Rainbow–free 3–colorings of Abelian Groups

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Abstract

A 3–coloring of the elements of an abelian group is said to be rainbow–free if there is no 3–term arithmetic progression with its members having pairwise distinct colors. We give a structural characterization of rainbow–free colorings of abelian groups. This characterization proves a conjecture of Jungić et al. on the size of the smallest chromatic class of a rainbow–free 3–coloring of cyclic groups.

1 Introduction

A *k–coloring* of a set X is a map $c : X \to [k]$ where $[k] = \{1, 2, \ldots, k\}$. A subset $Y \subset X$ is *rainbow* under c if the coloring assigns pairwise distinct colors to the elements of Y. The study of the existence of rainbow structures falls into the anti–Ramsey theory initiated by Erdős, Simonovits and Sós [3]. Arithmetic versions of this theory were initiated by Jungić, Licht, Mahdian, Nešetřil and Radoičić [5] where the authors study the existence of rainbow arithmetic progressions in colorings of cyclic groups and of intervals of integers.

In the case of colorings of the integers, it was shown by Axenovich and Fon der Flaas [1] that every 3–coloring of the integer interval $[1, n]$ such that each color class has cardinality at least $(n+4)/6$ contains a rainbow 3–term aritmetic progression, thus proving a conjecture stated in [5]. However, the same authors show that, no matter how large is the smaller color class, there are examples of k–colorings with no rainbow arithmetic progressions of length $k \geq 5$. Conlon, Jungić and Radoičić [2] gave a construction of

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equinumerous 4–colorings with no rainbow 4–term arithmetic progressions. The canonical version of van der Waerden's theorem by Erdős and Graham states that every coloring of the integers (with possibly infinitely many colors) contains either a monochromatic or a rainbow k–term arithmetic progression for each k. By the celebrated theorem of Szemerédi, if one of the color classes has positive density then one finds a monochromatic arithmetic progression of length k. In contrast, Jungić et al. [5] show that there are colorings with all color classes with positive density with no rainbow 3–term arithmetic progressions.

In the above mentioned reference of Jungić et al. [5] the authors also study the existence of rainbow 3–term arithmetic progressions in 3–colorings of finite cyclic groups. The authors characterize all integers n such that every 3–coloring of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ contains a rainbow 3-term arithmetic progression.

Theorem 1 (Jungić et al. [5]). *For every integer n, there is a rainbow–free 3–coloring of $\mathbb{Z}/n\mathbb{Z}$ with non–empty color classes, if and only if n does not satisfy any of the following conditions:*

(a) n is a power of 2.

(b) n is a prime and the multiplicative order of 2 is $n-1$.

(b) n is a prime, the multiplicative order of 2 is $(n-1)/2$ and $(n-1)/2$ is odd.

The above result motivates the following notation. We denote by $\mathcal{P}_0$ the set of primes p for which 2 has either multiplicative order $p-1$, or multiplicative order $(p-1)/2$ with $(p-1)/2$ odd. Let $\mathcal{P}_1$ be the set of remaining primes.

Following [5] we let $m(n)$ denote the largest integer m for which there is a rainbow–free 3–coloring of $\mathbb{Z}/n\mathbb{Z}$ such that the cardinality of the smallest color class is m. Among other results, the authors in [5] proved that if the smaller class in a 3–coloring of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ has size greater than $n/6$, then there exists a rainbow $AP(3)$. For n divisible by 6 this condition is tight, but for other values of n it is possible to obtain better bounds.

Theorem 2 (Jungić et al. [5]). *Let n be not a power of 2, q be the smallest prime factor of n, and r be the smallest odd prime factor of n, then:*

$$\left\lfloor \frac{n}{2r} \right\rfloor \leq m(n) \leq min(\frac{n}{6}, \frac{n}{q})$$

Motivated by the above result, Jungić, Nešetřil and Radoičić [6] mention that "Computing the exact value of $m(n)$ remains a challenge" and they formulate the following conjecture.

Conjecture 1 (Jungić, Nešetřil and Radoičić [6]). *Let n be an integer which is not a power of 2. Let p denote the smallest odd prime factor of n in $\mathcal{P}_0$ and let q be the smallest odd prime factor of n in $\mathcal{P}_1$. Then the largest cardinality of the smallest color class in a rainbow–free 3–coloring of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ satisfies*

$$m(n) = \left\lfloor \frac{n}{\min\{2p, q\}} \right\rfloor .$$

In this paper we give a structural characterization of 3–colorings of finite abelian groups of odd order with no rainbow 3–term arithmetic progressions. This characterization provides a proof of Conjecture 1 for general abelian groups of odd order. Combined with the study of the even case, we complete the proof of Conjecture 1 for *cyclic* groups of even order as well.

We remark that Conjecture 1 is only stated for cyclic groups. Actually the conjecture does not hold for general abelian groups of *even* order as illustrated by the following counterexample.

Let $G := H \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ where $|H|$ is not a power of 2. Consider the following 3–coloring of G: let the subgroup H be colored by A, color one of the three remaining H–cosets of G by B and the two cosets left by C. This coloring has no rainbow 3–term arithmetic progressions, since such a progression (a triple (x, y, z) such that $x-2y+z = 0$) must have two of its elements in the same H–coset. However the smaller color class has cardinality $|H| = |G|/4$ which can be arbitrarily larger than $\min\{\frac{|G|}{2p}, \frac{|G|}{q}\}$ according to the choice of H.

Our main result, Theorem 3 below, identifies the three possible kinds of rainbow-free colorings of an abelian group G of odd order which can be described as follows. There is a proper subgroup H of G such that, either the coloring of G is obtained by lifting a rainbow–free coloring with a color class of size one from the quotient group G/H, or there is one coset of H which is bichromatic and $G \setminus H$ is monochromatic, or a combination of the two possibilities above.

In order to state the main result, let us introduce some notation. Let G be a finite abelian group. Recall that the Minkowski sum of two nonempty subsets $X, Y \subset G$ is defined as

$$X + Y = \{x + y : x \in X, y \in Y\}.$$

The *period* (or stabilizer) of a subset $S \subseteq G$, denoted by $P(S)$, is the subgroup of G defined by:

$$P(S) = \{g \in G : S + g = S\}$$

Thus S is a union of cosets of $P(S)$ and it is a maximal subgroup with this property. We say that a set S is *H–periodic*, where H is a subgroup of G, if $S + H = S$, and S is *periodic* if $P(S)$ is a nontrivial subgroup of G (i.e. $P(S) \neq \{0\}$). If $P(S) = \{0\}$ we say that S is *aperiodic*.

For a subset $X \subset G$ we denote by $2 \cdot X = \{2x : x \in X\}$ and $-X = \{-x : x \in X\}$.

A 3–term arithmetic progression is an ordered triple (x, y, z) with $x, y, z \in G$ satisfying the equation $x + y = 2z$. In the present setting we say that a 3–coloring of the elements of G is *rainbow–free* if there are no rainbow 3–term arithmetic progressions. Note that the property of being rainbow–free is invariant by translations: c is a rainbow–free coloring of G if and only if, for each fixed $g \in G$, the coloring $c'(x) := c(x + g)$ is also rainbow–free. We will often use this remark without explicit reference. We identify a 3–coloring with the partition of G into its three color classes which we denote by $\{A, B, C\}$.

Theorem 3. *Let G be a finite abelian group of odd order n and let c be a 3–coloring of G with non–empty color classes A, B, C. Then c is rainbow–free if and only if, up to* *translation, there is a proper subgroup $H < G$ and a color class, say A, such that the following three conditions hold:*

(i) *$A \subseteq H$, and the 3–coloring induced in H is rainbow–free,*

(ii) *both $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ are H–periodic sets, and*

(iii) *$\widetilde{B} = -\widetilde{B} = 2 \cdot \widetilde{B}$ and $\widetilde{C} = -\widetilde{C} = 2 \cdot \widetilde{C}$.*

The 'if' part of Theorem 3 can be easily checked and its proof is detailed in Section 6, Proposition 2. For the "only if" part we use results by Kneser [8], Kemperman [7] and Grynkiewicz [9] which give the structure of sets with small sumset in an abelian group.

The paper is organized as follows. In Section 2 we recall some results in Additive Combinatorics and prove a simple Lemma which will be used in the remainder of the paper. Section 3 proves the main result for groups with odd prime order. Section 4 deals with colorings in which one color class is either small or an arithmetic progression. In Section 5 we consider colorings with structured color classes. The proof of Theorem 3 and the proof of Conjecture 1 for abelian groups of odd order is contained in Section 6. In Section 7 we give a structural characterization for the case of cyclic groups of even order (Theorem 7) which is analogous to Theorem 3. With this version of the characterization one can complete the proof of Conjecture 1 for cyclic groups of even order.

2 Some tools from Additive Combinatorics

We shall use the following well–known result of Kneser (see e.g. [12, Theorem 5.5])

Theorem 4 (Kneser)**.** *Let (A, B) be a pair of finite non-empty subsets of an abelian group G. Then, letting $H := P(A + B)$, we have:*

$$|A + B| \geq |A + H| + |B + H| - |H|.$$

Moreover, if $|A + B| \leq |A| + |B| - 1$ then we have equality.

It follows from Kneser's Theorem that, if $|A + B| \leq |A| + |B| - 1$, then either $A + B$ is periodic or $|A + B| = |A| + |B| - 1$. We shall use this remark in the following sections.

The structure of pairs of sets (X, Y) in an abelian group G verifying $|X + Y| = |X| + |Y| - 1$ is given by the Kemperman Structure Theorem (KST). We shall only use the following simplified version of Kemperman's theorem [7, Theorem 5.1] (see also the formulations of Lev [10, Theorem 2], Grynkiewicz [9, Theorem KST] and Hamidoune [4]). Let $H \neq \{0\}$ be a subgroup of G. A set $S \subset G$ is said to be H*–quasiperiodic* if it admits a decomposition $S = S_0 \cup S_1$, where each of S_0 and S_1 can be empty, S_1 is a maximal H–periodic subset of S and S_0 is (properly) contained in a single coset of H. Note that every set $S \subset G$ is quasiperiodic with $S_1 = \emptyset$ and $H = G$.

Theorem 5 (Kemperman [7]). *Let A and B be nonempty subsets of an abelian group G verifying*

$$|A+B| = |A| + |B| - 1 \leq |G| - 2.$$

If $A+B$ is aperiodic then one of the following holds:

(i) $\min\{|A|, |B|\} = 1$.

(ii) *Both A and B are arithmetic progressions with the same common difference.*

(iii) *Both A and B are H–quasiperiodic for some nontrivial proper subgroup $H < G$.*

We shall also use the following extension of KST, recently obtained by Grynkiewicz [9], which describes the structure of pairs of sets (X, Y) in an abelian group G verifying $|X+Y| = |X| + |Y|$. Again we only need a simplified version of the full result.

Theorem 6 (Grynkiewicz [9]). *Let A and B be nonempty subsets of an abelian group G of odd order n verifying*

$$|A+B| = |A| + |B| \leq |G| - 3.$$

If $A+B$ is aperiodic then one of the following holds:

(i) $\min\{|A|, |B|\} = 2$ *or* $|A| = |B| = 3$.

(ii) *Both A and B are H–quasiperiodic for some nontrivial proper subgroup $H < G$.*

(iii) *There are $a, b \in G$ such that $|A' + B'| = |A'| + |B'| - 1$ where $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$.*

It is well–known that, if A and B are subsets of a group G such that $|A| + |B| > |G|$ then $A+B = G$. The following lemma characterizes the structure of sets $A, B \subset G$ with $|A| + |B| = |G|$ and $A + B \neq G$. We include here a short proof for the benefit of the reader.

Lemma 1. *Let A, B be subsets of a finite abelian group G.*

(i) *If $|A| + |B| > |G|$ then $A+B = G$.*

(ii) *If $|A| + |B| = |G|$ then either $A+B = G$ or there is a subgroup H and $a \in G$ such that both A and B are H–periodic and*

$$A + B = G \setminus (a + H).$$

Proof. We only prove (ii). If $|A+B| = |G| - 1$ then the statement holds with $H = \{0\}$. Suppose that $|A+B| \leq |G| - 2$ and let $H = P(A+B)$ be the period of $A+B$. By Kneser's Theorem

$$|G| > |A+B| = |A+H| + |B+H| - |H| \geq |A| + |B| - |H| = |G| - |H|.$$

Since $A+B$ is H–periodic, equality holds in the second inequality. It follows that $A+B = G \setminus (a+H)$ for some $a \in G$ and that $A+H = A$ and $B+H = B$. □

One of the applications of Lemma 1 is the following result which will be often used. Let H be a proper subgroup of G. As usual we denote by G/H the quotient group. If X is a subset of G we write X/H for the image of X in G/H by the natural projection $\pi : G \to G/H$. We say that a triple (X,Y,Z) of H–cosets is in arithmetic progression if $(X/H) + (Y/H) = 2 \cdot (Z/H)$. For X an H–coset and U a color class of a coloring we write $X_U := X \cap U$.

Lemma 2 (The 3-cosets Lemma)**.** *Let $\{A, B, C\}$ be a rainbow–free 3–coloring of an abelian group G with odd order n. Let $H < G$ be a subgroup of G and let (X, Y, Z) be a triple of H–cosets in arithmetic progression.*

If each of X_A, Y_B and Z_C is non–empty, then

$$\max\{|X_A| + |Y_B|, |X_A| + |Z_C|, |Z_C| + |Y_B|\} \leq |H|. \tag{1}$$

In particular, none of the three cosets can be monochromatic.

Moreover, if equality holds then there is a proper subgroup $K < H$ such that two of the sets X_A, Y_B, Z_C are K–periodic (the two involved in the equality holding) and the third one is contained in a single coset of K.

Proof. Since the coloring is rainbow–free and the three cosets are in arithmetic progression we have

$$X_A + Y_B \subseteq (2 \cdot Z) \setminus (2 \cdot Z_C).$$

Hence $|X_A + Y_B| < |H|$ which, by Lemma 1 (i), implies $|X_A| + |Y_B| \leq |H|$. Similarly $X_A - (2 \cdot Z_C) \subsetneq -Y$ and $Y_B - (2 \cdot Z_C) \subsetneq -X$ imply $|X_A| + |Z_C| \leq |H|$ and $|Y_B| + |Z_C| \leq |H|$ respectively. This proves the first part of the statement.

Suppose that $|X_A| + |Y_B| = |H|$. By Lemma 1 (ii) there is a subgroup $K < H$ such that both X_A and Y_B are K–periodic and $(2 \cdot Z) \setminus (X_A + Y_B)$ consists of a single K–coset, which contains $2 \cdot Z_C$. A symmetric argument applies if $|X_A| + |Z_C| = |H|$ or $|Y_B| + |Z_C| = |H|$. □

3 The prime case

The proof of Theorem 3 (in Section 6) is by induction on the number of (not necessarily distinct) primes dividing $n = |G|$. In this section we prove Theorem 3 for groups of odd prime order, which is a direct consequence of Theorem 1 and Theorem 2.

Proposition 1. *Let G be a group of prime order p and let c be a 3–coloring of G with nonempty color classes A, B, C. Then c is rainbow–free if and only if, $p \in \mathcal{P}_1$ and, up to translation, there is a color class, say A, such that:*

(i) $A = \{0\}$,

(ii) $2 \cdot B = B = -B$ *and* $2 \cdot C = C = -C$.

Proof. Suppose first that c is rainbow–free. Since $G \simeq \mathbb{Z}/p\mathbb{Z}$, it follows from Theorem 2 that $m(p) \leq 1$ and, from Theorem 1, that if $p \in \mathcal{P}_0$ then there are no rainbow–free 3–coloring of $\mathbb{Z}/p\mathbb{Z}$ with non–empty color classes. That is, if c is a rainbow–free 3–coloring with nonempty color classes, then necessarily $p \in \mathcal{P}_1$ and there is a color class, say A, such that $|A| = 1$. We can assume that $A = \{0\}$ satisfying (i).

To prove (ii) note that, for every $x \in B$, since G is a cyclic group of prime order, then both $-x, 2x \in \{B, C\}$. Hence we must have $-x, 2x \in B$, otherwise we get a rainbow 3–term arithmetic progression of the form $(-x, 0, x)$ or $(0, x, 2x)$. Thus $2 \cdot B = B = -B$ and similarly $2 \cdot C = C = -C$.

Reciprocally, if the coloring satisfies (i) and (ii), then any 3–term arithmetic progression containing 0 has its remaining members in the same color class, thus c is rainbow–free. $\square$

4 Small color classes and color classes in arithmetic progresion

Throughout this section G denotes an abelian group of odd order n and c is a rainbow–free 3–coloring of G with non–empty color classes $\{A, B, C\}$. The coloring is said to be *H–regular* if, up to translation, it satisfies conditions (i), (ii) and (iii) of Theorem 3 for the subgroup $H < G$.

We begin with the case when there is a color class with just one element.

Lemma 3. *If $|A| = 1$ then the coloring is H–regular with $H = \{0\}$.*

Proof. We may assume that $A = \{0\}$. By choosing $H = \{0\}$, parts (i) and (ii) of Theorem 3 are satisfied.

To prove (iii) note that, since G is a group of odd order, for every $x \in B$ then both $-x, 2x \in \{B, C\}$. Hence we must have $-x, 2x \in B$, otherwise we get a rainbow 3–term arithmetic progression of the form $(-x, 0, x)$ or $(0, x, 2x)$. Thus $2 \cdot B = B = -B$ and similarly $2 \cdot C = C = -C$. $\square$

Lemma 3 provides the description given in Theorem 3 when one of the colors has cardinality one. The 3-cosets lemma (Lemma 2) can be used to show the analogous result if one of the color classes is contained in a single coset.

Lemma 4. *If A is contained in a single coset of a proper subgroup $H' < G$ and $|A| > 1$, then the coloring is H–regular for some proper subgroup $H < G$.*

Proof. We may assume that $0 \in A$. Let H be the the minimal proper subgroup of G which contains A. Suppose that $Y \neq H$ is an H–coset which intersects the two remaining color classes B and C. Let Z be a third coset such that $(X = H, Y, Z)$ are in arithmetic progression. Since A does not meet Y and Z, we have

$$|Y_B| + |Y_C| + |Z_B| + |Z_C| = 2|H|.$$

It follows from Lemma 2 that $|Y_B| + |Z_C| = |H|$ and $|Y_C| + |Z_B| = |H|$. Moreover, there is a subgroup $K \leq H$ such that $X_A = A$ is contained in a single coset of K and each of Y_B, Y_C, Z_B, Z_C is K–periodic. By the minimality of H we have $K = H$ contradicting the existence of the bichromatic coset Y. Thus parts (i) and (ii) of Theorem 3 are satisfied.

Now consider the 3–coloring of G/H with color classes $A' = \{0\}$, $B' = \widetilde{B}/H$ and $C' = \widetilde{C}/H$ where $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$. Note that, since the original coloring $\{A, B, C\}$ of G is rainbow–free, then so it is $\{A', B', C'\}$ in G/H.

If one of B' or C' is an empty set, then part (iii) of Theorem 3 is clearly satisfied.

If both B' and C' are nonempty sets, then $\{A', B', C'\}$ is a rainbow–free coloring of G/H with nonempty color classes and $|A'| = 1$. By Lemma 3, it follows that $2 \cdot B' = B' = -B'$ and $2 \cdot C' = C' = -C'$. Thus part (iii) is also satisfied for the coloring $\{A, B, C\}$. $\square$

We next handle the cases when there is a class with two elements or there are two classes with three elements.

Lemma 5. *If $|A| = 2$ then the coloring is H–regular for some $H < G$.*

Proof. By Lemma 4 we only have to show that one color is contained in a single coset of a proper subgroup $H < G$.

We may assume that $A = \{0, a\}$. Let us show that a generates a proper subgroup H of G. Suppose on the contrary that the cyclic group generated by a is the whole group $G = \langle a \rangle \cong \mathbb{Z}/n\mathbb{Z}$.

Since $\{-a, a, 3a\}$ can not be rainbow, we have $c(-a) = c(3a)$. Since $\{-3a, 0, 3a\}$ can not be rainbow we have $c(-3a) = c(3a)$. By iterating this argument, we have

$$c(-a) = c(3a) = c(-3a) = c(5a) = c(-5a) = \cdots = c((n-2)a) = c(-(n-2)a),$$

so that the color class of $-a$ has $n - 2$ elements. But then the third one is empty, a contradiction.

Hence $A \subset \langle a \rangle = H < G$ and, by Lemma 4, parts (i), (ii) and (iii) of Theorem 3 are satisfied. $\square$

Lemma 6. *If $|A| = |B| = 3$ and $|A + B| = 6$, then the coloring is H–regular for some $H < G$.*

Proof. Let $A = \{0, a, a'\}$. It can be shown (see e.g. [9]) that only two possibilities occur if $|A + B| = |A| + |B| = 6$: either B is a translate of A or one of the sets, say A, is an arithmetic progression, and the second one, B, is an arithmetic progression of length four with the same difference and with one element removed.

Suppose that $B = A + x = \{x, x + a, x + a'\}$ for some $x \in G$. Since $\{-x, 0, x\}$ can not be rainbow we have $-x \in A \cup B$. If $-x \in A$ then $0 \in A \cap B$, a contradiction. Thus $-x \in B$ and $a = -2x$. Since $\{0, x, 2x\}$ can not be rainbow we have $2x \in A \cup B$. If $2x \in B$ then $x \in A \cap B$ (since $A = B - x$). Thus $2x = a$ which implies $B = \{-x, x, 3x\}$ and $A = \{-2x, 0, 2x\}$ and both sets are arithmetic progressions with the same difference contradicting $|A + B| = |A| + |B|$.

Suppose now that A is an arithmetic progression with difference d. If A generates a proper subgroup H of G then the result follows from Lemma 4. Hence we may assume that $A = \{0, 1, 2\}$ and G is the cyclic group of order n. Moreover $B = \{x, x+2, x+3\}$ for some $x \in \mathbb{Z}/n\mathbb{Z} \setminus \{0, 1, 2, n-3, n-2, n-1\}$. If $\{0, x, -x\}$ is not rainbow, since $-x \in \{1, 2\}$ can not hold and n is odd, we must have $-x = x+3$. But then $x = (n-3)/2$ and $\{0, (n-1)/2, (n+1)/2\}$ is rainbow. This contradiction completes the proof. □

We next consider the case when two color classes are *almost* progressions. An *almost*–progression is an arithmetic progression with one point removed. Observe that, with this definition, the class of almost progressions contains the class of all arithmetic progressions except the ones whose length equals the order of the cyclic group generated by the difference.

Lemma 7. *Assume that $4 \leq |A| \leq |B| \leq |C|$. If A and B are almost–progressions with the same difference d, then the coloring is H–regular for some $H < G$.*

Proof. If d generates a proper subgroup H of G then A is contained in a single coset of H and the result follows by Lemma 4.

Thus we may assume that d generates the full group, so that G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and we may assume $d = 1$ (since the property of being rainbow–free is invariant by dilations). We will show that in this case c contains a rainbow 3–term arithmetic progression.

Let b be the minimum circular distance from elements in A to elements in B.

If $b = 1$ we may assume that $n-1 \in A$ and $0 \in B$. Since $\max\{|A|, |B|\} \leq (n-5)/2$ we have $(n-1)/2 \in C$ giving the rainbow $\{0, (n-1)/2, n-1\}$.

Suppose now that $b > 1$. Since $|A| \geq 4$ we may assume that $\{n-1, 0\} \subset A$ and $\{1, 2, \ldots, b'\} \subseteq C$ and $b'+1 \in B$ for some $b' \geq b$. If b' is odd then $\{0, (b'+1)/2, b'+1\}$ is rainbow and if b' is even then $\{n-1, (b'+2)/2, b'+1\}$ is rainbow. □

5 Periodic color classes

In this section we analyze the structure of the color classes when they are close to be periodic. The consideration of these cases arise from the discussion on the size of sumsets of the color classes in a ranbow–free 3–coloring and the KST and Grynkiewicz theorems.

Throughout the section we keep the notation of the previous one. Thus G denotes an abelian group of odd order n and c is a rainbow–free 3–coloring of G with non–empty color classes $\{A, B, C\}$. The coloring is H–regular if it satisfies conditions (i), (ii) and (iii) of Theorem 3 for some subgroup H.

Recall that, for a subset $X \subset G$ and a subgroup $H < G$, we denote by X/H the image of X by the natural projection $\pi : G \to G/H$.

The proof of Theorem 3 (in Section 6) is by induction on the number of (not necessarily distinct) primes dividing $n = |G|$, the initial step being proved in Section 3. Thus, in the remainder of this section we will assume that Theorem 3 holds for any group of order a proper divisor of $n = |G|$.

We start with the simplest case.

Lemma 8. *If the three color classes A, B, and C are K–periodic for some subgroup $K < G$, then the coloring is H–regular for some $H < G$.*

Proof. Consider the coloring $A' = A/K$, $B' = B/K$ and $C' = C/K$ of G/K. Note that, since $\{A, B, C\}$ is a rainbow–free 3–coloring with nonempty color classes, then so it is $\{A', B', C'\}$. Since G/K is a group of order a proper divisor of $n = |G|$, the coloring $\{A', B', C'\}$ is H'–regular for some $H' < G/K$. In particular, there is a color class, say A', such that $A' \subseteq H' < G/K$. Thus A is contained in a single coset of the proper subgroup $H' + K$ in G, and the statement follows from Lemma 4. □

We next consider the case where two of the color classes are quasiperiodic. Recall that a set $S \subset G$ is H*–quasiperiodic* if it admits a decomposition $S = S_0 \cup S_1$, where each of S_0 and S_1 can be empty, S_1 is a maximal H–periodic subset of S and S_0 is (properly) contained in a single coset of H.

Lemma 9. *If $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ are K–quasiperiodic decompositions of A and B with K a nontrivial proper subgroup of G, then the coloring is H–regular for some $H < G$.*

Proof. By Lemma 4 we may assume that none of the color classes is contained in a single coset of a proper subgroup of G. If two of the color classes are periodic then so is the third one and the result follows from Lemma 8. Therefore, up to renaming the color classes we may assume that each of the sets A_0, B_0, A_1 and B_1 are nonempty, and that $|C/K| > 1$. We also assume that $0 \in A_0$.

Let us show that $A_0/K = B_0/K$. Suppose the contrary and let Z be a K–coset such that $X = K, Y = B_0 + K$ and Z are in arithmetic progression (such a coset always exists since n is odd). Note that X intersects A and C, Y intersects B and C and that Z is monochromatic. This contradicts the 3–cosets Lemma (Lemma 2). Hence $A_0/K = B_0/K$.

Consider the 3–coloring c_K of G/K with color classes $\{A', B', C'\}$ where $A' = A/K$, $B' = B_1/K$ and $C' = G/K \setminus (A' \cup B')$. Note that $C' = (C \setminus K)/K$. Observe that c_K is a 3–coloring of G/K with non empty color classes. Moreover, c_K is rainbow–free, otherwise we have three K–cosets in G in arithmetic progression where at least two of them are monochromatic (since both B_1 and $C \setminus K$ are K–periodic) contradicting the 3–cosets Lemma (Lemma 2).

Since we assume that Theorem 3 holds in G/K, there is a proper subgroup $L < G$ containing K such that, up to translation, one of the three chromatic classes of c_K is contained in L/K and the remaining two are (L/K)–periodic outside L/K.

Suppose that $A' \subset L/K$. Then $A \subset L$ and the statement follows by Lemma 4.

Let us show now that C' can not be contained in a single coset of L/K in G/K. Suppose on the contrary that $C \backslash K$ is contained in a single L–coset X of G. Let Z be a L–coset in arithmetic progression with X and $Y = A_0 + L$. Since Y intersects the two colors, A and B, and Z is necessarily monochromatic with color A or B, we have $|Z| + |Y_A|, |Z| + |Y_B| > |L|$ contradicting Lemma 2.

Suppose now that B' is contained in a single coset of L/K in G/K. We may assume that B_1 is contained in a single L–coset $X \neq L$ in G, otherwise we are done by Lemma 4.

Consider the coloring c_L of G/L with color classes $\{A'', B'', C''\}$ where $A'' = (A_1 \setminus X)/H, B'' = B/L$, and $C'' = G/L \setminus (A'' \cup B'')$. Note that $|B''| = 2$ and $|C''| > 1$, otherwise $C \subset X$ and C' can not be contained in a single coset of L/K as shown in the paragraph above.

If $A'' = \emptyset$, consider Z an L–coset in arithmetic progression with L and X (exist since the order of G is odd). Note that L intersects A, X intersects B and Z is monochromatic of color C, a contradiction by Lemma 2.

If $|A''| > 1$, note that c_L is a 3–coloring of G/L with non empty color classes. Moreover, c_L is rainbow–free since otherwise we have three L–cosets in G in arithmetic progression where one of them is monochromatic (since C'' is L–periodic) a contradiction by Lemma 2. Since $|B''| = 2$, it follows from Lemma 5 that B'' is contained in a single coset of a proper subgroup $H/L < G/L$. Thus $B \subseteq H$ and the statement follows from Lemma 4. □

We next consider the case where $A + B$ is K–periodic for some subgroup K of G. Observe that, since c is rainbow–free, we also have $K \neq G$.

Lemma 10. *If $A + B$ is K–periodic for some proper nontrivial subgroup K of G, then the coloring is H–regular for some subgroup $H < G$.*

Proof. We show that, under the assumption of the Lemma, both sets A and B admit a K–quasiperiodic decomposition and thus the statement follows from Lemma 9.

By Kneser's theorem we have

$$|A/K + B/K| \geq |A/K| + |B/K| - 1. \tag{2}$$

Since $A + B \cap 2 \cdot C = \emptyset$ we have

$$(A + B)/2 \subset (G \setminus C) = A \cup B, \tag{3}$$

where $X/2$ denotes the image of $X \subset G$ by the inverse of the automorphism of G defined as $x \mapsto 2x$. This automorphism leaves all subgroups invariant so that $(A + B)/2$ is also K–periodic. Let

$$D = ((A \cup B) + K) \setminus (A + B)/2.$$

Note that the aperiodic parts of A and of B are contained in $D \cup (A \cap B)$. By (2) we have

$$\begin{aligned} |A/K| + |B/K| - |A/K \cap B/K| = |(A \cup B)/K| &= |(A + B)/K| + |D/K| \\ &\geq |A/K| + |B/K| - 1 + |D/K|, \end{aligned}$$

which implies

$$|D/K| + |A/K \cap B/K| \leq 1.$$

Hence each of A and B admits a K–quasiperiodic decomposition. □

Now we consider the case where two of the color classes are *almost* quasiperiodic. A set $X \subset G$ is *almost* H–quasiperiodic (resp. almost H–periodic) if there is $x \in G$ such that $X \cup \{x\}$ is H–quasiperiodic (resp. H–periodic).

Lemma 11. *If A and B are almost H–quasiperiodic for some proper nontrivial subgroup $H < G$, then the coloring is H'–regular for some proper subgroup $H' < G$.*

Proof. We say that a coset X of a subgroup $H < G$ is punctured if all but one of its elements are in the same color class $U \in \{A, B, C\}$. We then say that X is a punctured coset of color U.

Since A and B are almost H–quasiperiodic, they admit decompositions $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ where each of A_0 and B_0 are subsets of some H–coset and each of A_1 and B_1 are almost periodic so that each of them contains at most one punctured coset.

We may assume that at least one of A_1 or B_1 contains a punctured coset and that $0 < |A_0|, |B_0| < |H|$ since otherwise A and B are quasiperiodic and the result follows from Lemma 9. We may also assume that none of A, B and C are periodic since otherwise at least one of $A + B$ or $A + C$ is periodic and the result is implied by Lemma 10. Finally we may assume that $\min\{|A/H|, |B/H|, |C/H|\} > 1$ since otherwise the result is a consequence of Lemma 4.

We next show that the above assumptions lead to a contradiction, thus proving the Lemma. We consider two cases:

Case 1: $A_0+H \neq B_0+H$. Let Z be a coset in arithmetic progression with $X = A_0+H$ and $Y = B_0 + H$.

We may assume that one of X, Y, say X, intersects C, since otherwise X is the punctured coset of B_1 and Y is the punctured coset of A_1, which implies that C is periodic. In particular $X \cap B = \emptyset$. Moreover, whatever the colors present in Z, the conditions of Lemma 2 are satisfied and Z can not be a full coset. Since all H–cosets different from X and Y are either monochromatic or punctured, Z is a punctured coset. Moreover it can not be of color C since $Z \cap A_0 = Z \cap B_0 = \emptyset$.

Suppose that $|Z_A| = |Z \cap A| = |H| - 1$. Then, again by Lemma 2, $|Z_A| + |X_C| = |H|$, which implies $|X_C| = |X \cap C| = 1$ and $|Y_B| = |Y \cap B| = 1$. Thus both X and Z are punctured cosets of color A. Since A can not contain more than two partially filled cosets, Y is a punctured coset of color C. Finally, the other color in Z must also be C since Z is not the coset containing B_0.

Since $|B/H| > 1$ there is a coset $Y' \notin \{X, Y, Z\}$ which intersects B. Moreover, Y' is either a full coset or a punctured coset of B. Let Z' be a third coset in arithmetic progression with X and Y'. Whatever the colors present in Z', the conditions of Lemma 2 are satisfied, so that both Y' and Z' must be punctured cosets. Thus Z' must intersect C (there are no punctured cosets with colors A and B) and $|X_A| + |Y_B| > |H|$, contradicting Lemma 2.

Suppose now that $|Z_B| = |Z \cap B| = |H| - 1$. If $Y \cap A \neq \emptyset$ then Y is a punctured coset of color A and $|Y_A| + |Z_B| > |H|$ contradicting Lemma 2. Otherwise Y intersects C and application of Lemma 2 implies $|Y_C| = |X_A| = 1$. Thus both Y and Z are punctured cosets of B with second color C and X is a punctured coset of C with second color A, the same structure as in the case above with colors A and B exchanged, and we again obtain a contradiction with Lemma 2.

Case 2: $A_0 + H = B_0 + H$. We may assume that at least one of A_1 or B_1 contains a punctured coset which is not X, otherwise A and B are quasiperiodic and the results follows from Lemma 9. So let Y be a punctured coset of color A (observe that $Y_B = \emptyset$ since B_0 is contained in X, thus $|Y_C| = 1$). Let Z be a coset in arithmetic progression with X and Y.

We first prove that Z is not monochromatic. If Z is monochromatic of color B (resp. C or A) then $|Z_B|+|Y_C|>|H|$ (resp. $|Z_C|+|Y_A|>|H|$ or $|Z_A|+|Y_C|>|H|$) and we get a contradiction by Lemma 2 since X_A (resp. X_B) is not empty.

Thus Z must be a punctured coset of color B with $|Z_C|=1$. Since $|Y_A|+|Z_B|>|H|$ then $X_C=\emptyset$. Since $|Y_A|+|Z_C|=|H|$ then $|B_0|=1$, but also $|Y_C|+|Z_B|=|H|$ implies $|A_0|=1$ which is a contradiction. This completes the proof. □

6 Proof of Theorem 3

The next proposition proves the 'if' part of Theorem 3.

Proposition 2. *Let $\{A,B,C\}$ be a coloring of an abelian group G of odd order. If there is a proper subgroup H of G and a color class, say A, such that the three following conditions hold:*

(i) *$A\subseteq H$, and the 3–coloring induced in H is rainbow–free,*

(ii) *both $\widetilde{B}=B\setminus H$ and $\widetilde{C}=C\setminus H$ are H–periodic sets, and*

(iii) *$\widetilde{B}=-\widetilde{B}=2\widetilde{B}$ and $\widetilde{C}=-\widetilde{C}=2\widetilde{C}$.*

Then the 3-coloring is rainbow–free.

Proof. If $C=G\setminus H$ then $A+B$ is contained in H, and thus it is disjoint from $2\cdot C$. Moreover, each of $2\cdot A$ and $2\cdot B$ are contained in H and thus disjoint from $A+C=B+C=C$.

Suppose that $C\neq G\setminus H$. Since, by (i), a rainbow 3–term arithmetic progression in G can not be contained in H, it gives rise, by conditions (ii) and (iii), to a rainbow 3–term arithmetic progression in G/H with the coloring $\{A/H,\widetilde{B}/H,\widetilde{C}/H\}$. However, since every color class X in this last coloring verifies $X=-X=2\cdot X$ any 3–term arithmetic progression of G/H containing A/H has its remaining members in the same color class. Hence $\{A/H,\widetilde{B}/H,\widetilde{C}/H\}$ is rainbow–free and so it is c. □

It remains to prove that, if c is a rainbow–free coloring of an abelian group G of odd order, then the color classes verify conditions (i), (ii) and (iii) of Theorem 3 with some proper subgroup $H<G$. To prove this we use the results in sections 3, 4 and 5 together with the theorems of Kneser, Kemperman and Grynkiewicz.

Proposition 3. *Let $\{A,B,C\}$ be a rainbow–free coloring of an abelian group G of odd order.*

There is a proper subgroup H of G and a color class, say A, such that the three following conditions hold:

(i) *$A\subseteq H$, and the 3–coloring induced in H is rainbow–free,*

(ii) *both $\widetilde{B}=B\setminus H$ and $\widetilde{C}=C\setminus H$ are H–periodic sets, and*

(iii) $\widetilde{B} = -\widetilde{B} = 2\widetilde{B}$ *and* $\widetilde{C} = -\widetilde{C} = 2\widetilde{C}$.

Proof. The proof is by induction on the number of (not necessarily distinct) primes dividing $n = |G|$. If n is prime, the statement holds by Proposition 1. We assume that Theorem 3 holds for any group of order a proper divisor of $n = |G|$, so that we can use the results in Section 5.

We first note the following remark.

Remark 1. *For any pair of distinct color classes $X, Y \in \{A, B, C\}$ we have*

$$|X + Y| \leq |X| + |Y|.$$

Proof. Suppose on the contrary that $|X + Y| \geq |X| + |Y| + 1$ for some distinct color classes $X, Y \in \{A, B, C\}$. Since $n = |G|$ is odd, we have $|2 \cdot Z| = |Z|$ (where Z is the remaining color class). It follows from the condition $|X + Y| \geq |X| + |Y| + 1$ that $(X+Y) \cap (2 \cdot Z) \neq \emptyset$, which implies that there is a rainbow 3–term arithmetic progression. □

It follows from the above remark that

$$|A + B| \leq |A| + |B|. \tag{4}$$

By Lemma 10 we can assume that $A + B$ is aperiodic. Then it follows from Kneser's theorem that $|A + B| \geq |A| + |B| - 1$. According to (4) we have to consider two cases.

Case 1: $|A + B| = |A| + |B| - 1$. It follows from the simplified version of the KST, Theorem 5, that one of the following holds:

(i) $\min\{|A|, |B|\} = 1$. In this case the result follows by Lemma 3.

(ii) Both A and B are arithmetic progressions with the same common difference d. The result follows by Lemma 7.

(iii) Both A and B are H–quasiperiodic for some nontrivial proper subgroup $H < G$. The result follows by Lemma 9.

Case 2: $|A + B| = |A| + |B|$. Then the simplified version of the Theorem by Grynkiewicz, Theorem 6, implies that one of the following holds:

(i) $\min\{|A|, |B|\} = 2$ or $|A| = |B| = 3$. In this case the result follows by Lemmas 5 and 6 respectively.

(ii) Both A and B are H–quasiperiodic for some nontrivial proper subgroup $H < G$. The result follows by Lemma 11.

(iii) There are $a, b \in G$ such that $|A' + B'| = |A'| + |B'| - 1$ where $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$. According to Kemperman's Theorem 5, either $A' + B'$ is periodic, in which case $A + B$ is also periodic and the result follows by Lemma 10, or A', B' are both quasiperiodic, in which case A and B are almost periodic and we can apply Lemma 11, or A', B' are both arithmetic progressions and then A and B are almost arithmetic progressions, a case handled in Lemma 7.

This completes the proof of the Proposition and of Theorem 3. □

The description of rainbow–free 3–colorings of abelian groups of odd order can be used to prove Conjecture 1. Actually Conjecture 1 holds for general abelian groups of odd order as shown in the next Corollary.

Corollary 1. *Let G be an abelian group of odd order n. Let p denote the smallest prime factor of n in $\mathcal{P}_0$ and let q be the smallest prime factor of n in $\mathcal{P}_1$. If $\{A, B, C\}$ is a rainbow–free 3–coloring of G then*

$$\min\{|A|, |B|, |C|\} \leq \left\lfloor \frac{n}{\min\{2p, q\}} \right\rfloor. \tag{5}$$

Moreover, there are rainbow–free 3–colorings of G for which equality holds.

Proof. We first observe that (5) is equivalent to:

$$\min\{|A|, |B|, |C|\} \leq \max\left\{ \left\lfloor \frac{n}{2p} \right\rfloor, \frac{n}{q} \right\}. \tag{6}$$

Note that, since the smallest prime factor of n is either p or q, then the largest proper subgroup of G has size either $\frac{n}{p}$ or $\frac{n}{q}$.

By Theorem 3 (i), there is a proper subgroup $H < G$ and one color class, say A, contained in H. Hence, $|A| \leq |H|$.

If $|H| \leq \frac{n}{q}$ then (6) is satisfied.

Suppose that $|H| > \frac{n}{q}$. Then necessarily $p < q$ and the size of the largest proper subgroup of G is $\frac{n}{p}$. Hence $|H| \leq \frac{n}{p}$.

If $|H| < \frac{n}{p}$, since n is an odd number, then $|H| = \frac{n}{a}$ where $a > 2p$. Thus $|H| < \lfloor \frac{n}{2p} \rfloor$ and (6) is satisfied.

Suppose that $|H| = \frac{n}{p}$. Then G/H is a cyclic group of prime order $p \in \mathcal{P}_0$. By Theorem 3 (ii), each of the two sets $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ is a (possibly empty) union of H–cosets. Consider the 3–coloring of G/H with color classes $A' = \{0\}$, $B' = \widetilde{B}/H$ and $C' = \widetilde{C}/H$. Note that, if the original coloring A, B, C of G is rainbow–free, then the induced coloring A', B', C' of $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ is also rainbow–free. By Proposition 1, since $p \in \mathcal{P}_0$, one of the color classes B' or C' must be empty. Hence either B/H or C/H is an empty set which implies that $G \setminus H$ is monochromatic and thus H contains two colors. It follows that $\min\{|A|, |B|, |C|\} \leq \lfloor \frac{n}{2p} \rfloor$. This completes the first part of the Corollary.

Let us show now that there are rainbow–free 3–colorings of G for which equality holds in (5).

If $2p \leq q$ then choose a subgroup $H < G$ with cardinality $\frac{n}{p}$. Consider a partition $A \cup B = H$ where $|A| = \lfloor \frac{n}{2p} \rfloor$, and let $C = G \setminus H$. This coloring clearly satisfies parts (i), (ii) and (iii) of Theorem 3 and therefore, by Proposition 2, it is rainbow–free.

If $q < 2p$ then choose a subgroup $H < G$ with cardinality $\frac{n}{q}$. We define a coloring A, B, C of G as follows. Since $q \in \mathcal{P}_1$, by Theorem 1, there is a rainbow–free 3–coloring of $\mathbb{Z}/q\mathbb{Z}$ with nonempty color classes A', B' and C'. Let $\pi : G \to G/H \simeq \mathbb{Z}/q\mathbb{Z}$ denote the natural projection and define $A = \pi^{-1}(A')$, $B = \pi^{-1}(B')$ and $C = \pi^{-1}(C')$. By Proposition 1 this coloring satisfies parts (i), (ii) and (iii) of Theorem 3 and therefore, by Proposition 2, it is rainbow–free. $\square$

7 The even case

In this Section we shall prove Conjecture 1 for *cyclic* groups $\mathbb{Z}/n\mathbb{Z}$ of even order. We recall that, by Theorem 1, if $n = 2^m$ then there are no rainbow–free 3–colorings of $\mathbb{Z}/n\mathbb{Z}$. Therefore, throughout this Section, we assume that $n = 2^m l$ for some $m \geq 1$ and odd $l > 1$.

We start with a Lemma which will be useful later on.

Lemma 12. *Let $\{A, B, C\}$ be a rainbow–free 3–coloring of a cyclic group G. Suppose that there is a subgroup H such that one of the colors, say A, is contained in H, and each of $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ are H-periodic.*

There is a proper subgroup K of G containing H such that

$$B + C \supseteq G \setminus K$$

and each of $B \setminus K$ and $C \setminus K$ is K–periodic.

Proof. We consider two cases.

Case 1: $H = \{0\}$. We have $A = \{0\}$ and $|B| + |C| = |G| - 1$.

If $|B + C| = |B| + |C| = |G| - 1$ then we can choose $K = \{0\}$.

Suppose that $|B + C| = |B| + |C| - 1$ and let $\{0, x\} = G \setminus (B + C)$. Since the coloring is rainbow–free we have $-B = B$ and $-C = C$. Since $x \notin B + C$ we have $x - B \cap C = \emptyset$ and $x - C \cap B = \emptyset$. Hence

$$\{0, x\} + B = \{0, x\} - B \subset G \setminus C.$$

It follows that

$$|\{0, x\} + B| \leq |B| + 1. \tag{7}$$

Let $K = \langle x \rangle$ be the cyclic subgroup generated by x and let $B = B_1 \cup \cdots \cup B_t$ be a decomposition of B into maximal arithmetic progressions with difference x. By the maximality of each B_i we have

$$B_i + \{0, x\} = \min\{|K|, |B_i| + 1\}.$$

By (7), we see that all but at most one of the B_i's are cosets of K. Moreover, if there is one of the B_i's, say B_1, which is not a K–coset, then B_1 is an arithmetic progression of difference x properly contained in one K–coset.

Likewise $\{0, x\} + C \subset G \setminus B$ implies the analogous structure for C. Since none of B and C contains the whole subgroup K, the only coset where the proper arithmetic progressions can sit in is K itself. If both colors meet K then we have a rainbow–free 3–coloring of this cyclic group with all three colors arithmetic progressions. But we can not partition $K \setminus \{0\}$ into two arithmetic progressions B', C' with $B' = -B$ and $C' = -C$. Therefore only one of the two colors meets K. This shows that K is a proper subgroup of G. Moreover, $B \setminus K$ and $C \setminus K$ are K–periodic. By the definition of x, we also have $B + C \supset G \setminus K$.

Finally suppose that $|B+C| < |B|+|C|-1$. By Kneser's theorem there is a proper subgroup $K < G$ such that $B+C$ is K–periodic and $|B+C| = |B+K|+|C+K|-|K|$. Since $0 \notin B+C$ we have $B+C \subset G \setminus K$. Hence,

$$|G|-1 = |B|+|C| \leq |B+K|+|C+K| \leq |G|.$$

The last inequalities imply that one of the sets, say B, satisfies $B = B+K$ and the second one satisfies $|C| = |C+K|-1$. Thus B is K–periodic, $C+K = C \cup \{0\}$ and $C \setminus K$ is K–periodic or empty, and $B+C = G \setminus K$.

Case 2: $H \neq \{0\}$.

Suppose first that $\widetilde{C} = G \setminus H$, so that $\widetilde{B} = \emptyset$ and $B \subset H$. Then $B+C = C$ and the statement holds with $K = H$.

Suppose now that both $\widetilde{B}$ and $\widetilde{C}$ are nonempty. Then $\{A' = A/H, B' = \widetilde{B}/H, C' = \widetilde{C}/H\}$ is a rainbow–free 3–coloring of G/H. It follows from Case 1 that there is a proper subgroup K of G containing H such that $B' \setminus (K/H)$ and $C' \setminus (K/H)$ are K–periodic and $A'+B' \supseteq (G/H) \setminus (K/H)$. It follows that each of $B \setminus K$ and $C \setminus K$ are K–periodic (one of the two may be empty) and $B+C \supseteq G \setminus K$. □

Let $n = 2^m l$ with $l > 1$ odd and $m \geq 1$. Then the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ can be written as

$$G = L \oplus \mathbb{Z}/2^m\mathbb{Z}$$

where L has odd order l and $m \geq 1$.

As usual $\{A, B, C\}$ denotes a rainbow–free 3–coloring of G. Let $P_0 = 2 \cdot G$. Since the even factor of G is cyclic we have

$$P_0 \cong L \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}.$$

For each color $X \in \{A, B, C\}$ we write $X_0 = X \cap P_0$ and $X_1 = X \cap P_1$ where P_1 is the second coset of P_0 in G.

Lemma 13. *With the assumptions above, none of the two cosets of P_0 is monochromatic.*

Proof. Suppose the contrary and choose the minimal m for which there is a counterexample to the statement. We may assume that $P_1 = C_1$. Since $A+B \subset P_0 \setminus 2 \cdot C_1 = P_0 \setminus 2 \cdot P_1$ we have $m \geq 2$ (otherwise $2 \cdot P_1 = P_0$) and $A+B$ is contained in the proper subgroup $2 \cdot P_0$ of P_0. Thus $A \cup B$ is contained in one coset of $2 \cdot P_0$ and the second coset of this subgroup in P_0 must be colored only with C contradicting the minimality of m. □

We next give the structural result analogous to Theorem 3 for cyclic groups of even order.

Theorem 7. *Let $G = L \oplus \mathbb{Z}/2^m\mathbb{Z}$ where L has odd order and $m \geq 0$. There is a proper subgroup H' of L such that one of the colors, say A, is contained in one coset of $H = H' \oplus \mathbb{Z}/2^m\mathbb{Z}$ and each of $B \setminus H$ and $C \setminus H$ is H–periodic.*

Proof. By Lemma 13 we may assume that none of the two cosets of P_0 is monochromatic. The proof is by induction on m. By Theorem 3 the result follows for $m = 0$. Assume $m \geq 1$. We consider three cases.

Case 1: One of the two cosets of P_0 is bichromatic.

We may assume that $P_1 = B_1 \cup C_1$ and $0 \in A_0 = A$. Thus

$$|B_1| + |C_1| = |P_0| \text{ and } B_1 + C_1 \subset P_0.$$

Since c is rainbow–free we have $2 \cdot A \subset P_0 \setminus (B_1 + C_1)$. In particular, $B_1 + C_1 \neq P_0$. It follows from Lemma 1 (ii) that there is a proper subgroup H_1 of P_0 such that $2 \cdot A \subseteq H_1$, and that both B_1 and C_1 are H_1–periodic. Let

$$H_1 = H_1' \oplus \mathbb{Z}/2^{m_1}\mathbb{Z},$$

where $H_1' < L$ has odd order and $\mathbb{Z}/2^{m_1}\mathbb{Z}$ denotes the cyclic subgroup of $\mathbb{Z}/2^m\mathbb{Z}$ of order 2^{m_1} for some $m_1 \leq m$. We next consider two cases according to P_0 being bichromatic or trichromatic.

Case 1.1: P_0 is bichromatic. We may assume that $P_0 = A_0 \cup B_0$. Since $|A_0| + |B_0| = |P_0|$ and $2 \cdot C_1 \subset P_0 \setminus (A_0 + B_0)$, it follows from Lemma 1 (ii) that there is a proper subgroup H_0 of P_0 such that $2 \cdot C_1$ is contained in a single coset of H_0, and that both A_0 and B_0 are H_0–periodic. Let

$$H_0 = H_0' \oplus \mathbb{Z}/2^{m_0}\mathbb{Z},$$

with $H_0' < L$ of odd order.

Now C_1 being H_1–periodic and $2 \cdot C_1$ contained in a single coset of H_0 implies $H_1' \leq H_0'$. By the analogous argument on A_0 we get $H_0' \leq H_1'$. Thus $H_0' = H_1'$ and, by symmetry, we may assume $H_0 \leq H_1$. Therefore each color class is H_0–periodic.

It follows that H_0' is a proper subgroup of L since otherwise we get the rainbow–free 3–coloring $\{A/L, B/L, C/L\}$ of the cyclic group G/L of order 2^m, contradicting Theorem 1.

Consider the subgroup $H = H_0' \oplus \mathbb{Z}/2^m\mathbb{Z}$. Observe that H contains A_0 since $2 \cdot A_0 \subset H_1 \leq H$ and the two subgroups H_1 and H have the same odd factor. Similarly, since $2 \cdot C_1 \subset H_0$ the color C_1 is also contained in H. Hence B does not intersect H, since otherwise, as all color classes are H_0'–periodic, we would get the rainbow-free 3–coloring

$$\{(A_0 \cap H)/H_0', (B \cap H)/H_0', (C \cap H)/H_0')\},$$

of the cyclic group G'/H_0' of order 2^m contradicting again Theorem 1. Thus $B = G \setminus H$ and the statement of the Theorem holds with $H' = H_0'$.

Case 1.2: P_0 is trichromatic. By the induction hypothesis there is a subgroup

$$H_0 = H_0' \oplus \mathbb{Z}/2^{m-1}\mathbb{Z},$$

such that $A \subseteq H_0$ and $B_0 \setminus H_0$ and $C_0 \setminus H_0$ are H_0–periodic. Choose a minimal H_0' with this property. We may assume that $C_0 \setminus H_0$ is nonempty. Since

$$2 \cdot A \subset H_1' \oplus \mathbb{Z}/2^{m_1}\mathbb{Z},$$

we have $A \subset H' \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}$ with $H' = H_0' \cap H_1'$. By the minimality of H_0' we have $H' = H_0' \leq H_1'$. Thus,

$$A \subset H_0.$$

Suppose that, for some $x \in L \setminus \{0\}$, the coset $X = H_0 + (2x, 0)$ is colored B. Since $X \subset A_0 + B_0$ is disjoint from $2 \cdot C$, then the coset $H_0 + (x, 1)$ is also monochromatic of color B.

By switching the roles of B and C we conclude that B_1 and C_1 are also H_0–periodic. Let

$$H = H_0' \oplus \mathbb{Z}/2^m\mathbb{Z}.$$

If $C = G \setminus H$ the statement holds with H and we are done. Otherwise the 3–coloring

$$\{A/H_0, B/H_0, C/H_0\},$$

is rainbow–free with the color $A' = A/H_0$ consisting only of zero. Since the coloring is rainbow–free, every color X satisfies $2 \cdot X \subset X \cup \{0\}$. This implies that each of $B \setminus H$ and $C \setminus H$ are not only H_0–periodic but in fact H–periodic. This concludes this case.

Case 2: Both cosets of P_0 are trichromatic.

By the induction hypothesis there is a subgroup

$$H_0 = H_0' \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}$$

of P_0 such that A_0 is contained in a single coset H_0 and $B_0 \setminus H_0$ and $C_0 \setminus H_0$ are H_0–periodic. Choose a minimal H_0' with this property.

It follows from Lemma 12 that there is a proper subgroup

$$T_0 = T_0' \oplus \mathbb{Z}/2^{m-1}\mathbb{Z} < P_0$$

containing H_0 such that $B_0 + C_0 \supset P_0 \setminus T_0$ and each of $B_0 \setminus T_0$ and $C_0 \setminus T_0$ is T_0–periodic.

We have

$$2 \cdot A_1 \subset P_0 \setminus (B_0 + C_0),$$

so that $2 \cdot A_1 \subset T_0$. It follows that $A_1 \subset H$ with

$$H = T_0' \oplus \mathbb{Z}/2^m\mathbb{Z}.$$

Thus

$$A \subset H.$$

We now use a similar argument to Case 1.2. For each $x \in L \setminus T_0$ the coset $X = T_0 + (2x, 0) \subset P_0 \setminus T_0$ is monochromatic and, since the coloring is rainbow–free, so that $X = A_0 + X$ is disjoint from $T_0 + (2x, 1)$, the coset $T_0 + (x, 1)$ is also monochromatic. Hence each of $B_1 \setminus (T_0 + A_1)$ and $C_1 \setminus (T_0 + A_1)$ is also T_0–periodic. Hence, either $G \setminus H$ is monochromatic and we are done, or $\{A/H, B/H, C/H\}$ is a rainbow–free 3–coloring of G/H with the color class $A' = A/H$ consisting only of zero. Hence, every color X satisfies $2 \cdot X \subset X \cup \{0\}$. This implies that each of $B \setminus H$ and $C \setminus H$ are not only T_0–periodic but in fact H–periodic. This completes the proof. $\square$

Theorem 7 provides a proof of Conjecture 1. The proof is completely analogous to the one in Corollary 1 for the case of abelian groups of odd order except that we invoke Theorem 7 instead of Theorem 3.

Corollary 2. *Let G be cyclic group of order n. Let p denote the smallest odd prime factor of n in $\mathcal{P}_0$ and let q be the smallest odd prime factor of n in $\mathcal{P}_1$. If $\{A,B,C\}$ is a rainbow–free 3–coloring of G then*

$$\min\{|A|,|B|,|C|\} \leq \left\lfloor \frac{n}{\min\{2p,q\}} \right\rfloor. \tag{8}$$

Moreover, there are rainbow–free 3–colorings of G for which equality holds.

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