Properties of θ-super positive graphs

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Abstract

Let the matching polynomial of a graph G be denoted by $\mu(G,x)$. A graph G is said to be θ-super positive if $\mu(G,\theta) \neq 0$ and $\mu(G \setminus v,\theta) = 0$ for all $v \in V(G)$. In particular, G is 0-super positive if and only if G has a perfect matching. While much is known about 0-super positive graphs, almost nothing is known about θ-super positive graphs for $\theta \neq 0$. This motivates us to investigate the structure of θ-super positive graphs in this paper. Though a 0-super positive graph need not contain any cycle, we show that a θ-super positive graph with $\theta \neq 0$ must contain a cycle. We introduce two important types of θ-super positive graphs, namely θ-elementary and θ-base graphs. One of our main results is that any θ-super positive graph G can be constructed by adding certain type of edges to a disjoint union of θ-base graphs; moreover, these θ-base graphs are uniquely determined by G. We also give a characterization of θ-elementary graphs: a graph G is θ-elementary if and only if the set of all its θ-barrier sets form a partition of $V(G)$. Here, θ-elementary graphs and θ-barrier sets can be regarded as θ-analogue of elementary graphs and Tutte sets in classical matching theory.

KEYWORDS: matching polynomial, Gallai-Edmonds decomposition, elementary graph, barrier sets, extreme sets

1 Introduction

We begin by introducing matching polynomials with an interest in the multiplicities of their roots. This will lead us to a recent extension of the celebrated Gallai-Edmonds Strcuture Theorem by Chen and Ku [1] which will be useful later in our study of θ-super positive graphs. This result has been instrumental in recent investigations of the subject, see [5, 6, 7, 8, 9, 10].

All the graphs in this paper are simple and finite. The vertex set and edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively.

Definition 1.1. An *r-matching* in a graph G is a set of r edges, no two of which have a vertex in common. The number of r-matchings in G will be denoted by $p(G, r)$. We set $p(G, 0) = 1$ and define the *matching polynomial* of G by

$$\mu(G, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r p(G, r) x^{n-2r}$$

where $n = |V(G)|$. We denote the multiplicity of θ as a root of $\mu(G, x)$ by $\mathrm{mult}(\theta, G)$. Let $u \in V(G)$, the graph obtained from G by deleting the vertex u and all edges that contain u is denoted by $G \setminus u$. Inductively if $u_1, \dots, u_k \in V(G)$, $G \setminus u_1 \cdots u_k = (G \setminus u_1 \cdots u_{k-1}) \setminus u_k$. Note that the order in which the vertices are being deleted is not important, that is, if $i_1, \dots, i_k$ is a permutation of $1, \dots, k$, we have $G \setminus u_1 \cdots u_k = G \setminus u_{1_1} \cdots u_{i_k}$. Furthermore, if $X = \{u_1, \dots, u_k\}$, we set $G \setminus X = G \setminus u_1 \cdots u_k$. If H is a subgraph of G, by an abuse of notation, we have $G \setminus H = G \setminus V(H)$. For example, if $p = v_1 v_2 \dots v_n$ is a path in G then $G \setminus p = G \setminus v_1 v_2 \cdots v_n$. If e is an edge of G, let $G - e$ denote the graph obtained from G by deleting the edge e from G. Inductively, if $e_1, \dots, e_k \in E(G)$, $G - e_1 \cdots e_k = (G - e_1 \cdots e_{k-1}) - e_k$.

A graph G is said to have a *perfect matching* if it has an $n/2$-matching (n must be even). This is equivalent to $\mathrm{mult}(0, G) = 0$, that is, 0 is not a root of $\mu(G, x)$. Recall that in the literature $\mathrm{mult}(0, G)$ is also known as the *deficiency* of G which is the number of vertices of G missed by some maximum matching.

The following are some basic properties of $\mu(G, x)$.

Theorem 1.2. [2, Theorem 1.1 on p. 2]

(a) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$ *where G and H are disjoint graphs,*

(b) $\mu(G, x) = \mu(G - e, x) - \mu(G \setminus uv, x)$ *if $e = (u, v)$ is an edge of G,*

(c) $\mu(G, x) = x\mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x)$ *where $i \sim u$ means i is adjacent to u,*

(d) $\dfrac{d}{dx}\mu(G, x) = \sum_{i \in V(G)} \mu(G \setminus i, x)$ *where $V(G)$ is the vertex set of G.*

It is well known that all roots of $\mu(G, x)$ are real. Throughout, let θ be a real number. The multiplicity of a matching polynomial root satisfies the the following interlacing property:

Lemma 1.3. [2, Corollary 1.3 on p. 97] (Interlacing) *Let G be a graph and $u \in V(G)$. Let θ be a real number. Then*

$$\mathrm{mult}(\theta, G) - 1 \leq \mathrm{mult}(\theta, G \setminus u) \leq \mathrm{mult}(\theta, G) + 1.$$

Lemma 1.3 suggests that given any real number θ, we can classify the vertices of a graph according to an increase of 1 or a decrease of 1 or no change in the multiplicity of θ upon deletion of a vertex.

Definition 1.4. [3, Section 3] For any $u \in V(G)$,

(a) u is θ-*essential* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G) - 1$,

(b) u is θ-*neutral* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G)$,

(c) u is θ-*positive* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G) + 1$.

Furthermore, if u is not θ-essential but it is adjacent to some θ-essential vertex, we say u is θ-special.

It turns out that θ-special vertices play an important role in the Gallai-Edmonds Decomposition of a graph (see [1]). Godsil [3, Corollary 4.3] proved that a θ-special vertex must be θ-positive. Note that if $\mathrm{mult}(\theta, G) = 0$ then for any $u \in V(G)$, u is either θ-neutral or θ-positive and no vertices in G can be θ-special. Now $V(G)$ can be partitioned into the following sets:

$$V(G) = D_\theta(G) \cup A_\theta(G) \cup P_\theta(G) \cup N_\theta(G),$$

where

$D_\theta(G)$ is the set of all θ-essential vertices in G,

$A_\theta(G)$ is the set of all θ-special vertices in G,

$N_\theta(G)$ is the set of all θ-neutral vertices in G,

$P_\theta(G) = Q_\theta(G) \setminus A_\theta(G)$, where $Q_\theta(G)$ is the set of all θ-positive vertices in G.

Note that there are no 0-neutral vertices. So $N_0(G) = \varnothing$ and $V(G) = D_0(G) \cup A_0(G) \cup P_0(G)$.

Definition 1.5. [3, Section 3] A graph G is said to be θ-critical if all vertices in G are θ-essential and $\mathrm{mult}(\theta, G) = 1$.

The celebrated Gallai-Edmonds Structure Theorem describes the stability of a certain canonical decomposition of $V(G)$ with respect to the zero root of $\mu(G, x)$. In [1], Chen and Ku extended the Gallai-Edmonds Structure Theorem to any root $\theta \neq 0$, which consists of the following two theorems:

Theorem 1.6. [1, Theorem 1.5] (θ-Stability Lemma) *Let G be a graph with θ a root of $\mu(G, x)$. If $u \in A_\theta(G)$ then*

(i) $D_\theta(G \setminus u) = D_\theta(G)$,

(ii) $P_\theta(G \setminus u) = P_\theta(G)$,

(iii) $N_\theta(G \setminus u) = N_\theta(G)$,

(iv) $A_\theta(G \setminus u) = A_\theta(G) \setminus \{u\}$.

Theorem 1.7. [1, Theorem 1.7] (θ-Gallai's Lemma) *If G is connected and every vertex of G is θ-essential then* $\mathrm{mult}(\theta, G) = 1$.

Theorem 1.6 asserts that the decomposition of $V(G)$ into $D_\theta(G)$, $P_\theta(G)$, $N_\theta(G)$ and $A_\theta(G)$ is *stable* upon deleting a θ-special vertex of G. We may delete every such vertex one by one until there are no θ-special vertices left. Together with Theorem 1.7, it is not hard to deduce the following whose proof is omitted.

Corollary 1.8.

(i) $A_\theta(G \setminus A_\theta(G)) = \varnothing$, $D_\theta(G \setminus A_\theta(G)) = D_\theta(G)$, $P_\theta(G \setminus A_\theta(G)) = P_\theta(G)$, *and* $N_\theta(G \setminus A_\theta(G)) = N_\theta(G)$.

(ii) $G \setminus A_\theta(G)$ *has exactly* $|A_\theta(G)| + \mathrm{mult}(\theta, G)$ *θ-critical components.*

(iii) If H is a component of $G \setminus A_\theta(G)$ then either H is θ-critical or $\mathrm{mult}(\theta, H) = 0$.

(iv) The subgraph induced by $D_\theta(G)$ consists of all the θ-critical components in $G \setminus A_\theta(G)$.

This paper is devoted to the study of θ-super positive graphs. A graph is θ-super positive if θ is not a root of $\mu(G, x)$ but is a root of $\mu(G \setminus v, x)$ for every $v \in V(G)$. It is worth noting that G is 0-super positive if and only if G has a perfect matching. While much is known about graphs with a perfect matching, almost nothing is known about θ-super positive graphs for $\theta \neq 0$. This gives us a motivation to investigate the structure of these graphs.

The outline of this paper is as follows:

In Section 2, we show how to construct θ-super positive graphs from smaller θ-super positive graphs (see Theorem 2.2). We prove that a tree is θ-super positive if and only if $\theta = 0$ and it has a perfect matching (see Theorem 2.4). Consequently, a θ-super positive graph must contain a cycle when $\theta \neq 0$. For a connected vertex transitive graph G, we prove that it is θ-super positive for any root θ of $\mu(G \setminus v, x)$ where $v \in V(G)$ (see Theorem 2.8). Finally we prove that if G is θ-super positive, then $N_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$ (see Theorem 2.9).

In Section 3, we introduce θ-elementary graphs. These are θ-super positive graphs with $P_\theta(G \setminus v) = \emptyset$ for all $v \in V(G)$. We prove a characterization of θ-elementary graphs: a graph G is θ-elementary if and only if the set of all θ-barrier sets form a partition of $V(G)$ (see Theorem 3.13).

In Section 4, we apply our results in Section 3 to prove that an n-cycle C_n is 1-elementary if and only if $n = 3k$ for some $k \in \mathbb{N}$ (see Theorem 4.4). Furthermore, we prove that C_{3k} has exactly 3 1-barrier sets (see Corollary 4.5).

In Section 5, we introduce θ-base graphs which can be regarded as building blocks of θ-super positive graphs. We prove a characterization of θ-super positive graphs, namely a θ-super positive graph can be constructed from a disjoint union of θ-base graphs by adding certain type of edges; moreover, these θ-base graphs are uniquely determined by G (see Theorem 5.7 and Corollary 5.9).

2 θ-super positive graphs

Definition 2.1. A graph G is *θ-super positive* if θ is not a root of $\mu(G, x)$ and every vertex of G is θ-positive.

By Lemma 1.3, this is equivalent to $\text{mult}(\theta, G) = 0$ and $\text{mult}(\theta, G \setminus v) = 1$ for all $v \in V(G)$. There are a lot of θ-super positive graphs. For instance the three cycle C_3 and the six cycle C_6 are 1-super positive. In the next theorem, we will show how to construct θ-super positive graphs from smaller θ-super positive graphs.

Theorem 2.2. *Let G_1 and G_2 be two θ-super positive graphs and $v_i \in V(G_i)$ for $i = 1, 2$. Let G be the graph obtained by adding the edge (v_1, v_2) to the union of G_1 and G_2. Then G is θ-super positive.*

Proof. Let $e = (v_1, v_2)$. First we prove that $\mu(G, \theta) \neq 0$. By part (b) of Theorem 1.2, we have $\mu(G, x) = \mu(G - e, x) - \mu(G \setminus v_1 v_2, x)$. It then follows from part (a) of Theorem 1.2 that $\mu(G, x) = \mu(G_1, x)\mu(G_2, x) - \mu(G_1 \setminus v_1, x)\mu(G_2 \setminus v_2, x)$. Since G_1 and G_2 are θ-super positive, $\mu(G, \theta) = \mu(G_1, \theta)\mu(G_2, \theta) \neq 0$.

It is left to prove that $\mu(G \setminus v, \theta) = 0$ for all $v \in V(G)$. Let $v \in V(G_1)$. Suppose $v = v_1$. Then by part (a) of Theorem 1.2, $\mu(G \setminus v, x) = \mu(G_1 \setminus v_1, x)\mu(G_2, x)$, and thus $\mu(G \setminus v, \theta) = 0$. Suppose $v \neq v_1$. By part (b) of Theorem 1.2, $\mu(G \setminus v, x) = \mu((G\setminus v)-e, x) - \mu((G\setminus v)\setminus v_1v_2, x)$. Note that $(G\setminus v)-e = (G_1\setminus v)\cup G_2$ and $(G\setminus v)\setminus v_1v_2 = (G_1\setminus vv_1)\cup(G_2\setminus v_2)$. Hence $\mu(G\setminus v, \theta) = \mu(G_1\setminus v, \theta)\mu(G_2, \theta) - \mu(G_1\setminus vv_1, \theta)\mu(G_2\setminus v_2, \theta) = 0$ (part (a) of Theorem 1.2).

The case $v \in V(G_2)$ is proved similarly. □

The graph G in Figure 1 is constructed by using Theorem 2.2, with $G_1 = C_6$ and $G_2 = C_3$. Therefore it is 1-super positive graph.

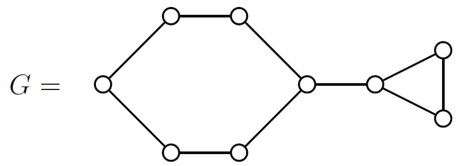

Figure 1.

It is clear that a 0-super positive need not contain any cycle. However, we will show later that if G is θ-super positive and $\theta \neq 0$, then it must contain a cycle (see Corollary 2.5). Note that any tree T with at least three vertices can be represented in the following form (see Figure 2), where u is a vertex with $n+1$ neighbors $v_1, \ldots, v_{n+1}$ such that all of them except possibly v_1 have degree 1 and T_1 is a subtree of T that contains v_1. Such a representation of T is denoted by $(T_1, u; v_1, \ldots, v_{n+1})$.

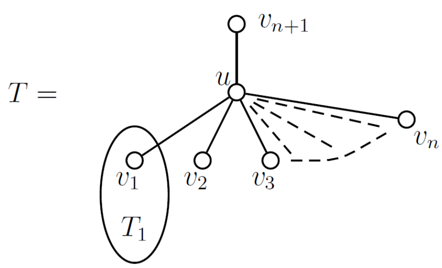

Figure 2.

Lemma 2.3. *Let T be a tree with at least three vertices. Suppose T has a representation $(T_1, u; v_1, \ldots, v_{n+1})$. Then θ is a root of $\mu(T, x)$ if and only if*

$$(n - \theta^2)\theta^{n-1}\mu(T_1, \theta) + \theta^n\mu(T_1 \setminus v_1, \theta) = 0.$$

Proof. By part (c) of Theorem 1.2, $\mu(T, \theta) = \theta\mu(T \setminus u, \theta) - \sum_{i=1}^{n+1} \mu(T \setminus uv_i, \theta)$ (see Figure 2), which implies (using part (a) of Theorem 1.2),

$$\mu(T, \theta) = (\theta^2 - n)\theta^{n-1}\mu(T_1, \theta) - \theta^n\mu(T_1 \setminus v_1, \theta).$$

Hence the lemma holds. □

Theorem 2.4. *Let T be a tree. Then T is θ-super positive if and only if $\theta = 0$ and it has a perfect matching.*

Proof. Suppose T is θ-super positive and $\theta \neq 0$. Then T must have at least three vertices. By Lemma 2.3,

$$(n - \theta^2)\theta^{n-1}\mu(T_1, \theta) + \theta^n \mu(T_1 \setminus v_1, \theta) \neq 0.$$

By part (a) of Theorem 1.2, $0 = \mu(T \setminus u, \theta) = \theta^n \mu(T_1, \theta)$ (see Figure 2). Therefore $\mu(T_1, \theta) = 0$ and $\mu(T_1 \setminus v_1, \theta) \neq 0$. Now $\mu(T \setminus v_{n+1}, \theta) = 0$. By part (c) of of Theorem 1.2, $\mu(T \setminus v_{n+1}, \theta) = \theta\mu(T \setminus uv_{n+1}, \theta) - \sum_{i=1}^{n} \mu(T \setminus uv_i v_{n+1}, \theta) = \theta^n \mu(T_1, \theta) - (n-1)\theta^{n-2}\mu(T_1, \theta) - \theta^{n-1}\mu(T_1 \setminus v_1, \theta)$. This implies that $\mu(T_1 \setminus v_1, \theta) = 0$, a contradiction. Hence $\theta = 0$. Since 0 is not a root of $\mu(T, x)$, T must have a perfect matching.

The converse is obvious. □

A consequence of Theorem 2.4 is the following corollary.

Corollary 2.5. *If G is θ-super positive for some $\theta \neq 0$, then G must contain a cycle.*

We shall need the following lemmas.

Lemma 2.6. [4, Theorem 6.3](Heilmann-Lieb Identity) *Let $u, v \in V(G)$. Then*

$$\mu(G \setminus u, x)\mu(G \setminus v, x) - \mu(G, x)\mu(G \setminus uv) = \sum_{p \in \mathcal{P}(u,v)} \mu(G \setminus p, x)^2,$$

where $\mathcal{P}(u, v)$ is the set of all the paths from u to v in G.

Lemma 2.7. [3, Lemma 3.1] *Suppose* $\mathrm{mult}(\theta, G) > 0$. *Then G contains at least one θ-essential vertex.*

Theorem 2.8. *Let G be connected, vertex transitive and $z \in V(G)$. If θ is a root of $\mu(G \setminus z, x)$ then G is θ-super positive.*

Proof. Since $G \setminus z$ is isomorphic to $G \setminus y$ for all $y \in V(G)$, $\mu(G \setminus z, x) = \mu(G \setminus y, x)$ for all $y \in V(G)$. So $\mathrm{mult}(\theta, G \setminus z) = \mathrm{mult}(\theta, G \setminus y)$. This implies that θ is a root of $\mu(G \setminus y, x)$ for all y.

Now it remains to show that $\mu(G, \theta) \neq 0$. Suppose the contrary. Then by Lemma 2.7, G has at least one θ-essential vertex. Since G is vertex transitive, all vertices in G are θ-essential. By Theorem 1.7, $\mathrm{mult}(\theta, G) = 1$. But then $\mathrm{mult}(\theta, G \setminus z) = 0$, a contradiction. Hence $\mu(G, \theta) \neq 0$ and G is θ-super positive. □

However, a θ-super positive graph is not necessarily vertex transitive (see Figure 1). Furthermore a θ-super positive graph is not necessary connected, for the union of two C_3 is 1-super positive.

Theorem 2.9. *Let G be θ-super positive. Then $N_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$.*

Proof. Suppose $N_\theta(G \setminus v) \neq \varnothing$ for some $v \in V(G)$. Let $u \in N_\theta(G \setminus v)$. By Lemma 2.6,

$$\mu(G \setminus u, x)\mu(G \setminus v, x) - \mu(G, x)\mu(G \setminus uv) = \sum_{p \in \mathcal{P}(u,v)} \mu(G \setminus p, x)^2.$$

Note that the multiplicity of θ as a root of $\mu(G \setminus u, x)\mu(G \setminus v, x)$ is 2, while the multiplicity of θ as a root of $\mu(G, x)\mu(G \setminus vu, x)$ is 1 since u is θ-neutral in $G \setminus v$. Therefore the multiplicity of θ as a root of the polynomial on the left-hand side of the equation is at least 1. But the multiplicity of θ as a root of the polynomial on the right-hand side of the equation is even and so, in comparison with the left-hand side, it must be at least 2. This forces the multiplicity of θ as a root of $\mu(G, x)\mu(G \setminus vu, x)$ to be at least 2, a contradiction. Hence $N_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. □

Now we know that for a θ-super positive graph G, $N_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. So it is quite natural to ask whether $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. Well, this is not true in general (see Figure 1). This motivates us to study the θ-super positive graph G, for which $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. We proceed to do this in the next section.

3 θ-elementary graphs

Definition 3.1. A graph G is said to be θ-*elementary* if it is θ-super positive and $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$.

The graph G in Figure 3 is 1-elementary. Not every θ-positive graph is θ-elementary. For instance, the graph in Figure 1 is not 1-elementary.

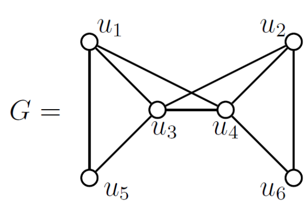

Figure 3.

Theorem 3.2. *A graph G is θ-elementary if and only if* $\operatorname{mult}(\theta, G) = 0$ *and* $P_\theta(G \setminus v) \cup N_\theta(G \setminus v) = \varnothing$ *for all $v \in V(G)$.*

Proof. Suppose $\operatorname{mult}(\theta, G) = 0$ and $P_\theta(G \setminus v) \cup N_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. For each $v \in V(G)$, $G \setminus v$ must consist entirely of θ-essential and θ-special vertices and must have at least one θ-essential vertex; therefore $\operatorname{mult}(\theta, G \setminus v) = 1$. Therefore G is θ-super positive and it is θ-elementary.

The other implication follows from Theorem 2.9. □

It turns out that the notion of a 0-elementary graph coincide with the classical notion of an elementary graph. Properties of elementary graphs can be found in Section 5.1 on p. 145 of [11].

The number of θ-critical components in G is denoted by $c_\theta(G)$.

Definition 3.3. A *θ-barrier set* is defined to be a set $X \subseteq V(G)$ for which $c_\theta(G \setminus X) = \text{mult}(\theta, G) + |X|$.

A *θ-extreme set* in G is defined to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$.

θ-barrier sets and θ-extreme sets can be regarded as θ-analogue of Tutte sets and extreme sets in classical matching theory. Properties of θ-barrier sets and θ-extreme sets have been studied by Ku and Wong [5]. In particular, the following results are needed.

Lemma 3.4. [5, Lemma 2.4] *A subset of a θ-extreme set is a θ-extreme set.*

Lemma 3.5. [5, Lemma 2.5] *If X is a θ-barrier set in G and $Y \subseteq X$ then $X \setminus Y$ is a θ-barrier set in $G \setminus Y$.*

Lemma 3.6. [5, Lemma 2.6] *Every θ-extreme set in G lies in a θ-barrier set in G.*

Lemma 3.7. [5, Lemma 2.7] *Let X be a θ-barrier set in G. Then X is a θ-extreme set in G.*

Lemma 3.8. [5, Lemma 3.1] *If X is a θ-barrier set in G then $X \subseteq A_\theta(G) \cup P_\theta(G)$.*

Lemma 3.9. [5, Theorem 3.5] *Let X be a θ-barrier set in G. Then $A_\theta(G) \subseteq X$.*

Lemma 3.10. *Let G be a graph. If X is a θ-barrier set in G, $x \in X$ and $P_\theta(G \setminus x) = \varnothing$, then $A_\theta(G \setminus x) = X \setminus \{x\}$.*

Proof. By Lemma 3.5, $X \setminus \{x\}$ is a θ-barrier set in $G \setminus x$. By Lemma 3.8, $X \setminus \{x\} \subseteq A_\theta(G \setminus x) \cup P_\theta(G \setminus x)$. Therefore $X \setminus \{x\} \subseteq A_\theta(G \setminus x)$. It then follows from Lemma 3.9 that $A_\theta(G \setminus x) = X \setminus \{x\}$. □

Definition 3.11. We define $\mathfrak{P}(\theta, G)$ to be the set of all the θ-barrier sets in G.

Note that in Figure 3, $\mathfrak{P}(1, G) = \{\{u_1\}, \{u_2\}, \{u_3, u_4\}, \{u_5\}, \{u_6\}\}$. Now Lemma 3.12 follows from part (c) of Theorem 1.2.

Lemma 3.12. *Suppose G is θ-super positive. Then for each $v \in V(G)$ there is a $u \in V(G)$ with $(u, v) \in E(G)$ and $\text{mult}(\theta, G \setminus uv) = 0$.*

Theorem 3.13. *A graph G is θ-elementary if and only if $\mathfrak{P}(\theta, G)$ is a partition of $V(G)$.*

Proof. Let $\mathfrak{P}(\theta, G) = \{S_1, \ldots, S_k\}$.
($\Rightarrow$) Suppose G is θ-elementary. Then for each $v \in V(G)$, $\{v\}$ is a θ-extreme set. By Lemma 3.6, it is contained in some θ-barrier set. Therefore $V(G) = S_1 \cup \cdots \cup S_k$. It remains to prove that $S_i \cap S_j = \varnothing$ for $i \neq j$. Suppose the contrary. Let $x \in S_i \cap S_j$. By Lemma 3.10, $S_i \setminus \{x\} = A_\theta(G \setminus x) = S_j \setminus \{x\}$ and so $S_i = S_j$, a contradiction. Hence $S_i \cap S_j = \varnothing$ for $i \neq j$ and $\mathfrak{P}(\theta, G)$ is a partition of $V(G)$.
($\Leftarrow$) Suppose $\mathfrak{P}(\theta, G)$ is a partition of $V(G)$. Let $v \in V(G)$. Then $v \in S_i$ for some θ-barrier set S_i. By Lemma 3.8, $v \in A_\theta(G) \cup P_\theta(G)$. Therefore $V(G) \subseteq A_\theta(G) \cup P_\theta(G)$. This implies that $\text{mult}(\theta, G) = 0$, for otherwise $D_\theta(G) \neq \varnothing$ by Lemma 2.7. Hence $A_\theta(G) = \varnothing$ and $V(G) = P_\theta(G)$, i.e., G is θ-super positive. It remains to show that $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$. Suppose the contrary. Then $P_\theta(G \setminus v_0) \neq \varnothing$ for some $v_0 \in V(G)$. We may assume $v_0 \in S_1$. By Corollary 1.8, $(G \setminus v_0) \setminus A_\theta(G \setminus v_0)$ has a component H for which $\text{mult}(\theta, H) = 0$. By Theorem 2.9, $N_\theta(G \setminus v_0) = \varnothing$. So we conclude that H is θ-super positive. Let $w \in H$. By Lemma 3.12, there is a $z \in V(H)$ with $(w, z) \in E(H)$ and $\text{mult}(\theta, H \setminus wz) = 0$. By part (a) of Theorem 1.2, and, (ii) and (iii) of Corollary 1.8, $\text{mult}(\theta, ((G \setminus v_0) \setminus A_\theta(G \setminus v_0)) \setminus wz) = 1 + |A_\theta(G \setminus v_0)|$.

On the other hand, by Lemma 3.5, $S_1 \setminus \{v_0\}$ is a θ-barrier set in $G \setminus v_0$. So by Lemma 3.9, $A_\theta(G \setminus v_0) \subseteq S_1 \setminus \{v_0\}$. By Lemma 3.5 again, $S_1 \setminus (\{v_0\} \cup A_\theta(G \setminus v_0))$ is a θ-barrier set in $(G \setminus v_0) \setminus A_\theta(G \setminus v_0)$. Note that w is θ-positive in $G \setminus v_0$ (by Corollary 1.8). Therefore $\{w, v_0\}$ is an θ-extreme set in G. By Lemma 3.6, $\{w, v_0\}$ is contained in some θ-barrier set in G. Since $\mathfrak{P}(\theta, G)$ is a partition of $V(G)$ and $v_0 \in S_1$, we must have $\{w, v_0\} \subseteq S_1$. Note also z is θ-positive in $G \setminus v_0$ (recall that H is θ-super positive). Using a similar argument, we can show that $\{z, v_0\} \subseteq S_1$. By Lemma 3.4 and Lemma 3.7, we conclude that $\{w, z\} \subseteq S_1 \setminus (\{v_0\} \cup A_\theta(G \setminus v_0))$ is a θ-extreme set in $(G \setminus v_0) \setminus A_\theta(G \setminus v_0)$. This implies that $\text{mult}(\theta, ((G \setminus v_0) \setminus A_\theta(G \setminus v_0)) \setminus wz) = 3 + |A_\theta(G \setminus v_0)|$, contradicting the last sentence of the preceding paragraph. Hence $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$ and G is θ-elementary. $\square$

Lemma 3.14. *Suppose G is θ-elementary. Then for each $\varnothing \neq X \subseteq S \in \mathfrak{P}(\theta, G)$, $A_\theta(G \setminus X) = S \setminus X$ and $P_\theta(G \setminus X) \cup N_\theta(G \setminus X) = \varnothing$.*

Proof. Let $x \in X$. Then $P_\theta(G \setminus x) = \varnothing$. By Theorem 2.9, $N_\theta(G \setminus x) = \varnothing$. Now by Lemma 3.10, $S \setminus \{x\} = A_\theta(G \setminus x)$ so that $X \setminus \{x\} \subseteq S \setminus \{x\} = A_\theta(G \setminus x)$. By Theorem 1.6, we conclude that $A_\theta(G \setminus X) = S \setminus X$ and $P_\theta(G \setminus X) \cup N_\theta(G \setminus X) = \varnothing$. $\square$

Corollary 3.15. *Suppose G is θ-elementary. Let $S \subseteq V(G)$. Then $S \in \mathfrak{P}(\theta, G)$ if and only if $G \setminus S$ has exactly $|S|$ components and each is θ-critical.*

Proof. Suppose $G \setminus S$ has exactly $|S|$ components and each is θ-critical. Then $c_\theta(G \setminus S) = |S|$ and S is a θ-barrier set in G. Hence $S \in \mathfrak{P}(\theta, G)$.

The other implication follows from Lemma 3.14 and Corollary 1.8. $\square$

4 1-elementary cycles

We shall need the following lemmas.

Lemma 4.1. [10, Corollary 4.4] *Suppose G has a Hamiltonian path P and θ is a root of $\mu(G,x)$. Then every vertex of G which is not θ-essential must be θ-special.*

Lemma 4.2. *Let p_n be a path with $n \geq 1$ vertices. Then*

$$\mu(p_n, 1) = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \mod 6; \\ -1, & \text{if } n \equiv 3 \text{ or } 4 \mod 6; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that for $t \geq 2$, $\mu(p_t, x) = x\mu(p_{t-1}, x) - \mu(p_{t-2}, x)$ (part (c) of Theorem 1.2), where we define $\mu(p_0, x) = 1$. Therefore $\mu(p_t, 1) = \mu(p_{t-1}, 1) - \mu(p_{t-2}, 1)$. Now $\mu(p_1, 1) = 1$. So, $\mu(p_2, 1) = 0$, and recursively we have $\mu(p_3, 1) = -1$, $\mu(p_4, 1) = -1$ and $\mu(p_5, 1) = 0$. By induction the lemma holds. □

Lemma 4.3. *Let C_n be a cycle with $n \geq 3$ vertices. Then*

$$\mu(C_n, 1) = \begin{cases} 1, & \text{if } n \equiv 1 \text{ or } 5 \mod 6; \\ -1, & \text{if } n \equiv 2 \text{ or } 4 \mod 6; \\ 2, & \text{if } n \equiv 0 \mod 6; \\ -2, & \text{if } n \equiv 3 \mod 6. \end{cases}$$

Proof. By part (c) of Theorem 1.2, $\mu(C_n, 1) = \mu(p_{n-1}, 1) - 2\mu(p_{n-2}, 1)$. The lemma follows from Lemma 4.2. □

Theorem 4.4. *A cycle C_n is 1-elementary if and only if $n = 3k$ for some $k \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Suppose C_n is 1-elementary. Then for any $v \in V(C_n)$, $C_n \setminus v = p_{n-1}$. By Lemma 4.2, $\text{mult}(1, p_{n-1}) > 0$ if and only if $n - 1 \equiv 2$ or $5 \mod 6$. Thus $n = 3k$ for some $k \in \mathbb{N}$.
($\Leftarrow$) Suppose $n = 3k$ for some $k \in \mathbb{N}$. By Lemma 4.3, $\text{mult}(1, C_n) = 0$. Note that $3k \equiv 3$ or $6 \mod 6$. Therefore $3k - 1 \equiv 2$ or $5 \mod 6$, and by Lemma 4.2 and Lemma 1.3, $\text{mult}(1, C_n \setminus v) = \text{mult}(1, p_{n-1}) = 1$ for all $v \in V(C_n)$. Thus C_n is 1-super positive. By Lemma 4.1, $P_1(C_n \setminus v) = \varnothing$ for all $v \in V(C_n)$. Hence C_n is 1-elementary. □

For our next result, let us denote the vertices of C_{3k} by $1, 2, 3, \ldots, 3k$ (see Figure 4).

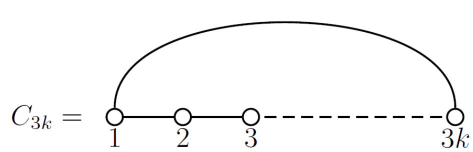

Figure 4.

Corollary 4.5. *C_{3k} has exactly 3 1-barrier sets, that is*

$$\mathfrak{P}(1, C_{3k}) = \{\{1, 4, 7, \ldots, 3k-2\}, \{2, 5, 8, \ldots, 3k-1\}, \{3, 6, 9, \ldots, 3k\}\}.$$

Proof. Note that $C_{3k} \setminus \{1, 4, 7, \ldots, 3k-2\}$ is a disjoint union of k number of K_2 and K_2 is 1-critical. So $\{1, 4, 7, \ldots, 3k-2\}$ is a 1-barrier set. Similarly $\{2, 5, 8, \ldots, 3k-1\}$ and $\{3, 6, 9, \ldots, 3k\}$ are 1-barrier sets. It then follows from Theorem 4.4 and Theorem 3.13 that these are the only 1-barrier sets. □

5 Decomposition of θ-super positive graphs

Definition 5.1. A set $X \subseteq V(G)$ with $|X| > 1$ is said to be *independent* in G if for all $u, v \in X$, u and v are not adjacent to each other. A graph G is said to be *θ-base* if it is θ-super positive and for all $S \in \mathfrak{P}(\theta, G)$, S is independent.

Note that the cycle C_{3k} is θ-base for $\theta = 1$. In fact a connected θ-base graph is θ-elementary.

Theorem 5.2. *A connected θ-base graph is θ-elementary.*

Proof. Let G be θ-base. Suppose it is not θ-elementary. Then $P_\theta(G \setminus v) \neq \varnothing$ for some $v \in V(G)$. By Lemma 2.7, $G \setminus v$ has at least one θ-essential vertex. Moreover, by Theorem 2.9, $N_\theta(G \setminus v) = \varnothing$.

If v is not a cut vertex of G, then there exists a path from $P_\theta(G \setminus v)$ to $D_\theta(G \setminus v)$, which implies that $A_\theta(G \setminus v) \neq \varnothing$. By Theorem 2.9 and Corollary 1.8, $(G \setminus v) \setminus A_\theta(G \setminus v)$ has a θ-super positive component, say H. Since $G \setminus v$ is connected, there exists $h \in V(H)$ that is adjacent to some element $w \in A_\theta(G \setminus v)$. Note that $\{h, w, v\}$ is a θ-extreme set in G. By Lemma 3.4, $\{h, w\}$ is a θ-extreme set in G. By Lemma 3.6, $\{h, w\}$ is contained in some $S \in \mathfrak{P}(\theta, G)$, contrary to the fact that S is independent.

If v is a cut vertex of G, then either $G \setminus v$ does not have any θ-super positive components or $G \setminus v$ contains a θ-super positive component (recall that $N_\theta(G \setminus v) = \varnothing$). In the former, $G \setminus v$ must contain a component H such that $A_\theta(H) \neq \emptyset$ and $P_\theta(H) \neq \emptyset$. Since H is connected, there is a vertex $h \in P_\theta(H)$ that is joined to some vertex $w \in A_\theta(H)$. Note that $\{h, w, v\}$ is a θ-extreme set in G. By Lemma 3.4, $\{h, w\}$ is a θ-extreme set in G. By Lemma 3.6, $\{h, w\}$ is contained in some $S \in \mathfrak{P}(\theta, G)$, contrary to the fact that S is independent. In the latter, some vertex in the θ-super positive component, say u, must be joined to v so that $\{u, v\}$ is a θ-extreme set in G. Again, by Lemma 3.6, $\{u, v\}$ is contained in some $S \in \mathfrak{P}(\theta, G)$, contrary to the fact that S is independent.

Hence $P_\theta(G \setminus v) = \varnothing$ for all $v \in V(G)$ and G is θ-elementary. □

Note that the converse of Theorem 5.2 is not true. Let G be the graph in Figure 3. Note that $\{u_3, u_4\} \in \mathfrak{P}(1, G)$ but it is not independent.

Lemma 5.3. *Let G be θ-super positive and $e = (u, v) \in E(G)$ such that $\{u, v\}$ is a θ-extreme set in G. Let G' be the graph obtained by removing the edge e from G. Then G' is θ-super positive.*

Proof. Now $\text{mult}(\theta, G \setminus uv) = 2$. By part (b) of Theorem 1.2, $\mu(G, x) = \mu(G', x) - \mu(G \setminus uv, x)$. This implies that $\mu(G', \theta) = \mu(G, \theta) \neq 0$.

It is left to show that $\mu(G' \setminus w, \theta) = 0$ for all $w \in V(G')$. Clearly if $w = u$ or v then $\mu(G' \setminus w, \theta) = \mu(G \setminus w, \theta) = 0$. Suppose $w \neq u, v$. By part (b) of Theorem 1.2 again, $\mu(G \setminus w, x) = \mu(G' \setminus w, x) - \mu(G \setminus wuv, x)$. By Lemma 1.3, $\text{mult}(\theta, G \setminus uvw) \geq 1$. Therefore $\mu(G' \setminus w, \theta) = \mu(G \setminus w, \theta) = 0$. Hence G' is θ-super positive. □

Note that after removing an edge from G as in Lemma 5.3, $\mathfrak{P}(\theta, G') \neq \mathfrak{P}(\theta, G)$ in general. In Figure 5, $\mathfrak{P}(1, G) = \{\{1, 4, 7\}, \{5, 8\}, \{6, 9\}, \{2\}, \{3\}\}$. After removing the edge $(1, 4)$ from G, the resulting graph $G' = C_9$. By Corollary 4.5, $\mathfrak{P}(1, G') = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}$.

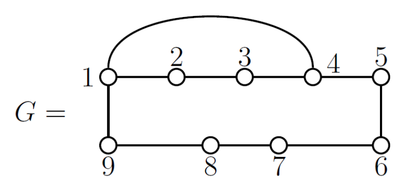

Figure 5.

We shall need the following lemma.

Lemma 5.4. [3, Corollary 2.5] *For any root θ of $\mu(G, x)$ and a path p in G,*

$$\text{mult}(\theta, G \setminus p) \geq \text{mult}(\theta, G) - 1.$$

Lemma 5.5. *Let G be θ-super positive and $e_1 = (u, v) \in E(G)$ with $\{u, v\}$ is a θ-extreme set. Let $G' = G - e_1$ and $e_2 = (w, z) \in E(G')$. Then $\{w, z\}$ is a θ-extreme set in G' if and only if it is a θ-extreme set in G.*

Proof. **Case 1**. Suppose e_1 and e_2 have a vertex in common, say $w = u$. Then $G' \setminus wz = G \setminus wz$.

($\Rightarrow$) Suppose $\{w, z\}$ is a θ-extreme set in G'. By Lemma 5.3, $\text{mult}(\theta, G') = 0$. Therefore $\text{mult}(\theta, G \setminus wz) = \text{mult}(\theta, G' \setminus wz) = 2$ and $\{w, z\}$ is a θ-extreme set in G.

($\Leftarrow$) The converse is proved similarly.

Case 2. Suppose e_1 and e_2 have no vertex in common. By part (b) of Theorem 1.2,

$$\mu(G \setminus wz, x) = \mu(G' \setminus wz, x) - \mu(G \setminus wzuv, x).$$

($\Rightarrow$) Suppose $\{w, z\}$ is a θ-extreme set in G'. Then $\text{mult}(\theta, G' \setminus wz) = 2$. Now $\text{mult}(\theta, G \setminus uv) = 2$ and by Lemma 5.4, $\text{mult}(\theta, G \setminus uvwz) \geq 1$. So we conclude that $\text{mult}(\theta, G \setminus wz) \geq 1$. On the other hand, $N_\theta(G \setminus w) = \varnothing$ (Theorem 2.9). Therefore either $\text{mult}(\theta, G \setminus wz) = 0$ or 2. Hence the latter holds and $\{w, z\}$ is a θ-extreme set in G.

($\Leftarrow$) Suppose $\{w, z\}$ is a θ-extreme set in G. Then $\text{mult}(\theta, G \setminus wz) = 2$. As before we have $\text{mult}(\theta, G \setminus uvwz) \geq 1$. So we conclude that $\text{mult}(\theta, G' \setminus wz) \geq 1$. On the other hand, by Lemma 5.3, G' is θ-super positive. Therefore $N_\theta(G' \setminus w) = \varnothing$ (Theorem 2.9), and then either $\text{mult}(\theta, G' \setminus wz) = 0$ or 2. Hence the latter holds and $\{w, z\}$ is a θ-extreme set in G'. □

Definition 5.6. Let G be θ-super positive. An edge $e = (u, v) \in E(G)$ is said to be θ-*extreme* in G if $\{u, v\}$ is a θ-extreme set.

The process described in Lemma 5.3, can be iterated. Let $Y_0 = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$ be the set of all θ-extreme edges. Let $G_1 = G - e_1$. Then G_1 is θ-super positive (Lemma 5.3). Let Y_1 be the set of all θ-extreme edges in G_1. Then by Lemma 5.5, $Y_1 = Y_0 \setminus \{e_1\}$. Now let $G_2 = G_1 - e_2$. By applying Lemma 5.3 and Lemma 5.5, we see that G_2 is θ-super positive and the set of all θ-extreme edges in G_2 is $Y_2 = Y_0 \setminus \{e_1, e_2\}$. By continuing this process, after k steps, we see that $G_k = G - e_1 e_2 \ldots e_k$ is θ-super positive and the set of all θ-extreme edges in G_k is $Y_k = \varnothing$. We claim that G_k is a disjoint union of θ-base graphs. Suppose the contrary. Let H be a component of G_k that is not θ-base. Since G_k is θ-super positive, by part (a) of Theorem 1.2, we deduce that H is θ-super positive. Therefore there is a $S \in \mathfrak{P}(\theta, H)$ for which S is not independent. Let $e = (u, v) \in E(H)$ with $\{u, v\} \subseteq S$. By Lemma 3.7 and Lemma 3.4, $\{u, v\}$ is a θ-extreme set in H. This means that e is θ-extreme in H, and by part (a) of Theorem 1.2, e is θ-extreme in G_k, contrary to the fact that $Y_k = \varnothing$. Hence H is θ-base and we have proved the following theorem.

Theorem 5.7. *Let G be θ-super positive. Then G can be decomposed into a disjoint union of θ-base graphs by deleting its θ-extreme edges. Furthermore, the decomposition is unique, i.e. the θ-base graphs are uniquely determined by G.*

The proof of the next lemma is similar to Lemma 5.3, and is thus omitted.

Lemma 5.8. *Let G be θ-super positive and $\{u, v\}$ is a θ-extreme set with $e = (u, v) \notin E(G)$. Let G' be the graph obtained by adding the edge e to G. Then G' is θ-super positive.*

Using the process described in Lemma 5.8, we can construct θ-super positive graph from θ-base graphs. Together with Theorem 5.7, we see that every θ-super positive can be constructed from θ-base graphs.

Corollary 5.9. *A graph is θ-super positive if and only if it can be constructed from θ-base graphs.*

In the next theorem, we shall extend Theorem 2.2.

Theorem 5.10. *Let G_1 and G_2 be two θ-super positive graphs and $S_i \in \mathfrak{P}(\theta, G_i)$ for $i = 1, 2$. Let G be the graph obtained by adding the edges $e_1, e_2, \ldots, e_m$ to the union of G_1 and G_2, where each e_j contains a point in S_1 and S_2. Then G is θ-super positive.*

Proof. We shall prove by induction on m. If $m = 1$, we are done by Theorem 2.2. Suppose $m \geq 2$. Assume that it is true for $m - 1$. Let G' be the graph obtained by adding the edges $e_1, e_2, \ldots, e_{m-1}$ to the union of G_1 and G_2. By induction G' is θ-super positive. Let $e_m = (v_1, v_2)$ where $v_i \in S_i$. Note that the number of θ-critical components in $G' \setminus (S_1 \cup S_2)$ is $c_\theta(G' \setminus (S_1 \cup S_2)) = c_\theta(G_1 \setminus S_1) + c_\theta(G_2 \setminus S_2) = |S_1| + |S_2|$. So $S_1 \cup S_2$ is a θ-barrier set in G'. By Lemma 3.7 and Lemma 3.4, $\{v_1, v_2\}$ is a θ-extreme set in G'. Therefore by Lemma 5.8, G is θ-super positive. □

In Figure 6, the graph G is obtained from two 1-base graphs by adding edges e_1 and e_2.

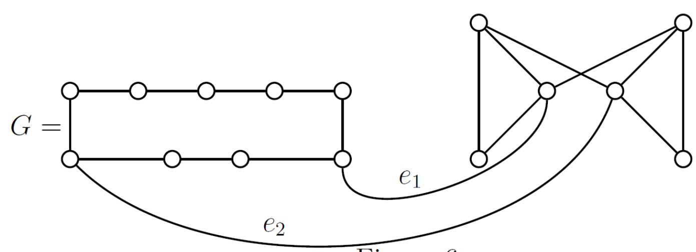

Figure 6.

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