The chromatic number and the
least eigenvalue of a graph

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Abstract

In this paper we get a structural property for a graph having the minimal least eigenvalue among all graphs of fixed order and given chromatic number, and characterize such graphs under the condition that the chromatic number is not larger than half the order of the graph. As a result, we obtain a lower bound on the least eigenvalue in terms of the chromatic number, and an upper bound on the chromatic number in terms of the least eigenvalue of a graph.

1 Introduction

Let $G = (V, E)$ be a simple graph of order n with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The *adjacency matrix* of G is defined to be a $(0, 1)$ matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are called the *eigenvalues of the graph G*. Since $A(G)$ is symmetric, its eigenvalues are real and can be arranged as: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$.

Recall that the *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colors assigned to the vertices of G such that each pair of adjacent vertices have different colors. There is a lot of work on the connection between eigenvalues and chromatic number of a graph. Wilf [18] proved that $\chi(G) \leq 1 + \lambda_1(G)$, with equality if and only if G is a complete graph or an odd cycle. Hoffman [10] shown that $\chi(G) \geq 1 + \frac{\lambda_1(G)}{-\lambda_n(G)}$. Cvetković [4] obtained that $\chi(G) \geq \frac{n}{n - \lambda_1(G)}$. One can also refer [9, 12] for other lower bounds of the chromatic number of a graph. In addition, Feng et al. [8] have characterized the unique

graph whose spectral radius attains the maximum among all graphs of fixed order and given chromatic number.

Denote by K_n, O_n a complete graph and an empty graph both of order n, respectively. Denote by $H_1 \vee H_2$ a graph obtained from two disjoint graphs H_1, H_2 by joining each vertex of H_1 and each of H_2. Let G be a graph of order n. Clearly $\chi(G) = n$ if and only if $G = K_n$; $\chi(G) = 1$ if and only if $G = O_n$. A graph G with $\chi(G) = 2$ is a bipartite graph. By [3] or [19], $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} := O_{\lceil n/2 \rceil} \vee O_{\lfloor n/2 \rfloor}$ is the unique graph whose least eigenvalue is minimum among all graphs (or all bipartite graphs) of order n.

A problem arises naturally what is the graph whose least eigenvalue is minimum among all graphs of order n with fixed chromatic number $\chi \geq 3$.

From the previous work, among all graphs of fixed order and fixed parameter, such as the vertex (edge) independence number or vertex (edge) cover number [14], the number of cut vertices or cut edges [15, 16, 17], the vertex or edge connectivity [19], the diameter [20], the graphs with minimum least eigenvalue are all bipartite. So it is necessary to discuss the non-bipartite graphs with minimum least eigenvalue. This work has been done in [5].

In this paper we will give a more detailed discussion on the non-bipartite graphs with minimum least eigenvalue among graphs of order n with fixed chromatic χ ($3 \leq \chi \leq \frac{n}{2}$), and give a characterization of such graph. Using the result we obtain a lower bound of the least eigenvalue in terms of chromatic number, and an upper bound of the chromatic number in terms of the least eigenvalue of a graph.

Other work on the least eigenvalues of graphs can be referred to Bell et.al. [1, 2], Liu et.al. [11], Petrović et.al. [13], and our workgroup [6, 7]. Through these work we can not only find the structures of graphs with extremal spectral properties but also establish some relationships between the graph parameters and the eigenvalues.

2 Preliminaries

Let $X = (X_1, X_2, \ldots, X_n)^T \in \mathbb{R}^n$, and let G be a graph on vertices $v_1, v_2, \ldots, v_n$. Then X can be considered as a function defined on $V(G)$, that is, each vertex v_i is mapped to $X_i = X(v_i)$; X_i is called a *value* of the vertex v_i given by X. If $U \subseteq V(G)$, denote by $X(U)$ the subvector of X corresponding to the vertices of U.

The least eigenvalue $\lambda_n(G)$ is now denoted by $\lambda_{\min}(G)$, and the corresponding eigenvectors are called *least vectors* of G. One can find that

$$X^T A(G) X = 2 \sum_{uv \in E(G)} X(u)X(v),$$

and λ is an eigenvector of G corresponding to an eigenvector X if and only if $X \neq 0$ and for each $v \in V(G)$

$$\lambda X(v) = \sum_{u \in N(v)} X(u),$$

where $N(v)$ denotes the neighborhood of v in G. In addition, for an arbitrary unit vector $X \in \mathbb{R}^n$,

$$\lambda_{\min}(G) \leq X^T A(G) X,$$

with equality if and only if X is a least vector of G.

We introduce a class of graphs. Given positive integer n, χ such that $3 \leq \chi \leq n - 1$, define a graph $G(a, a') := (K_a + O_{a'}) \vee (K_b + O_{b'})$, where $1 \leq a \leq \chi - 1$, $b = \chi - a$, a', b' are nonnegative integers such that $a' + b' = n - \chi$. Note that $\chi(G(a, a')) = \chi$, and $\lambda_{\min}(G(a, a')) < -1$ as $G(a, a')$ is connected and non-complete.

Let X be a least vector of $G(a, a')$. Then X has constant values, say α on $V(K_a)$, α' on $V(O_{a'})$, β on $V(K_b)$ and β' on $V(O_{b'})$. Hence $\lambda_{\min}(G(a, a'))$ is the least root of following equations:

$$\lambda\alpha = (a-1)\alpha + b\beta + b'\beta', \lambda\alpha' = b\beta + b'\beta', \lambda\beta = a\alpha + a'\alpha' + (b-1)\beta, \lambda\beta' = a\alpha + a'\alpha'. \quad (2.1)$$

So $\lambda_{\min}(G(a, a'))$ is the least root of the polynomial:

$$f(a, a', \lambda) := \lambda^2 (\lambda - a + 1)(\lambda - b + 1) - [(b + b')\lambda - b'(b - 1)][(a + a')\lambda - a'(a - 1)], \quad (2.2)$$

where $b = \chi - a, b' = n - \chi - a'$, and n, χ are considered fixed numbers. In addition, from (2.1) we also find that X contains no zero entries.

LEMMA 2.1 *Among all graphs of order n and with chromatic number χ, where $3 \leq \chi \leq n - 1$, if a graph G is one whose least eigenvalue attains the minimum, then $G = G(a, a')$ for some a and a'.*

Proof: Let X be a unit least vector of G. Surely x contains both positive and negative entries. Denote by $V_+ = \{v \in G : X(v) \geq 0\}$ and $V_- = \{v \in G : X(v) < 0\}$. Thus (V_+, V_-) forms a partition of the vertex set $V(G)$.

Let $\chi(G[V_+]) = \bar{a}, \chi(G[V_-]) = \bar{b}$. Then $\bar{a} \geq 1, \bar{b} \geq 1$ and $\bar{a} + \bar{b} \geq \chi$. Clearly the set V_+ can be partitioned into $\bar{a}$ subsets $V_+^1, V_+^2, \ldots, V_+^{\bar{a}}$ such that each edge of $G[V_+]$ joins two different subsets, and there exists at least one edge joining any two subsets. For each $i = 1, 2, \ldots, \bar{a}$, choose one vertex $v_i \in V_+^i$ with minimal value among all vertices in V_+^i. Now in the graph $G[V_+]$, adding all possible edges within the vertices $\{v_1, v_2, \ldots, v_{\bar{a}}\}$, and deleting all other edges, we get a graph $G_1 = K_{\bar{a}} + O_{\bar{a}'}$ with the same order as $G[V_+]$, where $\bar{a}' = |V_+| - \bar{a}$. One can find

$$X(V_+)^T A(G[V_+]) X(V_+) \geq X(V_+)^T A(G_1) X(V_+).$$

Similarly, we can get a graph $G_2 = K_{\bar{b}} + O_{\bar{b}'}$ with the same order as $G[V_-]$, where $\bar{b}' = |V_-| - \bar{b}$, which holds

$$X(V_-)^T A(G[V_-]) X(V_-) \geq X(V_-)^T A(G_2) X(V_-).$$

Let $G' = G_1 \vee G_2$. Then $\chi(G') = \bar{a} + \bar{b}$, and $X^T A(G) X \geq X^T A(G') X$. Now choose two positive integers a, b such that $1 \leq a \leq \bar{a}, 1 \leq b \leq \bar{b}$ and $a + b = \chi$. Arbitrarily choosing

a complete subgraph K_a of $K_{\bar{a}}$ (respectively, K_b of $K_{\bar{b}}$) and deleting all other edges in G_1 (respectively, G_2), we get a graph $G'' = (K_a + O_{a'}) \vee (K_b + O_{b'})$, where $a + a', b + b'$ are the orders of G_1, G_2 respectively. Then $\chi(G'') = \chi$, and

$$\lambda_{\min}(G) = X^T A(G) X \geq X^T A(G') X \geq X^T A(G'') X \geq \lambda_{\min}(G'').$$

By the definition of G, $\lambda_{\min}(G) = \lambda_{\min}(G'')$ and the above inequalities hold as equalities. Hence X is also a least vector of G''. Thus X contains no zero entries by the discussion prior to this lemma. By the equality $X^T A(G) X = X^T A(G'') X$, we get $G = (K_a + O_{a'}) \vee (K_b + O_{b'})$, as desired. $\blacksquare$

3   Results

Lemma 2.1 is very useful to establish a lower bound of the least eigenvalue of a graph in terms of chromatic number, and an upper bound of the chromatic number in terms of least eigenvalue. Returning to (2.2), we temporarily assume that a, a' are real variables of $f(a, a', \lambda)$, and will show that for an arbitrary given $\lambda \in (-\frac{n}{2}, -1)$, and given $\chi \leq \frac{n}{2}$,

$$f(a, a', \lambda) \geq f(\frac{\chi}{2}, \frac{n - \chi}{2}, \lambda),$$

with equality if and only if $a = \frac{\chi}{2}$ and $a' = \frac{n-\chi}{2}$. This implies the least root of $f(\frac{\chi}{2}, \frac{n-\chi}{2}, \lambda)$ is not greater than that of $f(a, a', \lambda)$.

Assume that $\lambda \in (-\frac{n}{2}, -1)$ and $\chi \leq \frac{n}{2}$ in the following. Without loss of generality, assume $a \geq b$. Taking derivative of f with respect to a, a' respectively, we have

$$\frac{\partial f}{\partial a} = (a - b)[a'b' - \lambda(a' + b')] + (a' - b')\lambda(\lambda + 1).$$
$$\frac{\partial f}{\partial a'} = (a - b)\lambda(\lambda + 1) + (\lambda - a + 1)(\lambda - b + 1)(a' - b').$$

If $a' \geq b'$ or equivalently $a' \geq \frac{n-\chi}{2}$, then

$$\frac{\partial f}{\partial a} \geq 0, \quad \frac{\partial f}{\partial a'} \geq 0, \tag{3.1}$$

and hence $f(a, a', \lambda) \geq f(\frac{\chi}{2}, \frac{n-\chi}{2}, \lambda)$.

Now assume $a' < b'$. If $b' - a' \geq a - b$, then

$$\frac{\partial f}{\partial a'} \leq 0, \tag{3.2}$$

and hence $f(a, a', \lambda) \geq f(a, a' + \gamma, \lambda)$, where $\gamma = \frac{(b'-a')-(a-b)}{2}$.

If $b' - a' \leq a - b$, then

$$\frac{\partial f}{\partial a} \geq (b' - a')[a'b' - \lambda(a' + b' + 1 + \lambda)] \geq 0, \tag{3.3}$$

since $-\frac{n}{2} < \lambda < -1$ and $a' + b' + 1 + \lambda = n - \chi + 1 + \lambda > \frac{n}{2} + 1 - \frac{n}{2} > 0$. Hence $f(a, a', \lambda) \geq f(a - \eta, a', \lambda)$, where $\eta = \frac{(a-b)-(b'-a')}{2}$. So in both cases we have $f(a, a', \lambda) \geq f(\bar{a}, \bar{a}', \lambda)$ for some $\bar{a}$ and $\bar{a}'$ such that $\bar{a} + \bar{a}' = \frac{n}{2}$.

Define
$$\varphi(\delta) := f(a - \delta, a' + \delta, \lambda), \tag{3.4}$$

where $\delta \in [0, \frac{a-b}{2}]$ and $a + a' = \frac{n}{2}$. Then by the derivatives of f with respect of a, a' and the chain rule of composite function,

$$\varphi'(\delta) = -(a - b - 2\delta)\{(a' + \delta)(b' - \delta) - \frac{n}{2}\lambda - 2\lambda(\lambda + 1) + [\lambda + 1 - (a - \delta)][\lambda + 1 - (b + \delta)]\} \leq 0.$$

So, if $a - b > 0$, or equivalently $a > \frac{\chi}{2}$, noting $a - b = b' - a'$,

$$f(a, a', \lambda) = \varphi(0) > \varphi(a - \frac{a-b}{2}, a' + \frac{a-b}{2}) = f(\frac{\chi}{2}, \frac{n-\chi}{2}, \lambda).$$

Now taking $a = b = \frac{\chi}{2}, a' = b' = \frac{n-\chi}{2}$ into (2.2), we get

$$\lambda^2 + (a' + 1)\lambda - a'(a - 1) = 0, \tag{3.5}$$

and the least root of the equation (3.5) is

$$\xi := -\frac{n - \chi + 2 + \sqrt{(n - \chi - 2)^2 + 4\chi(n - \chi)}}{4}.$$

If $\chi \geq 2$, then $\xi = \xi(\chi)$, as a function of χ, is strictly increasing on χ, and hence $-1 = \xi(n) \geq \xi \geq \xi(2) = -\frac{n}{2}$. Thus we get a sharper lower bound of the least eigenvalue of a graph in terms of the chromatic number.

THEOREM 3.1 *Let G be a graph of order n with chromatic number χ $(3 \leq \chi \leq \frac{n}{2})$. Then*

$$\lambda_{\min}(G) \geq -\frac{n - \chi + 2 + \sqrt{(n - \chi - 2)^2 + 4\chi(n - \chi)}}{4},$$

with equality if and only if n, χ are both even and $G = G(\chi/2, (n - \chi)/2)$.

Replacing a' by $\frac{n}{2} - a$, λ by ξ, rewrite (3.5) as follows:

$$a^2 - \left(\frac{n}{2} + 1 + \xi\right)a + (\xi + 1)\left(\xi + \frac{n}{2}\right) = 0.$$

Hence

$$\chi = 2a = \left(\frac{n}{2} + 1 + \xi\right) + \sqrt{\left(\frac{n}{2} + 1 + \xi\right)^2 - 4(\xi + 1)\left(\xi + \frac{n}{2}\right)}.$$

Noting that $\chi = \chi(\xi)$ is also increasing on ξ, $2 = \chi(-\frac{n}{2}) \leq \chi \leq \chi(-1) = n$. For a graph G with chromatic number χ $(3 \leq \chi \leq \frac{n}{2})$, $\lambda_{\min}(G) \geq \xi$ by Theorem 3.1. So we get an upper bound of the chromatic number of a graph in terms of the least eigenvalue.

THEOREM 3.2 *Let G be a graph of order n with chromatic number χ ($3 \le \chi \le \frac{n}{2}$) and least eigenvalue $\lambda = \lambda_{\min}(G)$. Then*

$$\chi \le \left(\frac{n}{2} + 1 + \lambda\right) + \sqrt{\left(\frac{n}{2} + 1 + \lambda\right)^2 - 4(\lambda + 1)\left(\lambda + \frac{n}{2}\right)},$$

with equality if and only if n, χ are both even and $G = G(\chi/2, (n - \chi)/2)$.

At the end of this paper, we characterize the unique graph whose least eigenvalue attains the minimum among all graphs with fixed order and chromatic number not greater than half of the order.

THEOREM 3.3 *Among all graphs of order n and with chromatic number χ, where $3 \le \chi \le \frac{n}{2}$, the graph $G(\lceil \chi/2 \rceil, \lfloor (n - \chi)/2 \rfloor)$ is the unique one whose least eigenvalue attains the minimum.*

Proof: By Lemma 2.1, it is enough to consider the graph $G(a, a')$. Without loss of generality, we assume $a \ge b$ or equivalently $a \ge \frac{\chi}{2}$. Assume that $\lambda \in (-\frac{n}{2}, -1)$ in the following discussion.

Case 1: $a' \ge b'$. By (3.1), the function f defined in (2.2) is strictly increasing on a and a' respectively. Note that a, a' now are integers. To make f attain the minimum, $a = \lceil \chi/2 \rceil$ and $a' = \lceil (n - \chi)/2 \rceil$. Hence, if $a \ge b + 2$ or $a' \ge b' + 2$ then $\lambda_{\min}(G(a, a')) > \lambda_{\min}(G(\lceil \chi/2 \rceil, \lceil (n - \chi)/2 \rceil))$.

To reach the final result, we need to discuss the case of $(n - \chi) =: 2m + 1$ being odd. If χ is even, then $G(\lceil \chi/2 \rceil, m + 1)$ is isomorphic to $G(\lceil \chi/2 \rceil, m)$; otherwise by a little calculation $f(\lceil \chi/2 \rceil, m + 1, \lambda) - f(\lceil \chi/2 \rceil, m, \lambda) > 0$, which implies that $\lambda_{\min}(G(\lceil \chi/2 \rceil, m + 1)) > \lambda_{\min}(G(\lceil \chi/2 \rceil, m))$.

Case 2: $b' > a'$. If $a = b$, the result follows from Case 1 by considering the graph $G(b, b')$. So we assume that $a > b$ in the following.

If $b' - a' \ge a - b$, then $\frac{\partial f}{\partial a'} \le 0$ by (3.2). Hence, if $b' - a' \ge (a - b) + 2$, then $f(a, a', \lambda) > f(a, a' + 1, \lambda)$. So $\lambda_{\min}(G(a, a')) > \lambda_{\min}(G(a, a' + 1)) \ge \cdots \ge \lambda_{\min}(G(a, a' + l))$, for some positive integer l such that $(b' - l) - (a' + l)$ equals $a - b$ or $a - b + 1$.

If $b' - a' \le a - b$, then $\frac{\partial f}{\partial a} > 0$ by (3.3). Hence, if $b' - a' \le (a - b) - 2$, then $f(a, a', \lambda) > f(a - 1, a', \lambda)$, and hence $\lambda_{\min}(G(a, a')) > \lambda_{\min}(G(a - 1, a')) \ge \lambda_{\min}(G(a - k, a'))$ for some positive integer k such that $b' - a'$ equals $(a - k) - (b + k)$ or $(a - k) - (b + k) - 1$.

By above discussion, we only need to consider the graph $G(a, a')$ with $b' - a'$ equal to one of the values $a - b - 1, a - b, a - b + 1$. If $b' - a' = a - b + 1$, by a little calculation $f(a, a', \lambda) > f(a, a' + 1, \lambda)$, and hence $\lambda_{\min}(G(a, a') > \lambda_{\min}(G(a, a' + 1)$. So it is enough to deal with the first two types of graphs $G(a, a')$.

If $b' - a' = a - b =: \zeta \ge 2$, using the function $\varphi(\delta)$ defined in (3.4),

$$f(a, a', \lambda) = \varphi(0) > \phi(a - \lfloor \zeta/2 \rfloor, a' + \lfloor \zeta/2 \rfloor) = f(a - \lfloor \zeta/2 \rfloor, a' + \lfloor \zeta/2 \rfloor, \lambda),$$

and hence $\lambda_{\min}(G(a, a')) > \lambda_{\min}(G(a - \lfloor \zeta/2 \rfloor, a' + \lfloor \zeta/2 \rfloor)) = \lambda_{\min}(G(\lceil \chi/2 \rceil, \lfloor (n - \chi)/2 \rfloor))$.

If $b' - a' = (a - b) - 1$ and $a - b \geq 2$, then $f(a, a', \lambda) - f(a - 1, a' + 1, \lambda) =$

$$(-\lambda^2 - 2b'\lambda + a'b') + (a - b - 2)[-\lambda^2 - 2(a' + a - 1)\lambda + a'b' + (a - 2)b] > 0,$$

where the first summand is positive as $2b' > a' + b' = n - \chi \geq \frac{n}{2}$, and the second is nonnegative as $2(a + a' - 1) = 2(b + b') = n - 1 > \frac{n}{2}$. So $\lambda_{\min}(G(a, a')) > \lambda_{\min}(G(a - 1, a' + 1))$. The result follows by repeated using this inequality. ∎

Remark: For the graph $G(\chi/2, (n - \chi)/2)$, where n, χ ($\chi \leq n/2$) are both even, the spectral radius is also the largest root of the $f(\chi/2, (n - \chi)/2, \lambda)$, and is given by

$$\lambda_1(G(\chi/2, (n - \chi)/2)) = \frac{2a - 1 + a' + \sqrt{(2a - 1)^2 + a'(a' + 2)}}{2} > 2a - 1 + \frac{a'}{2},$$

where $a = \chi/2, a' = (n - \chi)/2$. According to Wilf's bound,

$$\chi(G(\chi/2, (n - \chi)/2)) = \chi < \chi + \frac{n - \chi}{4} = 1 + \lambda_1(G(\chi/2, (n - \chi)/2)).$$

However, our bound in Theorem 3.2 for this graph is tight. For an odd cycle, Wilf's bound is tight but our bound is not good.

Theorems 3.1-3.3 deal with the graphs G of order n with $3 \leq \chi(G) \leq n/2$. By a number of numerical experiments on small graphs, we conjecture that they hold for all graphs G without limitation on the range of $\chi(G)$.

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