chi^\lambda_U$ are then determined by the same polynomial $f^\lambda(q)$: one has $\chi^\lambda_{\mathfrak{S}}(1) = f^\lambda(1)$, and

$$\chi^\lambda_U(1) = (-1)^{\binom{n}{2} - n(\lambda)} f^\lambda(-q). \tag{4.3}$$

The relation (4.3) is what we are calling here *Ennola duality*. We wish to extend it to explain (4.1), utilizing the *characteristic maps* isomorphisms for the three families of groups $G_n = \mathfrak{S}_n, GL_n, U_n$; for $\mathfrak{S}_n$ see Macdonald [6, Chap I §7], for $GL_n$ see Macdonald [6, Chap IV §4], and for $U_n$ see and Thiem and Vinroot [15, §4]. In each case, these are $\mathbb{C}$-linear isomorphisms $\mathcal{U}(G_n) \xrightarrow{\mathrm{ch}_G} \Lambda_n$ where $\Lambda_n$ denotes the space of *symmetric functions* with $\mathbb{C}$ coefficients which are homogeneous of degree $n$. The characteristic maps are defined by

$$\begin{aligned} \chi^\lambda_{\mathfrak{S}} &\xmapsto{\mathrm{ch}_{\mathfrak{S}}} s_\lambda \\ \chi^\lambda_{GL} &\xmapsto{\mathrm{ch}_{GL}} s_\lambda \\ \chi^\lambda_U &\xmapsto{\mathrm{ch}_U} (-1)^{\lfloor \frac{n}{2} \rfloor + n(\lambda)} s_\lambda. \end{aligned}$$

where $s_\lambda$ is the *Schur function* indexed by the partition $\lambda$ of $n$; see [6, Chap. I §3]. It is also worth mentioning that if $p_\lambda$ is the *power sum* symmetric function corresponding to the partition $\lambda$ of $n$, then

$$R^G_{T_\lambda}(1) \xmapsto{\mathrm{ch}_G} (-1)^{|\lambda| - \ell(\lambda)} p_\lambda \quad \text{for } G = GL_n \text{ or } U_n.$$

From the characteristic map one deduces the following extension of (4.3).

**Proposition 4.1.** *Given three class functions $\chi_{\mathfrak{S}}, \chi_{GL}, \chi_U$ in $\mathcal{U}(G_n)$ for the three families $G_n$ above, whenever they have the* **same** *symmetric function image*

$$\mathrm{ch}_{\mathfrak{S}}\,\chi_{\mathfrak{S}} = \mathrm{ch}_{GL}\,\chi_{GL} = \mathrm{ch}_U\,\chi_U.$$

*then the polynomial $f(q)$ giving the degree $\chi_{GL}(1)$ satisfies $\chi_{\mathfrak{S}}(1) = f(1)$ and*

$$\chi_U(1) = \pm f(-q).$$

*Proof.* Expand the symmetric function as $\sum_\lambda c_\lambda s_\lambda$ for some integers $c_\lambda$, and apply the inverse of the characteristic map isomorphism to get these virtual character expansions and degrees:

$$\begin{array}{rcl} \chi_{GL} & = & \sum_\lambda c_\lambda \chi^\lambda_{GL} \\ f(q) := \chi_{GL}(1) & = & \sum_\lambda c_\lambda f^\lambda(q) \end{array}$$

$$\begin{array}{rcl} \chi_{\mathfrak{S}} & = & \sum_\lambda c_\lambda \chi^\lambda_{\mathfrak{S}} \\ \chi_{\mathfrak{S}}(1) & = & \sum_\lambda c_\lambda f^\lambda(1) = f(1) \end{array}$$

$$\begin{array}{rcl} \chi_U & = & \sum_\lambda c_\lambda (-1)^{\lfloor \frac{n}{2} \rfloor + n(\lambda)} \chi^\lambda_U \\ \chi_U(1) & = & \sum_\lambda c_\lambda (-1)^{\lfloor \frac{n}{2} \rfloor + n(\lambda)} \cdot (-1)^{\binom{n}{2} - n(\lambda)} f^\lambda(-q) \\ & = & (-1)^{\lfloor \frac{n}{2} \rfloor + \binom{n}{2}} f(-q). \end{array}$$

□

To explain (4.1), we need one further fundamental fact (see [6, Chap. I (7.3), Chap. IV (4.1)], [15, Cor. 4.1]) about the characteristic maps $\mathrm{ch}_G$ for all $n$: taken together, they give a *ring* (and even Hopf algebra) isomorphism

$$\mathcal{U}(G) := \bigoplus_{n \geq 0} \mathcal{U}(G_n) \quad \xrightarrow{\mathrm{ch}_G} \quad \bigoplus_{n \geq 0} \Lambda_n =: \Lambda.$$

Here the ring of symmetric functions $\Lambda$ is given its usual product, and the $\mathbb{C}$-vector space $\mathcal{U}(G)$ is endowed with product structure

$$\begin{array}{rcl} \mathcal{U}(G_a) \otimes \mathcal{U}(G_b) & \longrightarrow & \mathcal{U}_{a+b} \\ \chi_a \otimes \chi_b & \longmapsto & R^{G_{a+b}}_{G_a \times G_b}(\chi_a \otimes \chi_b) \end{array}$$

where $R^{G_{a+b}}_{G_a \times G_b}(-)$ is interpreted in the three cases as

- *induction* of characters from $\mathfrak{S}_a \times \mathfrak{S}_b$ to $\mathfrak{S}_{a+b}$,
- *Harish-Chandra induction* of characters from $GL_a \times GL_b$ to $GL_{a+b}$, that is, *inflation* from $GL_a \times GL_b$ to $P_{a,b}$ by composing with the quotient map $P_{a,b} \to GL_a \times GL_b$, followed by usual induction from $P_{a,b}$ to $GL_{a+b}$, and

- *Deligne-Lusztig induction* of characters from $U_a \times U_b$ to $U_{a+b}$; see, for example, [3, Chapter 11] and [2, Chapter 7] for a precise definition.

A key point is that these three induction operations multiply the character degree $(\chi_a \otimes \chi_b)(1) = \chi_a(1) \cdot \chi_b(1)$ by a predictable factor, equal to the right side of (4.1):

- Induction from $\mathfrak{S}_a \times \mathfrak{S}_b$ to $\mathfrak{S}_{a+b}$ multiplies degrees by $[\mathfrak{S}_n : \mathfrak{S}_a \times \mathfrak{S}_b]$.
- Inflation from $GL_a \times GL_b$ to $P_{a,b}$ does not change degree, while induction from $P_{a,b}$ to $GL_{a+b}$ multiplies degrees by $[GL_{a+b} : P_{a,b}]$.
- Deligne-Lusztig induction from $U_a \times U_b$ to $U_{a+b}$ is known [3, Proposition 12.17] to multiply degrees by $\pm q^{-ab}[U_{a+b} : U_a \times U_b]$.

We now apply Proposition 4.1 to the trivial degree one character $\mathbf{1}_{G_k} \otimes \mathbf{1}_{G_{n-k}}$ so that $R^{G_n}_{G_k \times G_{n-k}}\left(\mathbf{1}_{G_k} \otimes \mathbf{1}_{G_{n-k}}\right)$ has degree given by the right side of (4.1). On the other hand, its image under $\mathrm{ch}_G$ is given by

$$\mathrm{ch}\,\mathbf{1}_{G_k} \cdot \mathrm{ch}\,\mathbf{1}_{G_{n-k}} = (\pm s_{1^k})\,(\pm s_{1^{n-k}}) = \pm s_{1^k} s_{1^{n-k}}$$

for any of the three families, so that (4.1) becomes a special case of Proposition 4.1.

# 5 A cyclic sieving phenomenon for nondegenerate subspaces

One original motivation for the $(q,t)$-binomial in [10] was its role in an instance of the *cyclic sieving phenomenon*, which we recall here. The *finite Grassmannian* of all $k$-dimensional $\mathbb{F}_q$-subspaces inside $(\mathbb{F}_q)^n$ carries an interesting action of a cyclic group $\mathbb{Z}/(q^n-1)\mathbb{Z} \cong \mathbb{F}_{q^n}^\times$: one embeds $\mathbb{F}_{q^n}^\times \hookrightarrow GL_n(\mathbb{F}_q)$ through any choice of an $\mathbb{F}_q$-linear isomorphism $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$. One can then prove [10, Theorem 9.4] that for an element $c$ in $\mathbb{F}_{q^n}^\times$ of multiplicative order $d$, the number of $k$-dimensional subspaces preserved by $c$ equals the $(q,t)$-binomial with $t$ evaluated at any primitive $d^{th}$ root-of-unity.

In light of this result, and the interpretation for the negative $q$-binomial in terms of *nondegenerate* unitary subspaces given in (4.1), one might ask for a similar cyclic sieving phenomenon involving the $(q,t)$-binomial at negative $q$. Our goal in this section is such a result when $n$ is odd, Theorem 5.5 below. It involves the action of a certain subgroup $C$ of the cyclic group $\mathbb{F}_{q^{2n}}^\times$, acting unitarily on $V = \mathbb{F}_{q^{2n}}$, and permuting nondegenerate subspaces.

We begin by introducing a compatible family of sesquilinear forms on $V = \mathbb{F}_{q^{2n}}$ for $n$ odd, that depend upon the choice of scalars $\mathbb{F}_{q^{2m}}$ over which one views $V$ as an $\mathbb{F}_{q^{2m}}$-vector space.

**Definition 5.1.** Let $q$ be a prime power, and $n$ an odd integer with $n \geq 1$. For each positive divisor $m$ of $n$, consider $V = \mathbb{F}_{q^{2n}}$ as an $\mathbb{F}_{q^{2m}}$-vector space, and recall that one

has the surjective *trace map*

$$\begin{array}{rcl} \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}} : \mathbb{F}_{q^{2n}} & \longrightarrow & \mathbb{F}_{q^{2m}} \\ \alpha & \longmapsto & \alpha + \alpha^{q^{2m}} + \alpha^{q^{4m}} + \cdots + \alpha^{q^{2(n-m)}} \end{array}$$

Use this to define maps

$$\begin{array}{rcl} V \times V & \longrightarrow & \mathbb{F}_{q^{2m}} \\ (\alpha, \beta) & \longmapsto & (\alpha,\beta)_{\mathbb{F}_{q^{2m}}} := \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\alpha \cdot \beta^{q^n}\right) \end{array}$$

Recall also that $\mathbb{F}_{q^{2m}}$ is a degree two Galois extension of $\mathbb{F}_{q^m}$, and hence the nontrivial element $\bar{\alpha} := \alpha^{q^m}$ of the Galois group defines conjugation $\alpha \mapsto \overline{\alpha}$ on $\mathbb{F}_{q^{2m}}$.

**Proposition 5.2.** *Fix the prime power $q$ and the odd integer $n \geq 1$.*
*Then for each divisor $m$ of $n$, the map $(\cdot,\cdot)_{\mathbb{F}_{q^m}}$ is*

- *$\mathbb{F}_{q^{2m}}$-Hermitian with respect to the conjugation on $\mathbb{F}_{q^{2m}}$, and*
- *nondegenerate as an $\mathbb{F}_{q^{2m}}$-valued form,*

*thus endowing $V$ with the structure of an $\frac{n}{m}$-dimensional unitary space over $\mathbb{F}_{q^{2m}}$.*

*Proof.* It is straightforward to check that $(\cdot,\cdot) := (\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$ is an additive function of both arguments, and an $\mathbb{F}_{q^{2m}}$-linear function of its first argument, that is,

$$\begin{array}{rcll} (\alpha+\alpha',\beta) & = & (\alpha,\beta)+(\alpha',\beta) & \\ (\alpha,\beta+\beta') & = & (\alpha,\beta)+(\alpha,\beta') & \\ (c\alpha,\beta) & = & c(\alpha,\beta) & \text{for } c \in \mathbb{F}_{q^{2m}}. \end{array}$$

In checking the other properties that define a Hermitian form, it is useful to note that elements of $c$ in $\mathbb{F}_{q^{2m}}$ satisfy $(c^{q^m})^{q^m} = c$, and since $\frac{n}{m}$ is odd, one also has

$$c^{q^n} = \underbrace{\left(\cdots\left(\left(c^{q^m}\right)^{q^m}\right)\cdots\right)^{q^m}}_{\frac{n}{m} \text{ times}} = c^{q^m}.$$

To check $\mathbb{F}_{q^{2m}}$-sesquilinearity in the second argument, given $c \in \mathbb{F}_{q^{2m}}$, one has

$$\begin{array}{rl} (\alpha, c\beta) & = \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\alpha \cdot (c\beta)^{q^n}\right) \\ & = \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(c^{q^n} \cdot \alpha \cdot \beta^{q^n}\right) \\ & = c^{q^n}\,\mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\alpha \cdot \beta^{q^n}\right) \\ & = c^{q^m}(\alpha,\beta) \\ & = \overline{c} \cdot (\alpha,\beta) \end{array}$$

where the middle equality used the $\mathbb{F}_{q^{2m}}$-linearity of $\mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}$. One also has

$$\begin{array}{rl} \overline{(\beta,\alpha)} & = \left(\mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\beta \cdot \alpha^{q^n}\right)\right)^{q^m} \\ & = \left(\mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\beta \cdot \alpha^{q^n}\right)\right)^{q^n} \\ & = \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\beta^{q^n} \cdot \left(\alpha^{q^n}\right)^{q^n}\right) \\ & = \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\alpha \cdot \beta^{q^n}\right) \\ & = (\alpha,\beta). \end{array}$$

Lastly, one needs to know that $(\cdot,\cdot)$ is nondegenerate as a pairing on $V$, or equivalently, that for any nonzero $\alpha$ in $V = \mathbb{F}_{q^{2n}}$, the $\mathbb{F}_{q^{2m}}$-linear functional

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{F}_{q^{2m}} \\ \beta & \longmapsto & (\alpha,\beta) = \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\alpha\cdot\beta^{q^n}\right) \end{array}$$

is surjective, or equivalently, not identically zero. This follows because a separable field extension $K/k$, such as $\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}$, always has nondegenerate $k$-bilinear pairing $\langle\alpha,\beta\rangle_k := \mathrm{Tr}_{K/k}(\alpha\cdot\beta)$, and $\beta\mapsto\beta^{q^n}$ is an automorphism of $K = \mathbb{F}^{q^{2n}}$. □

We note here the following compatibility between the the various forms $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$, which will be used in the proof of Theorem 5.5.

**Proposition 5.3.** *Fix the prime power $q$ and odd positive integers $\ell, m, n$, with $\ell$ dividing $m$ and $m$ dividing $n$.*

*Then an $\mathbb{F}_{q^{2m}}$-subspace $W$ of $\mathbb{F}_{q^{2n}}$, when regarded as an $\mathbb{F}_{q^{2\ell}}$-subspace, is nondegenerate with respect to the form $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$ if and only if it is nondegenerate with respect to the form $(\cdot,\cdot)_{\mathbb{F}_{q^{2\ell}}}$.*

*Proof.* As in the previous proof, $W$ is $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$-nondegenerate if and only if for every nonzero $\alpha$ in $W$, the $\mathbb{F}_{q^{2m}}$-linear functional $f_{m,\alpha} : W \to \mathbb{F}_{q^{2m}}$ given by

$$f_{m,\alpha}(\beta) = (\alpha,\beta)_{\mathbb{F}_{q^{2m}}}$$

is surjective, or equivalently, not identically zero. As the corresponding functional $f_{\ell,\alpha} : W \to \mathbb{F}_{q^{2\ell}}$ can be expressed as the composite $f_{\ell,\alpha} = \mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^{2\ell}}} \circ f_{m,\alpha}$, where the trace map $\mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^{2\ell}}} : \mathbb{F}_{q^{2m}} \longrightarrow \mathbb{F}_{q^{2\ell}}$ is well-known to be surjective, $f_{m,\alpha}$ is nonzero if and only if $f_{\ell,\alpha}$ is nonzero. □

We next describe the subgroup of the multiplicative group $\mathbb{F}_{q^{2n}}^\times$ which will act unitarily with respect to our chosen Hermitian forms. Let $\gamma$ be a generator for $\mathbb{F}_{q^{2n}}^\times \cong \mathbb{Z}/(q^{2n}-1)\mathbb{Z}$ as a cyclic group.

**Proposition 5.4.** *Fix $q, n$ as before, and any divisor $m$ of $n$.*

*Then the power $\gamma^{q^n-1}$ generates a cyclic subgroup $C \cong \mathbb{Z}/(q^n+1)\mathbb{Z}$ of $\mathbb{F}_{q^{2n}}^\times$ which acts on $V = \mathbb{F}_{q^{2n}}$ unitarily with respect to the $\mathbb{F}_{q^{2m}}$-Hermitian form $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$.*

*Proof.* The cardinality of $C = \langle\gamma^{q^n-1}\rangle$ should be clear. For the rest, calculate

$$\begin{aligned} (\gamma^{q^n-1}\alpha, \gamma^{q^n-1}\beta)_{\mathbb{F}_{q^{2m}}} &= \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\gamma^{q^n-1}\alpha\cdot(\gamma^{q^n-1}\beta)^{q^n}\right) \\ &= \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left((\gamma^{q^n-1})^{q^n+1}\cdot\alpha\cdot\beta^{q^n}\right) \\ &= \mathrm{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2m}}}\left(\gamma^{q^{2n}-1}\cdot\alpha\cdot\beta^{q^n}\right) \\ &= (\alpha,\beta)_{\mathbb{F}_{q^{2m}}}. \end{aligned}$$

□

To state our cyclic sieving phenomenon, fix $n$ odd as above, a prime power $q$, and a $k$ in the range $0 \leq k \leq n$. Consider the set

$$X := \left\{ \begin{array}{c} \text{all } (\cdot,\cdot)_{\mathbb{F}_{q^2}}\text{-nondegenerate } k\text{-dimensional} \\ \mathbb{F}_{q^2}\text{-subspaces of } V = \mathbb{F}_{q^{2n}} \end{array} \right\}.$$

Since the group

$$C := \langle \gamma^{q^n-1} \rangle \cong \mathbb{Z}/(q^n+1)\mathbb{Z}$$

acts unitarily with respect to this form, $C$ permutes the set $X$.

To define a polynomial $X(t)$ in $\mathbb{N}[t]$ we first define a $t$-version of $q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q$ by

$$Y(q,t) := \left( \prod_{i=0}^{k-1} \frac{1-t^{q^n-q^{n-k+i}}}{1-t^{q^k-q^i}} \right) \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}.$$

We define $X(t)$ as a polynomial version of $Y(-q,t)$, namely

$$X(t) := t^E \cdot \prod_{i=0}^{k-1} \frac{1-t^{q^n+(-q)^{n-k+i}}}{1-t^{q^k-(-1)^k(-q)^i}} \cdot \prod_{i=0}^{k-1} \frac{1-t^{q^n+(-q)^i}}{1-t^{q^k-(-1)^k(-q)^i}},$$

where one defines

$$E := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2\sum_{i=0}^{k-1}(q^k-(-q)^i) & \text{if } k \text{ is even.} \end{cases}$$

One may show that Theorem 3.1 implies $X(t)$ lies in $\mathbb{N}[t]$. It is also easily checked that, since $n$ is odd, one has

$$X(1) = (-q)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q}. \tag{5.1}$$

Furthermore, whenever $q$ is odd, all powers of $t$ in $X(t)$ occur with even exponents, and thus one has

$$X(-1) = X(1) \text{ for odd } q. \tag{5.2}$$

**Theorem 5.5.** *This triple $(X, X(t), C)$ exhibits a* cyclic sieving phenomenon *as in [10]: for any $c$ in $C$, the number of elements $x$ in $X$ having $c(x) = x$ is given by evaluating $X(t)$ with $t$ any complex root-of-unity whose multiplicative order is the same as $c$.*

*Proof.* Given $c$ in $C \subset \mathbb{F}_{q^{2n}}^{\times}$, we wish to count how many $x$ in $X$ have $c(x) = x$. Let $\mathbb{F}_{q^2}(c)$ denote the subfield of $\mathbb{F}_{q^{2n}}$ generated by $\mathbb{F}_{q^2}$ and $c$, so there exists a unique divisor $m$ of $n$ for which

$$\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}. \tag{5.3}$$

A $k$-dimensional $\mathbb{F}_{q^2}$-subspace $W \subset V = \mathbb{F}_{q^{2n}}$ is fixed by $c$ if and only if $cW \subset W$, that is, if and only if $W$ is actually a subspace over $\mathbb{F}_{q^2}(c)$. By (5.3), this is equivalent to $W$ being a $k'$-dimensional $\mathbb{F}_{q^{2m}}$-subspace, where $k' := \frac{k}{m}$.

According to Proposition 5.3, this $\mathbb{F}_{q^2}$-subspace $W$ is in addition nondegenerate for $(\cdot,\cdot)_{\mathbb{F}_{q^2}}$ if and only if it is nondegenerate for $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$. Therefore the number of $x$ in $X$ with $c(x) = x$ will be the number of $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$-nondegenerate $k'$-dimensional $\mathbb{F}_{q^{2m}}$-subspaces of $V = \mathbb{F}_{q^{2n}} \cong (\mathbb{F}_{q^{2m}})^{n'}$ where $n' := \frac{n}{m}$. By (4.1), this number is

$$(-Q)^{k'(n'-k')} \begin{bmatrix} n' \\ k' \end{bmatrix}_{-Q}, \qquad \text{where } Q := q^m. \tag{5.4}$$

On the other hand, assuming that $c$ has multiplicative order $A$, one can evaluate $X(t)$ at $t = \omega$ a primitive $A^{th}$ root-of-unity. This makes heavy use of Proposition 5.6 below, relating $A$ to the number $m$ defined by (5.3) above, and allowing one to analyze the locations of zeroes in the numerator and denominators appearing in the explicit formula for $X(t)$. In particular, it will be shown that $X(\omega)$ vanishes unless $m$ divides both $n$ and $k$, and then a limiting procedure will yield the predicted value (5.4) for $X(\omega)$ in this case. We proceed in cases based on the value of $A$.

Case 1. $A = 1$.

This case follows from equality (5.1) combined with (4.1).

Case 2. $A = 2$.

If $A = 2$ then $c = -1 \neq +1$ in $\mathbb{F}_{q^{2n}}$, forcing $q$ to be odd. In this case, $cW = W$ for all subspaces $W$, and (5.1) shows that $X(-1) = X(1)$, so the result follows as in the $A = 1$ case.

Case 3. $A \geq 3$.

Given $A$, let $m$ be as in (5.3). We first show $X(\omega) = 0$ if $m$ does not divide $k$. Note that the first product in the definition of $X(t)$, namely

$$\prod_{i=0}^{k-1} \frac{1 - t^{q^n + (-q)^{n-k+i}}}{1 - t^{q^k - (-1)^k (-q)^i}} \tag{5.5}$$

has each of its factors a polynomial in $t$, since $n$ being odd implies

$$q^n + (-q)^{n-k+i} = q^{n-k}(q^k - (-1)^k(-q)^i).$$

Thus (5.5) never has poles. As for the second product in the definition of $X(t)$, namely

$$\prod_{i=0}^{k-1} \frac{1 - t^{q^n + (-q)^i}}{1 - t^{q^k - (-1)^k (-q)^i}}, \tag{5.6}$$

it has powers of $t$ with exponents

| | |
|---|---|
| $q^n + 1,\ q^n - q,\ q^n - q^2, \ldots,\ q^n + (-1)^{k-1} q^{k-1}$ | in the numerator, |
| $q^k + 1,\ q^k - q,\ q^k + q^2,\ \ldots,\ q^k + q^{k-1}$ | in the denominator for odd $k$, |
| $q^k - 1,\ q^k + q,\ q^k - q^2,\ \ldots,\ q^k + q^{k-1}$ | in the denominator for even $k$. |

Since $n$ is an odd multiple of $m$, Proposition 5.6(iii,iv) implies that the choice $t = \omega$ yields $\lceil k/m \rceil$ numerator zeroes, from exponents $q^n+1, q^n-q^m, q^n+q^{2m}, q^n-q^{3m}, \ldots$, and $\lfloor k/m \rfloor$ denominator zeroes, from exponents $q^k + q^{k-m}, q^k - q^{k-2m}, q^k + q^{k-3m}, \ldots$, regardless of the parity of $k$. Thus when $m$ does not divide $k$, the numerator has more zeros than the denominator, and $X(\omega) = 0$.

When $m$ does divide $k$, we wish to evaluate $X(t)$ at $t = \omega$ a primitive $A^{th}$ root-of-unity, using this general L'Hôpital's rule calculation: if $r \equiv \pm s \bmod A$ then

$$\lim_{t\to\omega} \frac{1-t^r}{1-t^s} = \begin{cases} r/s & \text{if } r \equiv s \equiv 0 \bmod A \\ 1 & \text{if } r \equiv s \not\equiv 0 \bmod A \\ -\omega^{-s} & \text{if } r \equiv -s \not\equiv 0 \bmod A. \end{cases} \tag{5.7}$$

Pairing zeroes at $t = \omega$ in numerator, denominator of (5.6) and using (5.7) yields

$$\prod_{i=0}^{k/m-1} \frac{q^n - (-q^m)^i}{q^k - q^k(-q^{-m})^{i+1}} = \begin{bmatrix} n' \\ k' \end{bmatrix}_{-Q}.$$

One can do a similar analysis for the first factor (5.5) evaluated at $t = \omega$. This time one finds exponents on $t$ of $q^n + q^{n-1}, q^n - q^{n-2}, q^n + q^{n-3}, \ldots, q^n + q^{n-k}$ in the numerator, and $q^k + 1, q^k - q, q^k + q^2, \ldots, q^k + q^{k-1}$ in the denominator. The corresponding zeroes at $t = \omega$ come from exponents $q^n + q^{n-m}, q^n - q^{n-2m}, q^n + q^{n-3m}, \ldots$ in the numerator, and from $q^k + q^{k-m}, q^k - q^{k-2m}, q^k + q^{k-3m}, \ldots$ in the denominator, whose limit using (5.7) is

$$\prod_{i=0}^{k/m-1} \frac{q^n - q^n(-q^{-m})^{i+1}}{q^k - q^k(-q^{-m})^{i+1}} = q^{(n-k)k/m} = Q^{k'(n'-k')}.$$

It only remains to analyze the *nonzero* factors at $t = \omega$ in the numerator and denominators of the two products comprising $X(t)$. We treat this in two cases based on the parity of $k$.

For $k$ odd, we claim that these nonzero numerator and denominator factors in the second product (5.6) pair off to give 1 using (5.7). To see this claim, note that since $A$ divides $q^m + 1$, one has $\omega^{q^m+1} = 1$, and so one needs only check that the difference of the $t$-exponents

$$q^n + (-q)^i - (q^k + (-q)^i) = q^k(q^{n-k} - 1)$$

is divisible by $q^m + 1$. But $m$, which is odd, divides $n - k$, which is even, so one also has $2m$ dividing $n - k$. Thus $q^{2m} - 1$ divides $q^{n-k} - 1$, as does $q^m + 1$. We similarly claim that, for $k$ odd, the nonzero numerator and denominator factors in the first product (5.5) pair off to give factors of 1 using (5.7), because the difference of the exponents

$$q^n + (-q)^{n-k+i} - (q^k + (-q)^i) = (q^k + (-q)^i)(q^{n-k} - 1)$$

is again divisible by $q^m + 1$.

For $k$ even, we claim that the nonzero numerator and denominator factors in the second product (5.6) pair off in such a way that one can apply the third case of (5.7): one has as sum of numerator and denominator $t$-exponents

$$\left(q^n + (-q)^i\right) + \left(q^k - (-q)^i\right) = q^k(q^{n-k}+1) \equiv 0 \bmod q^m + 1$$

where the congruence follows as $m$ divides $n-k$ and both are odd. Each such factor contributes $-\omega^{-(q^k-(-q)^i)}$ by (5.7), and there are $k - \frac{k}{m}$ such factors, an even number since $k$ is odd and $m$ is even, giving a total contribution of $\omega^{-\sum_{i=0}^{k-1}(q^k-(-q)^i)}$. Similarly, when $k$ is even, we claim that these nonzero numerator and denominator factors in the first product (5.5) pair off with the sum of the numerator and denominator $t$-exponents

$$\left(q^n + (-q)^{n-k+i}\right) + (q^k + (-q)^i) = (q^k - (-q)^i)(q^{n-k}+1) \equiv 0 \bmod q^m + 1$$

where the congruence follows for the same reason. Again there are $k - \frac{k}{m}$ such factors, giving a total contribution of $\omega^{-\sum_{i=0}^{k-1}(q^k-(-q)^i)}$.

Together these contribute $\omega^{-2\sum_{i=0}^{k-1}(q^k-(-q)^i)}$, cancelled by $t^E$ for $k$ even. □

The following proposition collects technical facts used in the preceding proof.

**Proposition 5.6.** *Assume that $c$ in $\mathbb{F}_{q^{2n}}^{\times}$ has multiplicative order $A$ at least 3, and that $\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}$, where $m$ divides $n$.*

*(i) The order $A$ must divide $q^m + 1$.*

*(ii) The smallest positive integer $d$ such that $A$ divides $q^d + 1$ is $m$.*

*(iii) The order $A$ divides $q^s + q^t$ if and only if $s - t$ is an odd multiple of $m$.*

*(iv) The order $A$ divides $q^s - q^t$ if and only if $s - t$ is an even multiple of $m$.*

*Proof.* Assertions (i) and (ii). These will be deduced from the stronger

**Claim**: If $d|n$ and $A|q^n + 1$, then $A|q^{2d} - 1$ if and only if $A|q^d + 1$.

The "if" direction in the claim is clear, so we must only show that $A|q^{2d} - 1$ implies $A|q^d + 1$. If $d = n$ this is the hypothesis on $A$. So assume that $d < n$, and since $n$ is odd, we have $2d < n$. Clearly $A$ must divide $\gcd(q^n + 1, q^{2d} - 1)$, which we now prove is $q^d + 1$. Expressing

$$q^n + 1 = q^{n-2d}(q^{2d} - 1) + q^{n-2d} + 1$$

in order to use Euclidean algorithm, one has

$$\gcd(q^n + 1, q^{2d} - 1) = \gcd(q^{n-2d} + 1, q^{2d} - 1)$$

Since $d$ divides $n$, and $n$ is odd we have $n \equiv d \bmod 2d$, and therefore

$$\gcd(q^n + 1, q^{2d} - 1) = \gcd(q^d + 1, q^{2d} - 1) = q^d + 1.$$

Given the claim, assertions (i),(ii) follow, since $\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}$ means that $m$ is the smallest positive integer $d$ such that $c \in \mathbb{F}_{q^{2d}}$, i.e., such that $A$ divides $q^{2d} - 1$.

Assertion (iii). In one direction, if $s-t$ is an odd multiple of $m$, then $q^m+1$ divides $q^{s-t}+1$ and also $q^s + q^t$, so $A$ also divides $q^s + q^t$.

For the converse, suppose that $A$ divides $q^s+q^t$, and assume without loss of generality that $s \geq t$. Since $\gcd(A, q) = 1$, one has that $A$ also divides $q^{s-t}+1$ and $q^m+1$, so $A$ divides $\gcd(q^{s-t} + 1, q^m + 1)$. Since $A \geq 3$ we can assume that $s > t$ and write $s - t = m\alpha + \beta$ with $0 \leq \beta < m$. Expressing

$$q^{s-t} + 1 = (q^{m(\alpha-1)+\beta} - q^{m(\alpha-2)+\beta} + \cdots + (-1)^{\alpha-1}q^{\beta})(q^m + 1) + (-1)^{\alpha}q^{\beta} + 1$$

and using the Euclidean algorithm, one has that $A$ divides $(-1)^{\alpha}q^{\beta} + 1$.

If $\alpha$ is even, then $0 < \beta < m$ contradicts the minimality of $m$, while $\beta = 0$ contradicts $A \geq 3$. Thus $\alpha$ is odd, and $A$ divides $q^{\beta} - 1$.

Now if $\beta = 0$, then $s - t$ is an odd multiple of $m$, so we are done. Otherwise, if $\beta > 0$, then write $m = \gamma\beta + \delta$ with $0 \leq \delta < \beta < m$. Expressing

$$q^m + 1 = (q^{\beta(\gamma-1)+\delta} + q^{\beta(\gamma-2)+\delta} + \cdots + q^{\delta})(q^{\beta} - 1) + q^{\delta} + 1$$

and using the Euclidean algorithm, one concludes that $A$ divides $q^{\delta} + 1$. Again by minimality of $m$ this implies that $\delta = 0$, which contradicts $A \geq 3$.

Assertion (iv). In one direction, if $s - t$ is an even multiple of $m$, then $q^m + 1$ divides $q^{2m} - 1$, and hence also divides $q^{s-t} - 1$, and therefore divides $q^s - q^t$.

For the converse, suppose that $A$ divides $q^s - q^t$, and assume without loss of generality that $s \geq t$. Since $\gcd(A, q) = 1$, one has that $A$ also divides $q^{s-t} - 1$ and $q^m + 1$, so $A$ divides $\gcd(q^{s-t} - 1, q^m + 1)$. Again writing $s - t = m\alpha + \beta$ with $0 \leq \beta < m$, and expressing

$$q^{s-t} - 1 = (q^{m(\alpha-1)+\beta} - q^{m(\alpha-2)+\beta} + \cdots + (-1)^{\alpha-1}q^{\beta})(q^m + 1) + (-1)^{\alpha}q^{\beta} - 1$$

the Euclidean algorithm implies that $A$ divides $(-1)^{\alpha}q^{\beta} - 1$.

If $\alpha$ is odd, this contradicts the minimality $m$ for $\beta > 0$, so we can assume $\alpha$ is even. The argument proceeds as for Assertion (iii), $\beta = 0$, and $s - t$ is an even multiple of $m$. □

# 6 Remarks and further questions

## 6.1 Reformulating Theorem 2.1 via partitions

It is well-known (see e.g. [1, p. 40]) that $\Omega_{n,k}$ bijects with integer partitions $\lambda$ whose Ferrers diagram lie inside an $(n-k) \times k$ rectangle. One version of this bijection sends the word $\omega = (\omega_1, \ldots, \omega_n)$ to the partition $\lambda$ whose Ferrers diagram (drawn in the plane $\mathbb{Z}^2$ in English notation) has its northwest corner at $(0, n-k)$, and whose outer boundary is

the lattice path from $(n-k,k)$ to $(0,0)$ having its $i^{th}$ step go one down (resp. leftward) if $\omega_i = 0$ (resp. $\omega_i = 1$). One has $\mathrm{inv}(\omega) = |\lambda| = \sum_i \lambda_i$, the weight of $\lambda$.

We omit the details in verifying the following.

**Proposition 6.1.** *Under the above bijection, one has the following correspondences.*

*(i) The subset $\Omega'_{n,k} \subset \Omega_{n,k}$ corresponds to those partitions $\lambda$ inside $(n-k)\times k$ for which*

*(a) if $k$ is even, each odd part has even multiplicity,*

*(b) if $k$ is odd, each even part has even multiplicity, and moreover the number of parts has the same parity as $n-k$.*

*(ii) The statistic $p(\omega)$ counting occurrences of paired $\underline{10}$ in $\omega$ corresponds to the statistic $p(\lambda)$ counting the corner cells in $\lambda$ that are* **special** *in the following sense: they are the last cells in rows of $\lambda$ corresponding to the last occurrences of each part with the same parity as $k$.*

*(iii) Theorem 2.1 becomes*

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q = \sum_{\lambda} q^{|\lambda|-p(\lambda)}(q-1)^{p(\lambda)}.$$

*where the sum runs over $\lambda \subset (n-k)\times k$ satisfying condition (i) above.*

Here are three examples of assertion (iii), with the first compared to the example appearing just after Theorem 2.1:

$(n,k) = (5,2)$

| $\omega \in \Omega'_{5,2}$ | $\lambda$ | $\mathrm{wt}(\lambda)$ |
|---|---|---|
| $\underline{1}\ \underline{10}\ \underline{0}\ \underline{0}$ | 222 | $q(q-1)q^2q^2$ |
| $\underline{0}\ \underline{1}\ \underline{10}\ \underline{0}$ | 22 | $q(q-1)q^2$ |
| $\underline{1}\ \underline{00}\ \underline{10}$ | 211 | $q(q-1)q^2$ |
| $\underline{0}\ \underline{0}\ \underline{1}\ \underline{10}$ | 2 | $q(q-1)$ |
| $\underline{0}\ \underline{1}\ \underline{00}\ \underline{1}$ | 11 | $q^2$ |
| $\underline{0}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{1}$ | $\emptyset$ | 1 |

$(n,k) = (5,3)$

| $\lambda$ | $\mathrm{wt}(\lambda)$ |
|---|---|
| 33 | $q^2(q-1)q^3$ |
| 22 | $q^2q^2$ |
| 31 | $q^2(q-1)(q-1)$ |
| 11 | $(q-1)q^1$ |
| $\emptyset$ | 1 |

$(n,k) = (6,3)$

| $\lambda$ | $\mathrm{wt}(\lambda)$ |
|---|---|
| 333 | $q^2(q-1)q^3q^3$ |
| 322 | $q^2(q-1)q^2q^2$ |
| 331 | $q^2(q-1)q^3(q-1)$ |
| 221 | $q^2q^2(q-1)$ |
| 311 | $q^2(q-1)(q-1)q$ |
| 3 | $q^2(q-1)$ |
| 111 | $(q-1)q^2$ |
| 1 | $q-1$ |

## 6.2 Reformulating Theorem 2.1 via subspaces

When $q$ is a prime power, so the size of the finite field $\mathbb{F}_q$, one can also reformulate Theorem 2.1 as counting certain $k$-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$.

Recall that a $k$-dimensional subspace $V$ is the column-space of a matrix $A$ in $\mathbb{F}_q^{n\times k}$ in *column-echelon form*:

- each column ends with a string of 0's, preceded by a pivot entry 1,
- with only zeroes in the same row as any pivot, and
- where the row indices of the pivots decreasing from left-to-right.

The map $f$ sending $V$ to the word $\omega$ in $\Omega_{n,k}$ whose ones are in the same positions as the row indices of the pivots of $A$ corresponds (see e.g. [7], [12, Chapter 1]) to the Schubert cell decomposition of the Grassmannian $\mathbb{G}(k,\mathbb{F}_q^n)$.

If the word $\omega$ corresponds as above to the partition $\lambda$ inside $(n-k)\times k$, then there are $q^{\mathrm{inv}(\omega)}=q^\lambda$ elements in the pre-image $f^{-1}(\omega)$: deleting the $k$ pivot rows from $A$ gives an $(n-k)\times k$ matrix whose nonzero entries lie in the cells of $\lambda$. As an example, consider matrices $A$ with this column-echelon form for $(n,k)=(11,5)$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ * & * & * & 0 & 0\\ * & * & * & 0 & 0\\ * & * & * & 0 & 0\\ * & * & \underline{*} & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ * & 0 & 0 & 0 & 0\\ \underline{*} & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad \lambda = \begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & \underline{*} \\ * & & \\ \underline{*} & & \end{array} \quad \text{and} \quad \omega = \underline{10}\ \underline{0}\ \underline{1}\ \underline{10}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{1}.$$

The *special* entries in the matrix $A$, corresponding to the special corner cells of $\lambda$ from Proposition 6.1 and corresponding to the paired $\underline{10}$'s in $\omega$, are shown underlined.

One may then analogously reintepret Theorem 2.1 (or Theorem 6.1) as saying that the primed $q$-binomial counts those $k$-dimensional subspaces $V$ whose column-echelon form has all *special entries nonzero*.

Using echelon forms, it was shown in [9, §5.3] how to associate to each $k$-dimensional subspace $V$ a power of $t$ so that their generating function in $t$ interprets the $(q,t)$-binomial coefficient. Similarly, one can use Theorem 3.1 and its proof to associate a power of $t$ to each such subspace $V$ having special entries nonzero, so as to give a generating function interpretation to the $(q,t)$-binomial coefficient when $q$ is a negative integer. We omit this formulation here.

## 6.3 Geometry

Given a field $\mathbb{F}$, let $X_{\mathbb{F}}$ denote the Grassmannian of $k$-dimensional subspaces in $\mathbb{F}^n$. Aside from its interpretation when $q$ is a prime power as counting the points of the finite Grassmannian $X_{\mathbb{F}_q}$, the $q$-binomial coefficient has two well-known interpretations as the Poincaré polynomials

$$\sum_i \operatorname{rank}_{\mathbb{Z}} H^{2i}(X_{\mathbb{C}};\mathbb{Z})q^i$$

$$\sum_i \dim_{\mathbb{F}_2} H^i(X_{\mathbb{R}};\mathbb{F}_2)q^i$$

See, e.g., [14] for the second interpretation. These lead to the following interpretations for at least the $q=1$ specialization of the primed $q$-binomial

$$\begin{bmatrix} n \\ k \end{bmatrix}'_{q=1} = \#\{\omega \in \Omega' : p(\omega) = 0\}$$

- as the *signature* or *index* of $X_{\mathbb{C}}$ (see [8]), and
- as the *Euler characteristic* of $X_{\mathbb{R}}$, up to a $\pm$ sign.

*Question* 6.2. Can one generalize either of the above geometric interpretations for its $q=1$ specialization to a geometric interpretation for the full primed $q$-binomial?

## 6.4 Lack of monotonicity for $(q,t)$

As mentioned earlier, Theorem 2.1 makes inequality (2.2) transparent. Thus one might hope for an analogous inequality involving the $(q,t)$-binomial and its $-q$ relative, perhaps via Theorem 3.1.

Unfortunately, a naive guess along these lines fails already in the case of $(n,k,q) = (4,2,4)$: even though $\begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$ and $(-1)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q,t}$ both lie in $\mathbb{N}[t]$ and have the same degree $k(q^n - q^k) = 480$, their difference contains both positive and negative coefficients.

## 6.5 A conjecture on Schur functions

Conjecture 6.3 below is a generalization of Theorem 3.1 that applies to a $(q,t)$-analogue $S_\lambda(1,t,\ldots,t^n)$ of principally specialized Schur functions, discussed in [9, Definition 5.1]. Define for integer partitions $\lambda$, the statistic $b(\lambda) := \sum_i (i-1)\lambda_i$.

**Conjecture 6.3.** *If $q \leq -2$ is a negative integer, then*

$$(-1)^{n|\lambda|-b(\lambda)} S_\lambda(1,t,\ldots,t^n)$$

*is a Laurent polynomial in $t$, all of whose coefficients are non-negative integers.*

## 6.6 Generating function for $a, p$ on $\Omega'_{n,k}$

One may find an explicit product representation for the rational generating function

$$G_k(x, q, z) = \sum_{n \geq k} x^n \sum_{\omega \in \Omega'_{n,k}} q^{a(\omega)} z^{p(\omega)}.$$

We do not give the result here, but note that one of its specializations

$$G_k(x, 1, 1) = \sum_{n \geq k} x^n |\Omega'_{n,k}| = \frac{x^k}{(1-x)^{k+1}(1+x)^{\lfloor (k+1)/2 \rfloor}}$$

can be used to give an expression for $|\Omega'_{n,k}|$.

## 6.7 Lucasnomials

Sagan and Savage [11] recently introduced analogues of binomial coefficients, dubbed *lucasnomials*, defined as follows: for $0 \leq k \leq n$,

$$\begin{Bmatrix} n \\ k \end{Bmatrix} := \frac{\{n\}!}{\{k\}!\,\{n-k\}!}$$

where $\{n\}! := \{1\}\,\{2\}\cdots\{n\}$, and $\{n\}$ is defined as polynomials in variables $s, t$ recursively, via $\{0\} := 0, \{1\} := 1$, and

$$\{n\} = s\,\{n-1\} + t\,\{n-2\}$$

It is not hard to see that after substituting

$$s = q + 1, \quad t = -q$$

the lucasnomial is the $q$-binomial, and therefore after substituting

$$s = -q + 1 = -(q-1), \quad t = q \tag{6.1}$$

the lucasnomial is (up to sign) the primed $q$-binomial from (2.1). Since their main result [11, Theorem 3.1] expands the general lucasnomial as a sum of monomials $s^a t^b$, one might wonder how their expansion compares (after substituting as in (6.1)) with Theorem 2.1. It turns out that their expansion has more terms $s^a t^b = (-1)^a (q-1)^a q^b$ than Theorem 2.1, and not all terms in their expansion have the same sign $(-1)^a$.

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