On the number of F-matchings in a tree

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Abstract

We prove that for any integers $k, m > 0$ and any tree F with at least one edge, there exists a tree whose number of F-matchings is congruent to k modulo m as well as an analogous result for induced F-matchings. This answers a question of Alon, Haber and Krivelevich (The number of F-matchings in almost every tree is a zero residue, *Electron. J. Combin.* **18** (2011), #P30).

1 Introduction

An *independent set* in a graph is a set of vertices which does not induce any edges. Denote by $i(G)$ the number of independent sets in a graph G. The parameter was introduced by Prodinger and Tichy [11] who called it the *Fibonacci number* of G and proved that among trees of the same order, it is maximized by the star and minimized by the path. It has also been studied in mathematical chemistry where it is known as the *Merrifield-Simmons index* [9] which empirically relates physical properties of a compound with its chemical graphs. More generally, the whole profile of independent sets can be encoded as the *independent set polynomial* [4] which has connection to statistical physics, see for example [13].

In certain applications [6], it is of interest to know if an integer can be realized as the index of a chemical graph. To be precise, given an integer k, the inverse question asks whether there exists a graph G in a given family such that $i(G) = k$. In this aspect, Linek [8] proved that a bipartite graph can have any number of independent sets. Moreover, he conjectured that any sufficiently large integer is the number of independent sets in some tree. Although there has been extensive study of $i(G)$ from an extremal perspective (for example [7, 16, 14]), it appears that some new idea is required for the conjecture. We wrote a computer program to calculate $i(T)$ for trees up to order 27 and found that up to $i(P_{28}) = 832040$, there are 823 integers with no corresponding tree with the largest being 88013.

An interesting paper of Wagner [15] considered the parity of the parameter $i(G)$. Let $\mathcal{U}_n$ denote the space of unlabelled trees of order n and $m > 0$ be an integer. It was shown via singularity analysis that asymptotically almost every (a.a.e.) tree T sampled uniformly at random (u.a.r.) from $\mathcal{U}_n$ has $i(T)$ a multiple of m. A crucial fact is that there is a rooted tree (T, t) with $i(T)$ and $i(T - t)$ a multiple of m to begin with. Such a tree was constructed with the Fibonacci sequence and led to the conjecture [15] that any residue class is invertible as $i(T)$. The conjecture was resolved by Law [5] where it was shown that for any integers $k, m > 0$, there exists a tree T such that $i(T) \equiv k \pmod{m}$.

Searching for an intuitive explanation of Wagner's result, Alon, Haber and Krivelevich [1] considered F-matchings which generalize both independent sets and matchings. Let F, G be graphs. An *F-matching* is a subgraph of G consisting of vertex-disjoint copies of F; an *induced F-matching* is such a subgraph which does not induce any edges in G outside the copies of F. Since both versions are multiplicative over components, if there is a component whose number of F-matchings is a multiple of m, then the residue of the whole graph is necessarily 0. Developing this property, a *nullifying tree* was constructed in [1, Lemma 8] such that if contained in T, then the number of F-matchings in T is a zero residue. Then it was proved a.a.e. tree sampled u.a.r. from the space of labelled trees $\mathcal{T}_n$ of order n contains the nullifying tree, from which it follows that the number of F-matchings in a.a.e. tree is a zero residue. A tree whose number of F-matchings is a multiple of m is required in the proofs and was constructed using a linear recursion.

The authors of [1] raised the question: Given $k, m > 0$ and a tree F, is there a tree T whose number of F-matchings is congruent to $k \pmod{m}$? The analogous question for induced F-matchings was also raised. While the use of linear recursion seems a natural approach, the divisibility properties may be hard to analyze in general. We answer the two questions positively in a more general setting by constructing trees with no two disjoint copies of F. Indeed, for an integer $r \geq 1$, let an *(F, r)-matching* be a possibly empty collection of copies of F in G such that for any distinct copies F_1, F_2, $\min\{\mathrm{dist}_G(x, y) : x \in F_1, y \in F_2\} \geq r$ where dist_G denotes the graph distance in G. Thus an F-matching is an $(F, 1)$-matching and an induced F-matching is an $(F, 2)$-matching. Let $\mathcal{S}_r(F, G)$ be the set of (F, r)-matchings and $s_r(F, G) = |\mathcal{S}_r(F, G)|$.

Theorem 1. *For integers $k, m, r \geq 1$ and any tree F, there exist infinitely many trees T such that $s_r(F, T) \equiv k \pmod{m}$, except $\{s_1(K_1, T) \pmod{m}$: T a tree$\} = \mathbb{Z}_m$ iff $m \leq 2$.*

We prove Theorem 1 and some natural extensions in Section 2. In Section 3 we generalize the results in [15] and [1] on the asymptotic behaviour of $s_r(F, T)$. While zero is the dominating residue, we show that the other residue classes still grow exponentially in Section 4. The proof constructs finite trees that can control the residue in some limited sense.

It is interesting that the results given for F with diameter at least 3 are easier to prove. In fact for Lemma 9, we are unable to construct trees for $F = P_1, P_2$ or P_3 and general r. Perhaps copies of these trees occur rather too easily.

2 Inverse problems

In this section, we prove Theorem 1, i.e. with the exception of $s_1(K_1, T)$, we can represent any residue modulo an integer m as $s_r(F, T)$ for some tree T.

We first introduce some notation. Suppose F is a tree with at least two edges. Let $d = \operatorname{diam}(F) > 1$ be the diameter of F, i.e. the number of edges of a longest path in F. Take a path $u_0u_1 \cdots u_d$ of length d in F. Denote $F - u_d$ by D (for deficit). Also, let λ be the number of leaves adjacent to u_{d-1}.

Given integer $m > 0$, let $M = \Delta(F)!m$ where $\Delta(F)$ is the maximum degree of F. Then for any $l = 1, 2, \cdots, \Delta(F)$, the binomial coefficient $\binom{M}{l}$ is divisible by m. This will allow us to control the formation of copies of F.

Define a rooted tree $(W, w) = W(a)$ by identifying a copies of F and $M - a$ copies of D at the root $w = u_0$.

Since m is fixed, we usually omit '(mod m)' after '$\equiv$'. Recall that a *branch* at v is a maximal subtree of T such that v is a leaf.

Proof of Theorem 1. Let $d = \operatorname{diam}(F)$. Suppose $F = K_1$. If $r = 1$, then $s_1(K_1, T) = 2^{|T|}$ and $\{s_1(K_1, T) \pmod{m} : T \text{ tree } \} = \mathbb{Z}_m$ iff $m \le 2$. The case for $r = 2$ has been proved in [5]. For $r > 2$, since the leaves of $K_{1,n}$ are at distance 2 from each other, at most one vertex can be chosen in any (K_1, r)-matching. Thus, $s_r(K_1, K_{1,n}) = n + 2$. By varying n over $1, 2, \cdots, m$, we are done.

For other F, we construct various trees which do not contain disjoint copies of F.

If $d = 1$, then $F = K_2$. As $s_r(K_2, K_{1,n}) = n + 1$, the result holds.

Suppose $d > 1$, consider $(W, w) = W(a)$ where $a = k - 1$. There are three possibilities for a copy F_0 of F in W.

1. F_0 does not contain w. Then F_0 is contained in a component with strictly fewer vertices than F which is impossible. This case does not occur.

2. F_0 contains exactly one edge incident with w. Then F_0 lies in a branch of W at w. Clearly F_0 cannot be embedded into D, so this branch is isomorphic to F. There are a such copies of F_0.

3. F_0 contains at least two edges incident with w. Then F_0 does not contain any leaf adjacent to any copy of u_{d-1} for otherwise there is a path on more than d edges in F_0. Thus the number of such copies f depends only on M but not a. In fact, since the M branches of W within distance $d - 1$ of w are isomorphic, f can be written as a sum of terms each of which divisible by $\binom{M}{l}$ for some $1 < l \le \Delta(F)$. Indeed, we first choose $l > 1$ branches from the M at w, then we arrange these branches to form copies of F. By the choice of M, $f \equiv 0$.

As there are no disjoint copies of F in W, we have $s_r(F, T) \equiv 1 + a \equiv k$. □

The above proof motivates a further definition. We say that a rooted tree (T, t) is *symmetric with respect to* F if the isomorphism type of each branch within distance $d - 1$

of t occurs a multiple of $M = \Delta(F)!m$ times and that any copy of F in T contains t. Then as we have just seen, the number of copies F in T using at least two edges incident with t is divisible by m. With this notion, it is straightforward to generalize Theorem 1 to families of trees without pairwise embeddings.

Corollary 2. *Given integer $r \geq 1$ and trees $F_1, F_2, \cdots, F_l$ such that each F_i is not embeddable into F_j for any $j \neq i$, then for any integers $k_1, k_2, \cdots, k_l$, there exist infinitely many trees T such that $s_r(F_i, T) \equiv k_i$ (mod m).*

Proof. Let $d_i = \mathrm{diam}(F_i)$. Assume that $d_1 \leq d_2 \leq \cdots \leq d_l$. For each F_i, construct $W_i = W_i(a_i)$ by identifying a_i copies of F_i and $M - a_i$ copies of D_i where $M = (\max_{1 \leq i \leq l} \Delta(F_i))!m$. Identify the root of each W_i to form T. The nonembeddability condition implies that there are no disjoint copies of F_i in T.

As no copy of F_i contains a leaf u_{d_j} in F_j for $j > i$, by symmetry, $s_r(F_i, T)$ is independent of $a_{i+1}, a_{i+2}, \cdots, a_l$. Therefore, we can choose $a_1 = k_1 - 1, a_2, \cdots, a_l$ sequentially such that $s_r(F_i, T) \equiv k_i$ for all i. □

Theorem 1 answers the questions of Alon, Haber and Krivelevich [1] on the number of F-matchings and on induced F-matchings. Note also that by restricting ourselves to trees with no disjoint copies of F in the proof, we avoid multiplication in $\mathbb{Z}_m$ (and hence number theoretic difficulties). However it seems unavoidable if we want to control s_1, s_2 simultaneously. The idea of the following construction is to restrain the formation of a copy of F by supplying a power of m branches that we do not want F to be built on.

Theorem 3. *For integers $j, k > 2$, $m > 0$ and a tree F with at least one edge, there exist infinitely many trees T with $s_1(F, T) \equiv j, s_2(F, T) \equiv k$ (mod m).*

Proof. Let $d = \mathrm{diam}(F)$. Suppose $F = K_2$. For integers $a_1, a_2, \cdots, a_k > 0$, let T be a tree obtained by joining a vertex v by an edge to the centre of a copy of K_{1,a_i} for each i (see Figure 1). Then $s_1(K_2, T) = \prod_{i=1}^{k}(1 + a_i) + \sum_{i=1}^{k} \prod_{l \neq i}(1 + a_l)$ and $s_2(K_2, T) = \prod_{i=1}^{k}(1 + a_i) + k$. Choose a_i's such that $a_1 \equiv -1, a_2 \equiv j - 1, a_i \equiv 0$ for $i = 2, 3, \cdots, k$. Then $s_1(K_2, T) \equiv j, s_2(K_2, T) \equiv k$.

If $d = 2$, then $F = K_{1,n}$ for some $n > 1$. Let $M = n!m$. For any $p_1, \cdots p_M, q_1, \cdots, q_M > n$, define $Q(p_i, q_i)$ by identifying the centre of K_{1,q_i} with a leaf of p_i copies of $K_{1,n}$. Let $x_i = s_1(F, Q(p_i, q_i)) = 1 + \binom{p_i+q_i}{n} + p_i$. Define T by joining V to the centre of each $Q(p_i, q_i)$ (see Figure 1). Then we have

$$\begin{aligned} s_1(F,T) &= \prod_{i=1}^{M} x_i + \sum_{i=1}^{M} \binom{p_i + q_i}{n-1} \prod_{l \neq i} x_l + \sum_{A \subset [M], |A| = M-n} \prod_{i \in A} x_i, \\ s_2(F,T) &= \prod_{i=1}^{M} x_i + \sum_{i=1}^{M} \binom{p_i + q_i}{n-1} + \binom{M}{n}. \end{aligned}$$

Let $p_i + q_i \equiv n - 1$ for $1 \leq i \leq k$ and $p_i + q_i$ some multiple of m such that $\binom{p_i+q_i}{n-1} \equiv 0$ for

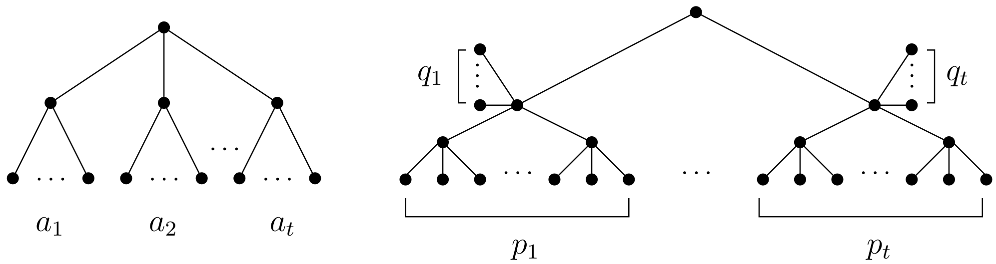

Figure 1: T for $F = K_2, K_{1,4}$ respectively in the proof of Theorem 3

$i > k$; and $x_i \equiv 1$, for $1 \leq i \leq M-n-1, x_{M-n} \equiv j, x_j \equiv 0$ for $j > M-n$. This is feasible because after we have chosen a value for $p_i + q_i$, we can adjust that of x_i by changing p_i. Then $s_1(F,T) \equiv j, s_2(F,T) \equiv k$.

Suppose $d > 2$. We construct a tree T whose constituents are X, Y and Z described as follows. For integers $a, b > 0$, let $(X, x) = W(a)$ and $(Y, y) = W(b)$.

To construct Z, we consider a poset. Let Z' be obtained by identifying $z = u_{d-1}$ of $c > \Delta(F)$ copies of F. Then any copy F_0 of F in Z' contains z. Define a *fragment* to be a branch at z of a copy F_0 in Z'. Note that the set $\mathcal{G}$ of fragments depends on F but not c. Define a partial order on $\mathcal{G}$ as follows: for any $G_i, G_l \in \mathcal{G}$, $G_i \leq G_l$ iff G_i can be embedded into G_l as a rooted subtree. Fix a linear extension $(G_1, G_2, \cdots, G_g)$ of $\mathcal{G}$ where $g = |\mathcal{G}|$. Thus $G_1 = K_2$ and G_g is F with all the leaves adjacent to u_{d-1} deleted.

Let $Z_g = Z'$. For $1 < i < g$, form Z_i by attaching copies of G_i to Z_{i+1} such that the number of copies of G_i is a multiple of M. Note that some copies may come from G_l for $l > i$ but not from $l < i$. Define Z by attaching leaves to z in Z_2 such that $\deg_Z(z) = Cm + \lambda$ for some integer C. Identify x with z and add the edge xy to form T (see Figure 2). Consider a copy F_0 in T.

1. F_0 lies completely in X or Y. The numbers of such copies are respectively congruent to a and b (mod m).

2. F_0 contains xy and vertices in $X - x$ or $Y - y$. Such a copy does not contain leaves adjacent to u_{d-1} within X or Y. By symmetry, the contribution is 0 (mod m).

3. F_0 contains vertices in both $X - x$ and $Z - z$ but not xy. Again, by symmetry, the contribution is again 0 (mod m).

4. F_0 lies completely in Z. Thus F_0 is formed by choosing fragments in $\mathcal{G}$. If F_0 uses any fragment other than G_1 or G_g, as there are a multiple of M such fragments, the contribution is 0 (mod m). Otherwise F_0 contains exactly one copy G_g so that we need to choose λ leaves at z. Since there are $Cm+\lambda-1$ available edges (one is used by G_g) incident with z in Z and $\binom{Cm+\lambda-1}{\lambda} \equiv 0$, the contribution is also 0 (mod m).

5. F_0 contains xy with the rest in Z. By the same reasoning as 4, we may assume F_0 chooses a copy of G_g. To complete F_0, we need to choose only $\lambda - 1$ edges incident

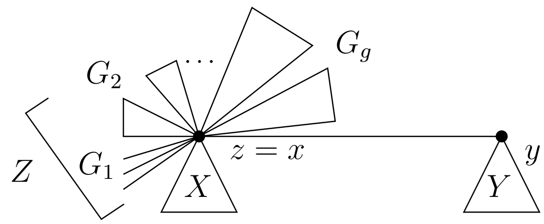

Figure 2: T for F with diameter $d > 2$ in the proof of Theorem 3

with z because xy accounts for one leaf already. Since $\binom{Cm+\lambda-1}{\lambda-1} \equiv 1$ and there are c copies of G_g to choose from, the number of copies is congruent to $c \pmod{m}$.

Since any copy of F contains either x or y, $s_1(F,T) \equiv (1+a)(1+b)+c$ and $s_2(F,T) \equiv 1+a+b+c$. By choosing $a \equiv -1, b \equiv k-j, c \equiv j$, we have $s_1(F,T) \equiv j, s_2(F,T) \equiv k$. $\square$

We conjecture that the theorem is true for general r.

Conjecture 4. *Given integers $r, m \geq 1$ and a tree F with at least one edge, for any $k_1, \cdots, k_r \in \mathbb{Z}_m$, there exists a tree T such that $s_i(F,T) \equiv k_i \pmod{m}$ for all $1 \leq i \leq r$.*

3 Asymptotic behaviour of divisibility properties

We first generalize the results in Wagner [15] and Alon, Haber and Krivelevich [1] to general r and a polynomial version. The proof is similar to [1] and consists of two parts as well. Given trees F, T, recall that $\mathcal{S}_r(F,T)$ denotes the set of (F,r)-matchings in T, define the (F,r)-*matching polynomial* as

$$S_r(F,T,z) = \sum_{S \in \mathcal{S}_r(F,T)} z^{|S|}$$

where $|S|$ denotes the number of copies of F in T. Thus $S_2(K_1,T,z)$ is the *independent set polynomial* and $S_1(K_2,T,z)$ is the *matching polynomial*.

Recall that a *limb* at v consists of a number of branches at v. For a subset $X \subset T$, let $\Gamma_{r-1}(X)$ denote the set of vertices within distance $r-1$ from X in T. The following is analogous to Lemma 8 in [1]; we include a proof for completeness.

Lemma 5. *Given trees F, H and integer $r > 0$, there exists a rooted tree L such that if a tree T contains L as a limb, then $S_r(F,H,z)$ divides $S_r(F,T,z)$.*

Proof. Let L be defined by joining a vertex v to a copy of H by an edge and $\Delta(F)+1$ copies by a path of length r. Suppose T contains L as a limb. Consider (F,r)-matchings not containing v. Since there is a component of H in $T-v$, these matchings contribute a multiple of $S_r(F,H,z)$. On the other hand, deleting a copy F_0 of F containing v along with $\Gamma_{r-1}(F_0)$ always leaves a forest with an H component. As there are finitely many possibilities for $L - F_0 - \Gamma_{r-1}(F_0)$, the contribution is also a multiple of $S_r(F,H,z)$. Thus, the sum $S_r(F,T,z)$ is divisible by $S_r(F,H,z)$. $\square$

The probabilistic part is a classical result of Schwenk's,

Theorem 6 ([12])**.** *Given any rooted tree L, a.a.e. tree in $\mathcal{U}_n$ contains L as a limb.*

The following analogue of Theorem 6 for labelled trees was proved in [1] by concentration inequalities. For the sake of variation, we sketch a proof using generating functions (for background, we refer to [3]). A rooted tree L occurs as a limb in a rooted tree T if L consists of branches of descendants at some vertex and $T \neq L$. For a power series $A(z) = \sum_{n\geq 0} a_n z^n$, let $\operatorname{rad}(A)$ be its radius of convergence.

Theorem 7. *Given rooted tree L, a.a.e. labelled tree in $\mathcal{T}_n$ contains L as a limb.*

Proof. Let $l = |L| > 1$ and ρ be the size of the automorphism group of L as a rooted tree. Let $T(z,u)$ be the exponential generating function (EGF) of rooted trees with u counting the occurrence of L as a limb. Then $T(z,u) = z\exp\left(T(z,u) + (u-1)\frac{z^l}{\rho}\right)$. Let $G(z) = T(z,0)$ be the EGF for rooted trees forbidding L as a limb. Then

$$G(z) = z\exp\left(G(z) - \frac{z^l}{\rho}\right).$$

Let $g(z) = z\exp\left(-\frac{z^l}{\rho}\right)$. As $T(g(z),1)$ is a function analytic around $z=0$ and satisfies the equation for G with $T(g(0),1) = G(0)$, it is the local solution for $G(z)$. It is well known (by Cayley's formula) that $\operatorname{rad}(T(z,1)) = 1/e$. Hence

$$\operatorname{rad}(G) = \operatorname{rad}(T(g(z),1)) = \sup_{z>0}\{g(x) < 1/e, \forall 0 < x < z\} > 1/e.$$

This implies that there are exponentially fewer rooted trees without L as a limb than the total number of rooted trees. As we can root a tree in n ways, the probability of a random unrooted tree without L as a limb is at most n times the corresponding probability for a random rooted tree sampled u.a.r., which tends to 0 exponentially fast. □

By Theorem 1, for any $m > 0$, there exists a tree H such that m divides $s_r(F,H) = S_r(F,H,1)$. Thus the following generalizes the results in [15] and [1].

Corollary 8. *Given trees F, H and integer $r > 0$, a.a.e. tree $T \in \mathcal{U}_n$ has the property that $S_r(F,H,z)$ divides $S_r(F,T,z)$. The same is true when T is sampled from $\mathcal{T}_n$.*

Proof. It follows directly from Lemma 5, Theorems 6 and 7. □

4 Trees in other residue classes

Returning to the modulo m case, it was shown in [15] that the number of trees with an odd number of independent sets is small compared to those with an even number but it still grows exponentially fast. While the use of 'nullifying trees' in [1] gives a quick proof that the number of F-matchings is a.a.s. a zero residue, it is not clear whether the other residue classes still grow exponentially. We show that this is the case for trees F with diameter at least 3.

Lemma 9. *Given a tree F with at least 3 edges and integers $r, k, m > 0$, then for any sufficiently large n, there exists a tree T of order n such that $s_r(F,T) \equiv k$ (mod m).*

Proof. Given F, for each k, let $W_k = W(k-1)$ be constructed as in Section 2 such that $s_r(F, W_k) \equiv k$. Note that $|W_k|$ may vary over k. Let $n > \max_k |W_k|$ be an integer.

If $\mathrm{diam}(F) = 2$, then $F = K_{1,n}$ for $n > 2$ since $|F| \geq 4$. By adding a path on $n - |W_k|$ vertices to any leaf of W_k, we obtain a tree T of order n such that $s_r(F,T) \equiv k$.

If $\mathrm{diam}(F) > 2$, then F is not a star. Then attaching $n - |W_k|$ leaves to the root does not change the residue of $s_r(F, W_k)$. Indeed, this follows from that any copy of F contains the root and by symmetry the number of copies of F containing an added leave is a multiple of m. Hence we can span all residues of m with trees of order n. $\square$

In the following, we restrict ourselves to $r = 1$ and F with diameter at least 3. Let $\mathcal{A}(k)$ denote the family of rooted trees T such that $s_1(F,T) \equiv k$ and $\mathcal{A}_n(k)$ the set of such trees of order n. Let $a_n(k) = |\mathcal{A}_n(k)|$ and $A(k,z) = \sum_{n\geq 0} a_n(k) z^n$. We will prove that $\mathrm{rad}(A(k,z)) < 1$ which implies that $a_n(k)$ grows exponentially fast. Let $U(z)$ be the ordinary generating function for the class of unlabelled rooted trees. It is well known (see for example [3, Chapter 1] for a derivation) that

$$U(z) = z \exp\left(\sum_{i\geq 1} \frac{U(z^i)}{i}\right). \tag{1}$$

For any rooted tree (T,t), let $s_1(F, T^+) = s_1(F,T) - s_1(F, T-t)$.

Theorem 10. *For a tree F of diameter at least 3 and integers $m \geq 1$, there exists $C > 1$ such that for any k and sufficiently large integer n, the number of trees T of order n such that $s_1(F,T) \equiv k$ (mod m) is at least C^n.*

Proof. We first reduce the general case to $k = 1$. Let $H \in \mathcal{A}(1)$ and $W_k \in \mathcal{A}(k)$ such that $|W_k| = n - M|H|$ as constructed as in Lemma 9 and M as defined in Section 2. Join the root of W_k to that of M copies of H by an edge to form T. It is easy to check that $T \in \mathcal{A}(k)$. In other words, given a tree $H \in \mathcal{A}(1)$, we can construct a tree of order $M|H| + |W_k|$ in $\mathcal{A}(k)$. Hence, $a_n(k) \geq a_{(n-|W_k|)/M}(1)$. Since M and $|W_k|$ are constant, to prove the result, it suffices to show that the number of choices for H grows exponentially fast.

To this end, we construct a rooted tree (V, v) such that if v is joined to any collection of trees in $\mathcal{A}(1)$, the resulting tree remains in $\mathcal{A}(1)$. We require (V, v) to satisfy:

1. $s_1(F, V - v) \equiv 0$;
2. $s_1(F, V^+) \equiv 1$;
3. V is symmetric.

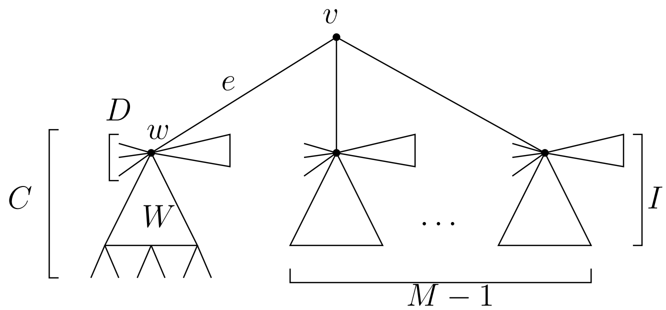

Figure 3: Construction of V in the proof of Theorem 10

Let $(W, w) = W(m-1) \in \mathcal{A}(0)$ and D be constructed as in Section 2. Identify u_{d-1} of D with w to form a tree (C, w) (for complete). Delete all vertices at distance d from w (all such vertices lie in W) to form (I, w) (for incomplete). Define V by adding a new vertex v joined to the root of a copy of C and $M-1$ copies of I (see Figure 3).

Clearly V is symmetric. Note that $s_1(F, C) \equiv (m-1)+1+\binom{\lambda-1}{\lambda} \equiv 0$, since $W \in \mathcal{A}(0)$ and D does not contain a copy of F. As $V-v$ contains C as a component, $s_1(F, V-v) \equiv 0$.

Consider a copy containing v. By symmetry of V, we may assume the copy also contains the edge e joining v and C but no other edges on v. Moreover, as W is symmetric as well, we may further assume no edge in W is involved. Thus there is exactly one such copy, namely, $D \cup e$. Hence, $s_1(F, V^+) \equiv 1$.

We claim that if we join any collection of trees in $\mathcal{A}(1)$ to $v \in V$ by an edge, the resulting tree remains in $\mathcal{A}(1)$. Indeed, for $H_1, \cdots, H_p \in \mathcal{A}(1)$, define H by adding a vertex h joined to each H_i. Furthermore, identify h and v to form a tree T. Then considering the edges incident with v in an (F, r)-matching, we have

$$s_1(F, T) \equiv s_1(F, V) \prod_{i=1}^{p} s_1(F, H_i) + s_1(F, V-v) s_1(F, H^+) \equiv 1.$$

To repeat, the tree defined by joining a collection of trees in $\mathcal{A}(1)$ to the large 'root' V is in $\mathcal{A}(1)$. Hence, the coefficients of $A(1, z)$ dominate those of $B(z)$ where

$$B(z) = z^{|V|} \exp\left(\sum_{i \geq 1} \frac{B(z^i)}{i}\right).$$

Suppose $U(z)$ satisfies (1), then the solution around $z = 0$ is $B(z) = U(z^{|V|})$. Thus

$$\mathrm{rad}(A(1, z)) \leq \mathrm{rad}(B) = \mathrm{rad}(U)^{1/|V|}.$$

It was proved by Pólya [10] that $\mathrm{rad}(U) < 1$. Hence, $\mathrm{rad}(A(1, z)) < 1$. It follows that $a_n(1)$ grows exponentially fast and so does $a_n(k)$ for any k. $\square$

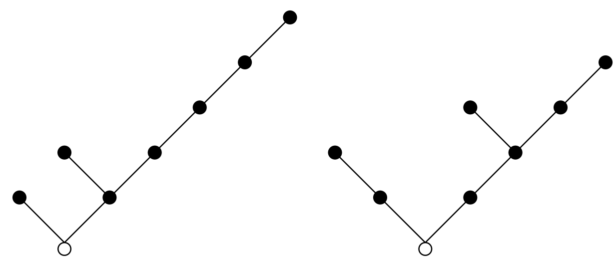

Figure 4: Pairs of rooted trees with identical local independent set polynomial data

Further questions

The original motivation of Schwenk [12] was to show that a.a.e. tree is cospectral to some other tree. He constructed a pair of rooted trees R_1, R_2 such that if a tree has R_1 as a limb, swapping it with R_2 preserves the characteristic polynomial. By Theorem 6 ([12]), a.a.e. tree contains R_1 as a limb, hence the cospectral result. Given the classical result that the characteristic polynomial of a tree carries the same information as $S_1(K_2, \cdot, z)$ (see for example [2]), it is natural to conjecture that swapping is possible for general (F, r)-matchings and hence a.a.e. tree is not distinguished by $S_r(F, \cdot, z)$.

Conjecture 11. *For any tree F and integer $r > 0$, there exist rooted trees R_1, R_2 with the property: if a tree T_1 has R_1 as a limb and T_2 is obtained from T_1 by swapping R_1 with R_2, then $S_r(F, T_1, z) = S_r(F, T_2, z)$.*

Thus we are asking for a pair of rooted trees with identical local versions of $S_r(F, R, z)$. For example, the rooted trees (R_i, r_i), $i = 1, 2$ in Figure 4 satisfy $S_2(K_1, R_1, z) = S_2(K_1, R_2, z)$ and $S_2(K_1, R_1 - r_1, z) = S_2(K_1, R_2 - r_2, z)$. Switching them does not alter $S_2(K_1, T, z)$ for any T containing R_1 as a limb. It follows that a.a.e. trees are not distinguished by the independent set polynomial.

We end with a question on the growth rate of the maximum number of (F, r)-matchings among trees. In [11] it was proved that the star and the path maximize the number of independent sets and matchings respectively for trees of order n. It would be interesting to study the extremal trees for other pairs (F, r). Here we give a start in this direction. Let $\mu(F, r, n) = \max\{s_r(F, T) : T \text{ is a tree of order } n\}$.

Proposition 12. *For any $r \geq 1$ and tree F, $\lambda(F, r) = \lim_{n\to\infty} \mu(F, r, n)^{1/n}$ exists and $1 < \lambda(F, r) \leq 2$.*

Proof. Let $\mu_n = \mu(F, r, n)$. We may assume $f = |F| > 2$. Since joining two trees by a path of length r preserves (F, r)-matchings, it follows that $\mu_{n+k+r-1} \geq \mu_n \mu_k$ for all integers $n, k \geq 0$. By an adaptation of Fekete's Lemma, $\lambda(F, r) = \lim_{n\to\infty} \mu_n^{1/n}$ exists.

By taking disjoint copies of F and joining them to a vertex with a path of length r, we obtain a tree T with $s_r(F, T) > c^n$ for some constant $c > 1$. Hence $\lambda > 1$. Moreover, as an (F, r)-matching corresponds to an edge subset, $\mu_n \leq 2^{n-1}$. Hence, $\lambda \leq 2$. $\square$

We remark that while $1 \notin \Lambda$, it can be shown that $\lim_{p\to\infty} \lambda(K_1, 2p) = 1$. Thus in the spirit of inverse problems, we ask the following.

Question 13. *Let $\Lambda = \{\lambda(F, r) : F \text{ tree}, r \geq 1\}$. Is Λ dense in $[1, 2]$?*

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