

On-line list colouring of complete multipartite graphs

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Abstract

The Ohba Conjecture says that every graph G with $|V(G)| \leq 2\chi(G) + 1$ is chromatic choosable. This paper studies an on-line version of Ohba Conjecture. We prove that unlike the off-line case, for $k \geq 3$, the complete multipartite graph $K_{2\star(k-1),3}$ is not on-line chromatic-choosable. Based on this result, the on-line version of Ohba Conjecture is modified as follows: Every graph G with $|V(G)| \leq 2\chi(G)$ is on-line chromatic choosable. We present an explicit strategy to show that for any positive integer k , the graph $K_{2\star k}$ is on-line chromatic-choosable. We then present a minimal function g for which the graph $K_{2\star(k-1),3}$ is on-line g -choosable.

1 Introduction

A *list assignment* of a graph G is a mapping L which assigns each vertex v a set $L(v)$ of colours. An L -*colouring* of G is a proper vertex colouring c of G such that $c(v) \in L(v)$ for each v . We say G is L -*colourable* if there exists an L -colouring of G . For a mapping $f : V(G) \rightarrow \mathbb{N}$, a graph G is called f -*choosable* if for every list assignment L with $|L(v)| = f(v)$, G is L -colourable. For a positive integer k , we say G is k -choosable if G is f -choosable for the constant function $f(v) = k$. The *choice number* $ch(G)$ of G is the

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minimum k for which G is k -choosable. List colouring of graphs was introduced in the 1970s by Vizing [12] and independently by Erdős, Rubin and Taylor [1], and has been studied extensively in the literature [11].

Assume L is a list assignment of G . Without loss of generality, we may assume that $\cup_{v \in V(G)} L(v) = \{1, 2, \dots, q\}$ for some integer q . For $i = 1, 2, \dots, q$, let $V_i = \{v : i \in L(v)\}$. The sequence $(V_1, V_2, \dots, V_q)$ is just another way of specifying the list assignment. An L -colouring of G is equivalent to a sequence $(X_1, X_2, \dots, X_q)$ of independent sets that form a partition of $V(G)$ and such that $X_i \subseteq V_i$ for $i = 1, 2, \dots, q$. This alternate definition motivates the definition of the following list colouring game on a graph G , which was introduced in [7].

Definition 1.1. *Given a graph G and a mapping $f : V(G) \rightarrow \mathbb{N}$. Two players play the following game. In the i th step, Player A chooses a non-empty subset V_i of $V(G)$, and Player B chooses an independent set X_i contained in V_i . A vertex v is coloured if $v \in X_i$ for some i , and is finished if v is contained in $f(v)$ of the V_i 's. When Player A chooses the set V_i , it is required V_i contains only uncoloured non-finished vertices. If for some integer m , at the end of the m th step, there is a finished vertex v that is uncoloured, then Player A wins the game. Otherwise, at some step, all vertices are coloured. In this case, Player B wins the game.*

In this game, Player A is required to give $f(v)$ permissible colours to vertex v and Player B needs to colour v with a permissible colour, under the restriction that no colour is assigned to two adjacent vertices. Player B wins the game if every vertex v is successfully coloured. The game is called the Painter and Correct game in [9, 7]. In some sense, one can view it as an on-line version of list colouring: it is the same as a list colouring of a graph, except that the list assignment is given on-line, and the colouring is constructed on-line. We shall call such a game the *on-line (G, f) -list colouring game*.

Definition 1.2. *Suppose $f : V(G) \rightarrow \mathbb{N}$. We say G is on-line f -choosable if Player B has a winning strategy in the on-line (G, f) -list colouring game.*

For a positive integer k , G is *on-line k -choosable* means that G is on-line f -choosable for the constant function $f(v) = k$. The *on-line choice number* $\text{ch}^{\text{OL}}(G)$ of G is the minimum k for which G is on-line k -choosable. Thus $\text{ch}^{\text{OL}}(G) \geq \text{ch}(G)$ for all graphs G .

It is shown in [2, 8, 9, 7] that many upper bounds for the choice number of a graph remain upper bounds for its on-line choice number. For example, the on-line choice number of planar graphs is at most 5, the on-line choice number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$, and if G has an orientation in which the number of even Eulerian subgraphs differs from the number of odd Eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is on-line f -choosable. For these upper bounds for the choice number,

either the original proofs work directly for the on-line case, or a minor modification of the original proofs work for the on-line case. Nevertheless, there are several upper bounds for the choice number whose proofs do not work for the on-line case, and some of them fail to be an upper bound for the on-line case.

A graph G is called *chromatic-choosable* (resp. *on-line chromatic-choosable*) if $ch(G) = \chi(G)$ (resp. $ch^{OL}(G) = \chi(G)$). The following conjecture of Ohba [6] concerning chromatic-choosable graphs received a lot of attention.

Conjecture 1.3 (Ohba 2002). *If $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic-choosable.*

To prove Ohba's conjecture, it suffices to consider complete multipartite graphs. Let $k_1, k_2, \dots, k_s, n_1, n_2, \dots, n_s$ be non-negative integers with $k = k_1 + k_2 + \dots + k_s$. Let $K_{n_1 \star k_1, \dots, n_s \star k_s}$ denote the complete k -partite graph with k_i parts of cardinality n_i for $i = 1, 2, \dots, s$. If $k_i = 1$, then $n_i \star 1$ in the subscript will be shortened as n_i . For example, $K_{2 \star 4, 3} = K_{2 \star 4, 3 \star 1}$. Some partial results on Ohba Conjecture are obtained (we refer to [3, 10] for a survey of such partial results). In particular, it is shown in [10] that the conjecture is true for complete multipartite graphs with each partite set of cardinality at most 3. For example, $K_{2 \star (k-1), 3}$ is k -choosable. Recently, Kostochka, Stiebitz and Woodall [4] proved that the conjecture is true for complete multipartite graphs such that each partite set has cardinality at most 5.

For any result or conjecture concerning the choice number of graphs, one naturally wonders if the same result or conjecture applies to on-line choice number. In [3], an on-line version of Ohba Conjecture was considered. It is natural to ask if $ch^{OL}(G) = \chi(G)$ for graphs G with $|V(G)| \leq 2\chi(G) + 1$. This is true if $\chi(G) \leq 2$. However, we shall show in this paper that if $k \geq 3$, the complete multipartite graph $K_{2 \star (k-1), 3}$ is not on-line k -choosable. Based on this result, the on-line version of Ohba's conjecture was modified as follows in [3]:

Conjecture 1.4. *If $|V(G)| \leq 2\chi(G)$, then G is on-line chromatic-choosable.*

All the proofs of the special cases of Ohba Conjecture use Hall's Theorem to obtain a matching between vertices and colours under certain conditions. This means that one needs to know the whole list assignment before colouring the vertices. Therefore the proofs do not work for on-line list colouring. It was proved in [9] that if G has an orientation in which the number of even Eulerian subgraphs differs from the number of odd Eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is on-line f -choosable. So if an upper bound for the choice number of a graph is proven by using Combinatorial Nullstellensatz, then the upper bound holds true for its on-line choice number.

In [3], Combinatorial Nullstellensatz method was used to verify Conjecture 1.4 for some special cases. In particular, it was proved in [3], by using Combinatorial Nullstellensatz,

that $K_{2\star k}$ is on-line chromatic-choosable. However, the Combinatorial Nullstellensatz method leads to existence proofs, and it does not provide a simple strategy for Player B to win the on-line list colouring game.

In this article, we provide a simple strategy to show that for any positive integer k , the graph $K_{2\star k}$ is on-line chromatic-choosable. We then consider the problem as for which functions g , the graph $K_{2\star(k-1),3}$ is on-line g -choosable. Assume the partite sets of $G = K_{2\star(k-1),3}$ are $A_i = \{v_i, u_i\}$ for $i = 1, 2, \dots, k-1$ and $A_k = \{v_k, u_k, w_k\}$. Write a function g in the form $g(v_1)g(u_1)||g(v_2)g(u_2)|| \dots ||g(v_{k-1})g(u_{k-1})||g(v_k)g(u_k)g(w_k)$. We prove that for the function g of the form $(kk)^{k-1}||x_1x_2x_3 = \underbrace{kk||kk|| \dots ||kk||}_{k-1}||x_1x_2x_3$, if $x_1, x_2, x_3 \geq k$ and $x_1 + x_2 + x_3 \geq 4k - 2$, then G is on-line g -choosable. On the other hand, for g of the form $(kk)^{k-1}||kk(2k-3)$, G is not on-line g -choosable.

2 On-line list coloring for $K_{2\star k}$

Suppose G is a graph and f is a mapping which assigns to each vertex v a positive integer $f(v)$. We call the pair (G, f) a *configuration*. We say a configuration (G, f) is *feasible* if G is on-line f -choosable. Otherwise we say (G, f) is *infeasible* (or, f is infeasible for G). If the graph G is clear from the context, sometimes we simply say f is feasible (or infeasible) if (G, f) is feasible (or infeasible).

For a subset U of $V(G)$, let $\delta_U : V(G) \rightarrow \{0, 1\}$ be the characteristic function of U defined as

$$\delta_U(v) = \begin{cases} 1, & \text{if } v \in U; \\ 0, & \text{otherwise.} \end{cases}$$

As observed in [7], the following lemma can be viewed as an alternate definition of on-line f -choosability of graphs.

Lemma 2.1. *If G is an edgeless graph and $f(v) \geq 1$ for all $v \in V(G)$, then (G, f) is feasible. If G has at least one edge, then (G, f) is feasible if and only if for every subset U of $V(G)$, there is an independent set I of G contained in U such that $(G - I, f - \delta_U)$ is feasible. (More precisely, $(G - I, (f - \delta_U)|_{V \setminus I})$ is feasible.)*

Theorem 2.2. *Assume $k_1, k_2 \geq 0, k = k_1 + k_2 > 0$ and $G = K_{1\star k_1, 2\star k_2}$ is a complete k -partite graph in which each partite set has cardinality at most 2. Let $A_1, \dots, A_k$ be the k -partite sets of G such that $|A_i| = 1$ for $1 \leq i \leq k_1$ and $|A_j| = 2$ for $k_1 + 1 \leq j \leq k_1 + k_2 = k$. Let $f : V(G) \rightarrow \mathbb{N}$ be a function satisfying the following conditions:*

1. *For $1 \leq i \leq k_1$, if $v \in A_i$, then $f(v) \geq k_2 + i$.*

2. For $k_1 + 1 \leq i \leq k - 1$, if $v \in A_i$, then $f(v) \geq k_2$ and $\sum_{v \in A_i} f(v) \geq |V(G)|$.

3. If $k_2 > 0$, let $A_k = \{v_1, v_2\}$ with $1 \leq f(v_1) \leq f(v_2)$ and $f(v_1) + f(v_2) \geq |V(G)|$.

Then (G, f) is feasible.

Proof. We prove the result by induction on $|V(G)|$. The cases $k \leq 2$ are easily verified.

Assume $k \geq 3$. If $f(v_1) = 1$, then $f(v_2) \geq |V| - 1$. Since v_2 has $|V| - 2$ neighbours in G , (G, f) is feasible if and only if $(G \setminus \{v_1, v_2\}, f')$ is, where $f'(v) = f(v) - 1$ for $v \in V(G) \setminus \{v_1, v_2\}$. By induction hypothesis $(G \setminus \{v_1, v_2\}, f')$ is feasible.

In the following we assume $f(v_1) \geq 2$. Given a subset U of $V(G)$, we shall find an independent set I of G contained in U so that $(G - I, (f - \delta_U))$ is feasible. Let $G' = G - I$ and $f' = f - \delta_U$. Note that G' is again a complete multipartite graph with each partite set of cardinality at most 2. Let k'_1, k'_2 be integers such that $G' = K_{1 \star k'_1, 2 \star k'_2}$. Let A'_i be the corresponding partite sets of G' . We shall show that G', f' satisfy the conditions of Theorem 2.2, and hence by induction hypothesis, (G', f') is feasible.

Observe that in the conditions of Theorem 2.2, the partite sets of cardinality 1 are ordered. This ordering is important, because if the order is changed, then condition of Theorem 2.2 may no longer be satisfied. On the other hand, the ordering of the partite sets of cardinality 2, except the last part A_k , is irrelevant, and we may reorder them by our convenience.

The remaining of the proof is divided into several cases.

Case 1. U contains A_j for some $k_1 + 1 \leq j \leq k$.

By a reordering we may assume that $j \in \{k - 1, k\}$. Let $I = A_j$, $G' = G - I$ and $f' = f - \delta_U$. Then $k'_1 = k_1, k'_2 = k_2 - 1, k' = k - 1$ and $|V(G')| = |V(G)| - 2$. For each vertex v of G' , we have $f'(v) \geq f(v) - 1$.

For $1 \leq i \leq k'_1$, if $v \in A'_i = A_i$, then $f'(v) \geq f(v) - 1 \geq k'_2 + i$. For $k'_1 + 1 \leq i \leq k' - 1$, if $v \in A'_i = A_i$, then

$$f'(v) \geq f(v) - 1 \geq k'_2, \text{ and } \sum_{v \in A'_i} f'(v) \geq |V(G)| - 2 = |V(G')|.$$

If $v \in A'_{k'} = A_k$ or $v \in A_{k-1}$, then

$$f'(v) \geq f(v) - 1 \geq 1, \text{ and } \sum_{v \in A'_i} f'(v) \geq |V(G)| - 2 = |V(G')|.$$

Hence conditions of Theorem 2.2 are satisfied, and (G', f') is feasible.

Case 2. $|A_i \cap U| \leq 1$ for all $1 \leq i \leq k$.

Sub-case 2.1: For some $k_1 + 1 \leq j \leq k$, $A_j = \{x, y\}$, $U \cap A_j = \{x\}$ and $f(x) \leq k_2$.

By a reordering, we may assume $j = k_1 + 1$. By our assumption, $f(x) + f(y) \geq |V(G)| = k_1 + 2k_2$. As $f(x) \leq k_2$, we have $f(y) \geq k = k_1 + k_2$. Let $I = \{x\}$, $G' = G - I$ and $f' = f - \delta_U$. Now $k'_1 = k_1 + 1$, $k'_2 = k_2 - 1$, $k' = k$. The partite sets of G' are ordered as before (i.e., $A'_i = A_i$), except that $A'_{k'_1} = A_{k'_1} \setminus \{x\} = \{y\}$.

For any $1 \leq i \leq k_1 = k'_1 - 1$ and for $u \in A'_i = A_i$, we have

$$f'(u) \geq f(u) - 1 \geq k_2 - 1 + i = k'_2 + i.$$

For $y \in A'_{k'_1}$, $f(y) \geq k = k'_2 + k'_1$. For $k'_1 + 1 \leq i \leq k' - 1 = k - 1$ and for any $u \in A'_i = A_i$, we have $f'(u) \geq f(u) - 1 \geq k_2 - 1 = k'_2$. For $u \in A'_{k'}$, since $f(v_2) \geq f(v_1) \geq 2$, it follows that $f'(u) \geq f(u) - 1 \geq 1$. Moreover, for $k'_1 + 1 \leq i \leq k' = k$, we have $\sum_{x \in A'_i} f'(x) \geq |V(G)| - 1 = |V(G')|$, since $|A_i \cap U| \leq 1$ for any $1 \leq i \leq k$. Hence the conditions of Theorem 2.2 are satisfied and (G', f') is feasible.

Sub-case 2.2: For all $v \in U$, $f(v) \geq k_2 + 1$.

Let j be the smallest index such that $A_j \cap U \neq \emptyset$. Assume $\{x\} = A_j \cap U$. Let $I = \{x\}$, $G' = G - I$ and $f' = f - \delta_U$. Consider the following three possibilities for the value of j .

Sub-case 2.2.1: Assume $1 \leq j \leq k_1$. Then $k'_1 = k_1 - 1$, $k'_2 = k_2$, $k' = k - 1$ and $|V(G')| = |V(G)| - 1$. The ordering of the A_i 's is the same as before (i.e., $A'_i = A_i$), except that the A_j is gone, and for $j \leq i \leq k'_1$, $A'_i = A_{i+1}$.

For $1 \leq i \leq j - 1$, for $u \in A'_i = A_i$, we have $f'(u) = f(u) \geq k_2 + i = k'_2 + i$. For $j \leq i \leq k'_1$ and $u \in A'_i = A_{i+1}$, we have

$$f'(u) \geq f(u) - 1 \geq k_2 - 1 + i + 1 = k'_2 + i.$$

For $k'_1 + 1 \leq i \leq k' - 1$ and $u \in A'_i = A_{i+1}$, we have

$$f'(u) \geq f(u) - 1 \geq k_2 + 1 - 1 = k_2 = k'_2, \text{ and } \sum_{x \in A'_i} f'(x) \geq |V(G)| - 1 = |V(G')|.$$

For $u \in A'_{k'} = A_k$, recall $f(v_1) \geq 2$, so we have

$$f'(u) \geq f(u) - 1 \geq 1, \text{ and } \sum_{x \in A'_{k'}} f'(x) \geq |V(G)| - 1 = |V(G')|.$$

Hence G', f' satisfy the condition of Theorem 2.2. So (G', f') is feasible.

Sub-case 2.2.2: Assume $k_1 + 1 \leq j \leq k - 1$, and $A_j = \{x, y\}$. By a reordering, we may assume that $j = k_1 + 1$. Then $k'_1 = k_1 + 1$, $k'_2 = k_2 - 1$, $k' = k$ and $|V(G')| = |V(G)| - 1$. The ordering of the A_i 's is defined as follows: $A'_1 = \{y\}$, and for $2 \leq i \leq k'_1$, $A'_i = A_{i-1}$,

and for $k'_1 + 1 \leq i \leq k'$, $A'_i = A_i$. By our assumption, $f'(y) = f(y) \geq k_2 = k'_2 + 1$. For $2 \leq i \leq k'_1$ and $u \in A'_i = A_{i-1}$, we have $f'(u) = f(u) \geq k_2 + i - 1 = k'_2 + i$. For $k'_1 + 1 \leq i \leq k' - 1$, and $u \in A'_i = A_i$, we have

$$f'(u) \geq k_2 = k'_2 + 1, \text{ and } \sum_{x \in A'_i} f'(x) \geq |V(G)| - 1 = |V(G')|.$$

For $u \in A'_{k'} = A_k$, we have

$$f'(u) \geq f(u) - 1 \geq 1, \text{ and } \sum_{x \in A'_{k'}} f'(x) \geq |V(G)| - 1 = |V(G')|.$$

Hence G', f' satisfy the conditions of Theorem 2.2. So (G', f') is feasible.

Sub-case 2.2.3: Assume $j = k$. Then $I = U \cap A_k = \{x\} \subseteq \{v_1, v_2\}$, $G' = G \setminus \{x\}$, and $f'(v) = f(v)$ for all $v \in V(G')$. Let $y = \{v_1, v_2\} \setminus \{x\}$. Define a function g on $V(G')$ by $g(v) = f'(v)$ for all $v \neq y$, and $g(y) = 1$. Then (G', g) is feasible if and only if (G'', f'') is, where $G'' = G' \setminus \{y\}$, $f''(v) = f'(v) - 1$ for all $v \in V(G'')$. By induction hypothesis, (G'', f'') is feasible. Hence (G', f') is feasible, as $g(v) \leq f'(v)$ for all $v \in V(G')$. $\square$

Corollary 2.3. For any positive integer k , $\text{ch}^{\text{OL}}(K_{2\star k}) = k$.

Proof. Since $\chi(K_{2\star k}) = k$, we have that $\text{ch}^{\text{OL}}(K_{2\star k}) \geq k$. By Theorem 2.2, $K_{2\star k}$ is on-line k -choosable. So, $\text{ch}^{\text{OL}}(K_{2\star k}) = k$. $\square$

3 On-line choice number of $K_{2\star(k-1),3}$

For two mappings $f, g : V(G) \rightarrow \{1, 2, \dots\}$, we say f dominates g , written as $f \geq g$, if $f(v) \geq g(v)$ for all $v \in V(G)$. We write $f \geq q$ (where q is an integer) if $f(v) \geq q$ for all $v \in V(G)$. Fix a graph G . A configuration (G, f) is called *minimal feasible* if f is feasible for G , but g is not feasible for G for any $g < f$.

As our main focus are complete multipartite graphs G we introduce a simple scheme to express a configuration for such graphs: Assume G has partite sets $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ for $i = 1, 2, \dots, q$, we record (G, f) in a sequence as follows:

$$f(v_{1,1}) \dots f(v_{1,n_1}) || f(v_{2,1}) \dots f(v_{2,n_2}) || \dots || f(v_{q,1}) \dots f(v_{q,n_q}).$$

If there are p partite sets A_i of the same cardinality and with $f(A_i)$ being the same sequence, then instead of listing the same sequence p times, we may replace it with $(f(v_{i,1}) \dots f(v_{i,n_i}))^p$.

For example, $33||33||333$ represents the configuration (G, f) , where $G = K_{2,2,3}$ with $f(v) = 3$ for each vertex v . As the first partite set A_1 and the second partite set A_2 have the same cardinality and $f(A_1) = f(A_2)$, we may write the configuration as $(33)^2||333$.

In the following we shall frequently use the fact that (K_n, f) is infeasible if each vertex has at most $n - 1$ permissible colours.

Lemma 3.1. *For $k \geq 2$, $(kk)^k$ is minimal feasible for $K_{2 \star k}$.*

Proof. We prove this by induction on k . It is easy to check that $22||22$ is minimal feasible. Assume $k \geq 3$. It is enough to show that $(k - 1)k||(\overline{kk})^{k-1}$ is infeasible. Player A's first move is $(k - 1)\overline{k}||(\overline{kk})^{k-1}$, i.e., Player A chooses one vertex v with $f(v) = k$ from each part. In order to avoid a copy of K_k in which each vertex has at most $k - 1$ permissible colours, Player B has only one choice, resulting in $(k - 1)(k - 1)||k||((k - 1)k)^{k-2}$. Then, Player A chooses $\overline{(k - 1)(k - 1)}||\overline{k}||((k - 1)\overline{k})^{k-2}$. This again leaves Player B with only one choice, resulting in $(k - 2)(k - 1)||((k - 1)(k - 1))^{k-2}$. The result follows by the induction hypothesis. $\square$

Lemma 3.2. *For $k \geq 3$, the following is infeasible for $K_{1,2 \star (k-2),3}$:*

$$k||((k - 1)k)^{k-2}||(\overline{(k - 1)kk}).$$

Proof. We prove by induction on k . To show $3||23||233$ is infeasible (or Player A has a winning strategy), we let Player A's first move be: $\overline{3}||\overline{23}||\overline{233}$. Player B has three choices which lead to the following three configurations:

$$13||222, \quad 2||3||222, \quad 2||13||2.$$

The configuration $2||13||2$ is infeasible, because there is a copy of K_3 in which each vertex has at most 2 permissible colours. For configuration $13||222$, Player A's choice is $\overline{13}||\overline{222}$, and Player B's response leads to $3||111$, which is easily verified to be an A win configuration (that is, infeasible). In the 2nd configuration, Player A's choice is $\overline{2}||\overline{3}||\overline{222}$, and Player B's response would lead to $2||112$ or $1||2||2$ which are easily seen to be A win configurations.

Assume $k \geq 4$. Player A's first move is $\overline{k}||((k - 1)\overline{k})^{k-2}||(\overline{(k - 1)kk})$. Avoiding a copy of K_k in which each vertex has at most $k - 1$ permissible colours, Player B has only one choice which leads to the configuration

$$((k - 1)(k - 1))^{k-2}||(\overline{(k - 1)(k - 1)(k - 1)}).$$

Player A's next move would be $\overline{((k - 1)(k - 1))^{k-2}}||\overline{(k - 1)(k - 1)(k - 1)}$. Then Player B gets two possible results:

$$k - 1||((k - 2)(k - 1))^{k-3}||(\overline{(k - 2)(k - 1)(k - 1)}) \quad \text{and} \quad ((k - 2)(k - 1))^{k-2}||(\overline{(k - 1)(k - 1)}),$$

which are infeasible by induction hypothesis and by Lemma 3.1, respectively. $\square$

Theorem 3.3. *For any integer $k \geq 3$, $\text{ch}^{\text{OL}}(K_{2\star(k-1),3}) = k + 1$.*

Proof. Since $K_{2\star(k-1),3}$ is a subgraph of $K_{2\star(k+1)}$ and $\text{ch}^{\text{OL}}(K_{2\star(k+1)}) = k + 1$, by Corollary 2.3, we have $\text{ch}^{\text{OL}}(K_{2\star(k-1),3}) \leq k + 1$. Therefore it suffices to show that $\text{ch}^{\text{OL}}(K_{2\star(k-1),3}) > k$.

We will show the configuration $(kk)^{k-1}||kkk$ is infeasible. Player A's first move is $(\bar{k}k)^{k-1}||\bar{k}kk$. Player B has two choices which lead to:

$$((k-1)k)^{k-1}||kk \quad \text{and} \quad k||((k-1)k)^{k-2}||((k-1)kk).$$

Both are infeasible, according to Lemmas 3.1 and 3.2, respectively. $\square$

Corollary 3.4. *For any positive integer $k \geq 3$, $\text{ch}^{\text{OL}}(K_{2\star(k-1),4}) = k + 1$.*

Proof. Since $K_{2\star(k-1),3}$ is a subgraph of $K_{2\star(k-1),4}$, we have

$$\text{ch}^{\text{OL}}(K_{2\star(k-1),4}) \geq k + 1.$$

On the other hand, $\text{ch}^{\text{OL}}(K_{2\star(k-1),4}) \leq \text{ch}^{\text{OL}}(K_{2\star(k+1)}) = k + 1$. $\square$

4 A minimal feasible configuration on $K_{2\star(k-1),3}$

To approach Conjecture 1.4 by induction, it seems unavoidable to consider feasible and infeasible configurations for complete multipartite graphs in general. In particular, we need to consider complete multipartite graphs for which the number of vertices is greater than twice of its chromatic number. The graph $G = K_{2\star(k-1),3}$ is a simplest example of such graphs. Since G is not on-line k -choosable, a natural question is what are the minimal feasible configurations for G ?

In this section, we prove that $(kk)^{k-1}||kk(2k-2)$ is minimal feasible.

Lemma 4.1. *Assume $k_1 \geq 0$, $k_2 \geq 1$, $k = k_1 + k_2$, and $G = K_{1\star k_1, 2\star(k_2-1), 3}$. Let $A_1, \dots, A_k$ be the k -partite sets of G such that $|A_i| = 1$ for $1 \leq i \leq k_1$, $|A_j| = 2$ for $k_1 + 1 \leq j \leq k_1 + k_2 - 1 = k - 1$, and $|A_k| = 3$. Let $f : V(G) \rightarrow \mathbb{N}$ be a function satisfying the following conditions:*

1. *For $1 \leq i \leq k_1$, if $v \in A_i$, then $f(v) \geq k_2 + i$.*
2. *For $k_1 + 1 \leq i \leq k - 1$, if $v \in A_i$, then $f(v) \geq k_2$ and $\sum_{v \in A_i} f(v) \geq |V(G)| - 1$.*
3. *$A_k = \{v_1, v_2, v_3\}$ where $1 \leq f(v_1) \leq f(v_2) \leq f(v_3)$, $f(v_1) + f(v_2) \geq |V| - 1$ and $f(v_1) + f(v_2) + f(v_3) \geq 2|V| - 3$.*

Then (G, f) is feasible.

Proof. We prove that (G, f) is feasible by induction on $|V(G)|$. For $k = 2$, the conclusion can be checked easily. Assume $k \geq 3$.

Note that v_3 has exactly $|V| - 3$ neighbours. If $f(v_3) > |V| - 3$, then G is on-line f -choosable if (and only if) $G - v_3$ is on-line f -choosable. It follows from Theorem 2.2 that $G - v_3$ is on-line f -choosable. So we may assume that $f(v_3) \leq |V| - 3$, which implies $f(v_1) + f(v_2) \geq |V|$ and $f(v_1) \geq 3$ (as $f(v_2) \leq f(v_3) \leq |V| - 3$).

Given a subset U of $V(G)$, we shall find an independent set I of G contained in U so that $G' = G - I$ is on-line f' -choosable with $f' = f - \delta_U$. Note that G' is again a complete multi-partite graph with cardinality at most 2 for each partite set, except possibly one partite set of size 3. Let k'_1, k'_2 be integers such that $G' = K_{1 \star k'_1, 2 \star k'_2}$ or $G' = K_{1 \star k'_1, 2 \star (k'_2 - 1), 3}$. Let A'_i be the corresponding partite sets of G' . We shall show that G', f' satisfy the condition of Lemma 4.1 or Theorem 2.2, and hence by induction hypothesis or by Theorem 2.2, (G', f') is feasible. The proof is basically the same as the proof of Theorem 2.2, and we shall omit some details.

If U contains A_k , let $I = A_k$. By Theorem 2.2, (G', f') is feasible. So we assume $|U \cap A_k| \leq 2$. If U contains A_j for some $k_1 + 1 \leq j \leq k - 1$, let $I = A_j$. It follows from induction hypothesis that (G', f') is feasible. Hence, in the following we assume $|U \cap A_i| \leq 1$ for $1 \leq i \leq k - 1$.

If for some $k_1 + 1 \leq j \leq k - 1$, $A_j = \{x, y\}$, $U \cap A_j = \{x\}$ and $f(x) = k_2$, then let $I = \{x\}$. If, in addition, $|U \cap A_k| \leq 1$, then by a similar argument used in the proof of Theorem 2.2 (Sub-case 2.1) one can verify that by the induction hypothesis, (G', f') is feasible. Assume $|A_k \cap U| = 2$. We relabel the vertices in A_k by $A_k = \{v'_1, v'_2, v'_3\} = \{v_1, v_2, v_3\}$ where $f'(v'_1) \leq f'(v'_2) \leq f'(v'_3)$. Note that $f(v_1) + f(v_2) \geq |V|$ by the second paragraph of this proof and $f(v_1) + f(v_2) + f(v_3) \geq 2|V| - 3$ by assumption. Therefore

$$f'(v'_1) + f'(v'_2) \geq f(v_1) + f(v_2) - 2 \geq |V| - 2 = |V'| - 1, \text{ and}$$

$$f'(v'_1) + f'(v'_2) + f'(v'_3) = f(v_1) + f(v_2) + f(v_3) - 2 \geq 2|V| - 5 = 2|V'| - 3.$$

By induction hypothesis, (G', f') is feasible.

Therefore we assume, for all $1 \leq i \leq k - 1$, $|U \cap A_i| \leq 1$, and $f(v) \geq k_2 + 1$ for $v \in U \cap A_i$. Let $I = U \cap A_j$ where j is the smallest index such that $A_j \cap U \neq \emptyset$. If $1 \leq j \leq k - 1$, the same argument as above shows that (G', f') is feasible.

Assume $j = k$. If $|U \cap A_k| = 1$, it follows from Theorem 2.2 that (G', f') is feasible. Assume $U \cap A_k = \{x, y\}$ for some $x, y \in \{v_1, v_2, v_3\}$. Then $G' = G \setminus \{x, y\}$ and $f'(v) = f(v)$ for all $v \in V(G) \setminus \{x, y\}$. Denote $\{z\} = A_k \setminus \{x, y\}$. Define a function on G' by $g(v) = f'(v)$ for all $v \neq z$, and $g(z) = 1$. Then (G', g) is feasible if (and only if) $(G' \setminus \{z\}, g')$ is, where $g'(v) = f'(v) - 1$ for all $v \neq z$. Since $f'(v) = f(v)$, by Theorem 2.2, $(G' \setminus \{z\}, g')$ is feasible. Hence, (G', f') is feasible, as $g \leq f'$. $\square$

Corollary 4.2. *For $k \geq 2$, the configuration $(kk)^{k-1}||x_1x_2x_3$ is feasible for $x_1, x_2, x_3 \geq k$ and $x_1 + x_2 + x_3 \geq 4k - 2$. In particular, $(kk)^{k-1}||kk(2k - 2)$ is feasible.*

Proof. We prove the result by induction on k . It is clear that the result holds for $k = 2$. Assume $k \geq 3$. We denote by $A_1, A_2, \dots, A_k$ the partite sets of G where $A_k = \{v_1, v_2, v_3\}$ and $f(v_i) = x_i$.

Assume U is a subset of $V(G)$. We shall find an independent set $I \subseteq U$ so that $(G - I, f - \delta_U)$ is feasible. If U contains a whole partite set, then let I be that partite set, by induction hypothesis or by Theorem 2.2, $(G - I, f - \delta_U)$ is feasible.

Assume U does not contain a whole partite set. If U contains two vertices of A_k , then let $I = U \cap A_k$. By Theorem 2.2, $(G - I, f - \delta_U)$ is feasible.

Therefore we assume that U contains at most one vertex from each partite set. Let $I = U \cap A_j$ where j is the smallest such that $U \cap A_j \neq \emptyset$. If $1 \leq j \leq k - 1$, by Lemma 4.1, $(G - I, f - \delta_U)$ is feasible. If $j = k$, then the result follows by Theorem 2.2. $\square$

Theorem 4.3. *For $k \geq 2$, $(kk)^{k-1}||kk(2k - 2)$ is a minimal feasible configuration.*

Proof. By Corollary 4.2, $(kk)^{k-1}||kk(2k - 2)$ is feasible. By Lemma 3.1, the configuration $(kk)^{k-1}||k(k - 1)$ is infeasible, which implies that both $(k - 1)k||((kk)^{k-2}||kk(2k - 2))$ and $(kk)^{k-1}||k(k - 1)(2k - 2)$ are infeasible. To prove the theorem, it remains to show that $(kk)^{k-1}||kk(2k - 3)$ is infeasible. For this purpose, we shall present a winning strategy for Player A in the on-line list colouring game.

Let Player A's first move be: $(\bar{k}k)^{k-1}||\bar{k}k(2k - 3)$. Then Player B has two choices which lead to the following configurations:

$$((k - 1)k)^{k-1}||k(2k - 3) \quad \text{and} \quad k||((k - 1)k)^{k-2}||k(k - 1)(2k - 3).$$

For two configurations X, Y , we write $X \rightarrow Y$ if Player A has a sequence of moves, starting from X , so that for each of Player A's move, Player B has only one possible move (any other choice leads to a losing configuration for him), and after these moves, we arrive at Y .

Claim 4.4. *Assume a, b, q are positive integers, and $A_1, A_2, \dots, A_q$ are sets of positive integers. For an integer s , let $A_i - s = \{x - s : x \in A_i\}$. Then*

$$(ab)^a||A_1||A_2||\dots||A_q \rightarrow (b - a + 1)||b - a + 2||\dots||b||A_1 - a||A_2 - a||\dots||A_q - a.$$

Proof. Player A's first choice is $(\bar{a}b)^a||\bar{A}_1||\bar{A}_2||\dots||\bar{A}_q$. In order to avoid a complete graph on a vertices in which each vertex has only $a - 1$ permissible colours, Player B must colour a vertex that leads the following configuration: $b||((a - 1)b)^{a-1}||A_1 - 1||A_2 - 1||\dots||A_q - 1$. By induction hypothesis, i.e., apply the Claim to the case $a' = a - 1, b' = b, q' = q + 1, A'_i = A_i - 1$ and $A'_{q+1} = \{b\}$, we obtain the required configuration. $\square$

By Claim 4.4, $((k-1)k)^{k-1}||k(2k-3) \rightarrow 2||3||4|| \dots ||k||1(k-2) \rightarrow 1||2|| \dots ||(k-1)||k(2k-3)$. The last configuration is infeasible, as each vertex of a clique of size k has at most $k-1$ permissible colours.

It remains to consider the configuration $k||((k-1)k)^{k-2}||k(2k-3)$. The following claim shows that this configuration is infeasible.

Claim 4.5. *For $k \geq 2$ and for any $0 \leq j \leq k-2$, the configuration*

$$(k-j)||k-j+1|| \dots ||k||((k-j-1)k)^{k-j-2}||k(2k-3-j)$$

is infeasible.

Proof. We prove the result by induction on k . For $k = j+2$, we show the configuration $2||3||4|| \dots ||k||2(k-1)(k-1)$ is infeasible by induction on k . If $k = 2$, the configuration $2||211$ is infeasible. For $k \geq 3$, Player A has the first move: $\overline{2}||\overline{3}||\overline{4}|| \dots ||\overline{k}||2(k-1)(k-1)$. In order to avoid a K_k in which each vertex has at most $k-1$ permissible colours, Player B has only one choice, leading to $2||3||4|| \dots ||k-1||2(k-2)(k-2)$, which is infeasible by the induction hypothesis.

Assume $k \geq j+3$. Player A's choice is

$$\overline{(k-j)}||\overline{(k-j+1)}|| \dots ||\overline{k}||((k-j-1)k)^{k-j-2}||k(2k-3-j).$$

Player B has two choices, which result in the following:

$$(k-j-1)||k-j|| \dots ||k||((k-j-2)k)^{k-j-3}||k(2k-4-j), \text{ or} \\ (k-j)|| \dots ||k-1||((k-j-2)k)^{k-j-2}||k(2k-4-j).$$

For the former configuration, Player A chooses

$$\overline{(k-j-1)}||\overline{(k-j)}|| \dots ||\overline{k}||((k-j-2)k)^{k-j-3}||k(2k-4-j).$$

In order to avoid a copy of K_k in which each vertex has at most $k-1$ permissible colours, Player B has only one choice, leading to

$$(k-j-1)||k-j|| \dots ||k-1||((k-j-2)(k-1))^{k-j-3}||k(2k-5-j),$$

which is infeasible by the induction hypothesis.

For the latter configuration, by Claim 4.4,

$$\begin{aligned} & (k-j)|| \dots ||k-1||((k-j-2)k)^{k-j-2}||k(2k-4-j) \\ \rightarrow & 2||3|| \dots ||(j+1)||k||1(k-2) \\ \rightarrow & 1||2|| \dots ||j||k||1(k-2) \\ \rightarrow & \dots \dots \dots \\ \rightarrow & 1||3||4|| \dots ||k-j||2(k-j-1) \\ \rightarrow & 2||3||4|| \dots ||k-j-1||1(k-j-2) \\ \rightarrow & 1||2||3|| \dots ||k-j-2||k||1(k-j-2), \end{aligned}$$

which is infeasible. □

This finishes the proof of Theorem 4.3. □

Remark Recently, the on-line version of Ohba’s conjecture is verified in [5] for graphs of independence number at most 3.

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