Cubic vertex-transitive non-Cayley graphs of order $8p^*$

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Abstract

A graph is *vertex-transitive* if its automorphism group acts transitively on its vertices. A vertex-transitive graph is a *Cayley graph* if its automorphism group contains a subgroup acting regularly on its vertices. In this paper, the cubic vertex-transitive non-Cayley graphs of order $8p$ are classified for each prime p. It follows from this classification that there are two sporadic and two infinite families of such graphs, of which the sporadic ones have order 56, one infinite family exists for every prime $p > 3$ and the other family exists if and only if $p \equiv 1 \pmod 4$. For each family there is a unique graph for a given order.

Keywords: Cayley graphs, vertex-transitive graphs, automorphism groups

1 Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, $u \sim v$ means that u is adjacent to v and denote by $\{u, v\}$ the edge incident to u and v in X. A graph X is said to be *vertex-transitive*, and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.

It is well known that a vertex-transitive graph is a Cayley graph if and only if its automorphism group contains a subgroup acting regularly on its vertex set (see, for example, [25, Lemma 4]). There are vertex-transitive graphs which are not Cayley graphs

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and the smallest one is the well-known Petersen graph. Such a graph will be called a *vertex-transitive non-Cayley graph*, or a *VNC-graph* for short. Many publications have been put into investigation of VNC-graphs from different perspectives. For example, in [13], Marušič asked for a determination of the set NC of non-Cayley numbers, that is, those numbers for which there exists a VNC-graph of order n, and to settle this question, a lot of VNC-graphs were constructed in [9, 11, 14, 19, 15, 16, 17, 20, 22, 26]. In [6], Feng considered the question to determine the smallest valency for VNC-graphs of a given order and it was solved for the graphs of odd prime power order. By [19, Table 1], the total number of vertex-transitive graphs of order n and the number of VNC-graphs of order n were listed for each $n \leq 26$. It seems that, for small orders at least, the great majority of vertex-transitive graphs are Cayley graphs. This is true particularly for small valent vertex-transitive graphs (see [21]). This suggests the problem of classifying small valent VNC-graphs. From [3, 12] all VNC-graphs of order $2p$ are known for each prime p. Recently, in [30] all tetravalent VNC-graphs of order $4p$ were classified, and in [28, 29, 31], all cubic VNC-graphs of order $2pq$ were classified, where p and q are primes. In this paper we shall classify all cubic VNC-graphs of order $8p$ for each prime p. As a result, there are two sporadic and two infinite families of such graphs, of which the sporadic ones have order 56, one infinite family exists for every prime $p > 3$ and the other family exists if and only if $p \equiv 1 \pmod 4$. For each family there is a unique graph for a given order.

2 Preliminaries

In this section, we introduce some notations and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph X, use $d(X)$ to represent the valency of X, and for any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. Let X be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(X)$ be vertex-transitive on X. For a G-invariant partition $\mathcal{B}$ of $V(X)$, the *quotient graph* $X_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(X)$ is a G-invariant partition of $V(X)$. In this case, the symbol $X_{\mathcal{B}}$ will be replaced by X_N.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. We call C_n a *n-cycle*.

For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 1. [10, Chapter I, Theorem 4.5] *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of* H.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. For any $g \in G$, g is said to be *semiregular* if $\langle g \rangle$ is semiregular. The following proposition gives a characterization for Cayley graphs in terms of their automorphism groups.

Proposition 2. [25, Lemma 4] *A graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X.*

3 Double generalized Petersen graphs

In [28, 29, 31], the generalized Petersen graphs (see [27]) were used to construct cubic VNC-graphs with special orders. Let $n \geq 3$ and $1 \leq t < n/2$. The *generalized Petersen graph* $P(n,t)$ (GPG for short) is the graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set the union of the *out edges* $\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the *inner edges* $\{\{y_i, y_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the spokes $\{\{x_i, y_i\} \mid i \in \mathbb{Z}_n\}$. Note that the subgraph of $P(n,t)$ induced by the out edges is an n-cycle. In this section, we modify the generalized Petersen graph construction slightly so that the subgraph induced by the out edges is a union of two n-cycles.

Definition 3. Let $n \geq 3$ and $t \in \mathbb{Z}_n - \{0\}$. The *double generalized Petersen graph* $DP(n,t)$ (DGPG for short) is defined to have vertex set $\{x_i, y_i, u_i, v_i \mid i \in \mathbb{Z}_n\}$ and edge set the union of the *out edges* $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the *inner edges* $\{\{u_i, v_{i+t}\}, \{v_i, u_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the *spokes* $\{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$ (See Fig. 1 for DP(10,2)).

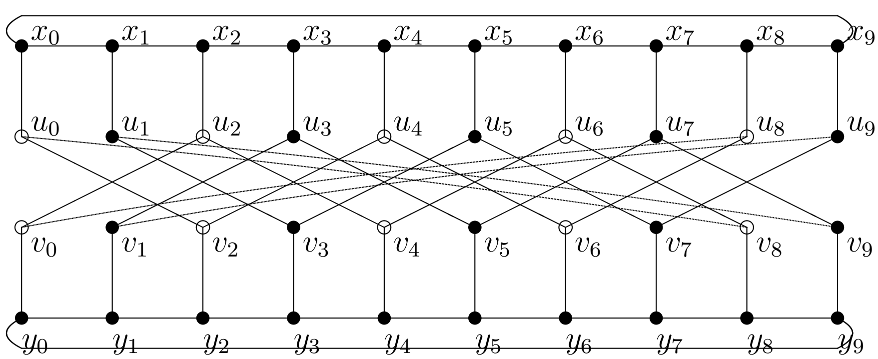

Figure 1: The graph $DP(10,2)$

Note that the complete classification of the automorphism groups of GPGs has been worked out in [8], and Nedela and Škoviera [23] have determined all Cayley graphs among GPGs. It is natural to consider the problem of determining all vertex-transitive graphs and all VNC-graphs among DGPGs. The complete solution of this problem may be a topic for our future effort. Here, we just give a sufficient condition for a DGPG being vertex-transitive non-Cayley. To do this, we introduce some notations.

In the remainder of this section, we always use p to represent a prime congruent to 1 modulo 4. It is easy to see that $\lambda \in \mathbb{Z}_p$ is a solution of the equation

$$x^2 \equiv -1 \pmod p \tag{1}$$

if and only if λ has order 4 in $\mathbb{Z}_p^*$. Since $p \equiv 1 \pmod 4$, $\mathbb{Z}_p^*$ has exactly two elements, say $\lambda, p-\lambda$, of order 4. So, in $\mathbb{Z}_p$, Eq. (1) has exactly two solutions that are λ and $p-\lambda$. Note that every solution of Eq. (1) in $\mathbb{Z}_{2p}$ is also a solution of Eq. (1) in $\mathbb{Z}_p$. This implies that in $\mathbb{Z}_{2p}$, Eq. (1) has exactly four pairwise different solutions that are $\lambda, 2p-\lambda$, $p-\lambda$ and $p+\lambda$.

Lemma 4. *For any $\lambda_1, \lambda_2 \in \{\lambda, 2p-\lambda, p-\lambda, p+\lambda\}$, we have $DP(2p, \lambda_1) \cong DP(2p, \lambda_2)$.*

Proof. By Definition 3, it is easy to see that if either $\{\lambda_1, \lambda_2\} = \{\lambda, 2p-\lambda\}$ or $\{\lambda_1, \lambda_2\} = \{p-\lambda, p+\lambda\}$, then $DP(2p, \lambda_1) = DP(2p, \lambda_2)$. To complete the proof, it suffices to show $DP(2p, \lambda) \cong DP(2p, p+\lambda)$.

Define a map from $V(DP(2p, \lambda))$ to $V(DP(2p, p+\lambda))$ as following:

$$f : x_i \mapsto x_i, y_i \mapsto y_{i+p}, u_i \mapsto u_i, v_i \mapsto v_{i+p}, \forall i \in \mathbb{Z}_{2p}.$$

It is easy to see that f is a bijection. Furthermore, f maps each of the out edges and the spokes of $DP(2p, \lambda)$ to an edge of $DP(2p, p+\lambda)$. For the inner edges, $\{u_i, v_{i+\lambda}\}^f = \{u_i, v_{i+\lambda+p}\} \in E(DP(2p, \lambda_2))$. Similarly, $\{v_i, u_{i+\lambda}\}^f = \{v_{i+p}, u_{i+\lambda}\} = \{v_{i+p}, u_{i+p+p+\lambda}\} \in E(DP(2p, \lambda_2))$. Thus, f is an isomorphism from $DP(2p, \lambda)$ to $DP(2p, p+\lambda)$. $\square$

Theorem 5. *Suppose that $VNC^1_{8p} := DP(2p, \lambda)$, where λ is a solution of Eq. (1) in $\mathbb{Z}_{2p}$. Then VNC^1_{8p} is a connected cubic VNC-graph of order $8p$.*

Proof. Let $X = VNC^1_{8p}$ and $A = \mathrm{Aut}(X)$. By the definition, it is easy to see that X is connected and has order $8p$. Since $p \equiv 1 \pmod 4$, one has $p \geq 5$. If $p = 5$, with the help of MAGMA [1], X is a cubic VNC-graph. In what follows, assume $p > 5$. Remember that Eq. (1) has exactly four solutions, namely, $\lambda, 2p-\lambda$, $p-\lambda$ and $p+\lambda$, in $\mathbb{Z}_{2p}$. By Lemma 4, we may assume that λ is even.

One can easily see that the following maps are permutations on the vertex set of X:

$$\begin{aligned}
\alpha:\ & x_i \mapsto x_{i+1}, y_i \mapsto y_{i+1}, u_i \mapsto u_{i+1}, v_i \mapsto v_{i+1}, i \in \mathbb{Z}_{2p},\\
\beta:\ & x_i \mapsto y_i, y_i \mapsto x_i, u_i \mapsto v_i, v_i \mapsto u_i, i \in \mathbb{Z}_{2p},\\
\gamma:\ & x_{2i+1} \mapsto v_{(2i+1)\lambda}, x_{2i} \mapsto u_{(2i)\lambda}, y_{2i+1} \mapsto v_{(2i+1)\lambda+p}, y_{2i} \mapsto u_{(2i)\lambda+p},\\
& u_{2i} \mapsto x_{(2i)\lambda}, v_{2i} \mapsto x_{(2i)\lambda+p}, u_{2i+1} \mapsto y_{(2i+1)\lambda}, v_{2i+1} \mapsto y_{(2i+1)\lambda+p}, i \in \mathbb{Z}_{2p}.
\end{aligned}$$

Also, one may easily see that α and β map each edge of X to an edge. So, $\alpha, \beta \in \mathrm{Aut}(X)$. Since λ is an even solution of Eq. (1), one has $p + \lambda^2 + 1 \equiv 0 \pmod{2p}$. For each $i \in \mathbb{Z}_{2p}$, we have

$$\begin{aligned}
&\{x_{2i}, x_{2i+1}\}^\gamma = \{u_{(2i)\lambda}, v_{(2i+1)\lambda}\}, \{y_{2i}, y_{2i+1}\}^\gamma = \{u_{(2i)\lambda+p}, v_{(2i+1)\lambda+p}\},\\
&\{u_{2i}, v_{2i+\lambda}\}^\gamma = \{x_{(2i)\lambda}, x_{(2i+\lambda)\lambda+p}\} = \{x_{(2i)\lambda}, x_{(2i)\lambda-1}\},\\
&\{v_{2i}, u_{2i+\lambda}\}^\gamma = \{x_{(2i)\lambda+p}, x_{(2i+\lambda)\lambda}\} = \{x_{(2i)\lambda+p}, x_{(2i)\lambda+p-1}\},\\
&\{u_{2i+1}, v_{2i+1+\lambda}\}^\gamma = \{y_{(2i+1)\lambda}, y_{(2i+1+\lambda)\lambda+p}\} = \{y_{(2i+1)\lambda}, y_{(2i+1)\lambda-1}\},\\
&\{v_{2i+1}, u_{2i+1+\lambda}\}^\gamma = \{y_{(2i+1)\lambda+p}, y_{(2i+1+\lambda)\lambda}\} = \{y_{(2i+1)\lambda+p}, y_{(2i+1)\lambda+p-1}\},\\
&\{x_{2i}, u_{2i}\}^\gamma = \{u_{(2i)\lambda}, x_{(2i)\lambda}\}, \{x_{2i+1}, u_{2i+1}\}^\gamma = \{v_{(2i+1)\lambda}, y_{(2i+1)\lambda}\},\\
&\{y_{2i}, v_{2i}\}^\gamma = \{u_{(2i)\lambda+p}, x_{(2i)\lambda+p}\}, \{y_{2i+1}, v_{2i+1}\}^\gamma = \{v_{(2i+1)\lambda+p}, y_{(2i+1)\lambda+p}\}.
\end{aligned}$$

This implies that $\gamma \in \mathrm{Aut}(X)$. Clearly, $\{x_i, y_i \mid i \in \mathbb{Z}_{2p}\}$ and $\{u_i, v_i \mid i \in \mathbb{Z}_{2p}\}$ are two orbits of $\langle \alpha, \beta \rangle$ on $V(X)$, and γ interchanges these two orbits. Hence, $\langle \alpha, \beta, \gamma \rangle$ is transitive on $V(X)$. We shall show that $A = \langle \alpha, \beta, \gamma \rangle$. Since X has valency 3, the vertex-stabilizer A_v is a $\{2, 3\}$-group. So, $|A| \mid 2^{3+i}3^j p$ for some integers i, j. As $p > 5$, the group $P = \langle \alpha^2 \rangle$ is a Sylow p-subgroup of A.

We claim that P is normal in A. By [7, Theorem 5.1 and Corollary 3.8], this is true for the case when X is symmetric. Suppose X is non-symmetric. Then, $3 \nmid |A_v|$, and so A is a $\{2, p\}$-group. By Burnside's $\{p, q\}$-theorem [24, 8.5.3], A is solvable. So, we can take a maximal normal 2-subgroup, say N, of A. Then $PN/N \trianglelefteq A/N$, namely, $PN \trianglelefteq A$. To show $P \trianglelefteq A$, it suffices to prove $P \trianglelefteq PN$ because then P is characteristic in PN and hence it is normal in A.

Consider the quotient graph X_N of X relative to the orbit set of N, and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and so $K = NK_v$ is a 2-group. The maximality of N gives that $K = N$. So, $A/N \leq \mathrm{Aut}(X_N)$. Clearly, X_N has valency 2 or 3, namely, $d(X_N) = 2$ or 3. Let $B \in V(X_N)$. If either $d(X_N) = 3$ or $d(X_N) = 2$ and $d(X[B]) = 1$, then the stabilizer N_v of $v \in V(X)$ fixes each neighbor of v. By the connectivity of X, N_v fixes all vertices of X. It follows that $N_v = 1$, and hence N is semiregular on $V(X)$. Consequently, $|N| \mid 8$. Since $p \geq 13$, by Sylow theorem, one has $P \trianglelefteq PN$, as required. Suppose $d(X_N) = 2$ and $d(X[B]) = 0$. Let B_0 and B_1 be two orbits adjacent to B. Since X is cubic, one may assume that $d(X[B \cup B_0]) = 1$ and $d(X[B \cup B_1]) = 2$. Since p is odd, it follows that $|B| = 2$ or 4. If $|B| = 2$ then $X[B \cup B_1] \cong C_4$. However, since the set of vertices of X at distance 2 from x_0 is $N_2(x_0) = \{x_2, u_1, v_\lambda, v_{-\lambda}, x_{2p-2}, u_{2p-1}\}$ which has cardinality 6, passing through x_0 there is no cycles of less than 5 in X. This implies that X has girth greater than 4 because it is vertex-transitive. A contradiction occurs. If $|B| = 4$, then $X[B \cup B_1] \cong C_8$ since X has girth greater than 4. So, the subgroup N^* of N fixing B pointwise also fixes B_0 and B_1 pointwise. By the connectivity of X, N^* fixes all vertices of X, forcing $N^* = 1$. It follows that $N \leq \mathrm{Aut}(X[B \cup B_1]) \cong D_{16}$. Since $p > 5$ and $p \equiv 1 \pmod 4$, by Sylow Theorem, one has $P \trianglelefteq PN$, as required.

Now we know the claim is true, that is, $P \trianglelefteq A$. Consider the quotient graph X_P of X relative to the orbit set of P, and let K be the kernel of A acting on $V(X_P)$. From the construction of X, $X_P \cong C_8$ and the subgraph of X induced by any two adjacent orbits of P is either pK_2 or C_{2p}. This implies that K acts faithfully on each orbit of P, and hence $K \leq \mathrm{Aut}(C_{2p}) \cong D_{4p}$. Since K fixes each orbit of P, one has $K \leq D_{2p}$. Clearly, A/K is not edge-transitive on X_P. It follows that $A/K \cong D_8$, and hence $|A| \leq 16p$. This implies that $A = \langle \alpha, \beta \rangle \rtimes \langle \gamma \rangle \cong (\mathbb{Z}_{2p} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$.

Now we are ready to finish the proof. Suppose that X is a Cayley graph. By Proposition 2, A has a regular subgroup, say G. Clearly, $|A : G| = 2$, and so G is maximal in A. Let $Q = \langle \alpha^p, \beta, \gamma \rangle$. Then Q is a Sylow 2-subgroup of A. So, $Q \subsetneq G$, and hence $A = GQ$. From $|A| = \frac{|G||Q|}{|Q \cap G|}$ we get that $|Q \cap G| = 8$, and hence $Q/(Q \cap G) \cong \mathbb{Z}_2$. It follows that $\gamma^2 \in Q \cap G$. However, since γ^2 fixes x_0, one has $\gamma^2 \notin G$, a contradiction. $\square$

4 Graphs associated with lexicographic products

Let n be a positive integer. The *lexicographic product* $C_n[2K_1]$ is defined as the graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{y_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\}$. In this section, we introduce a class of cubic vertex-transitive graphs which can be constructed from the lexicographic product $C_n[2K_1]$. Note that these graphs belong to a large family of graphs constructed in [5, Section 3].

Definition 6. For integer $n \geq 2$, let $X(n, 2)$ be the graph of order $4n$ and valency 3 with vertex set $V_0 \cup V_1 \cup \ldots V_{2n-2} \cup V_{2n-1}$, where $V_i = \{x_i^0, x_i^1\}$, and adjacencies $x_{2i}^r \sim x_{2i+1}^r (i \in \mathbb{Z}_n, r \in \mathbb{Z}_2)$ and $x_{2i+1}^r \sim x_{2i+2}^s (i \in \mathbb{Z}_n; r, s \in \mathbb{Z}_2)$.

Note that $X(n, 2)$ is obtained from $C_n[2K_1]$ by expending each vertex into an edge, in a natural way, so that each of the two blown-up endvertices inherits half of the neighbors of the original vertex.

Definition 7. Let $EX(n, 2)$ be the graph obtained from $X(n, 2)$ by blowing up each vertex x_i^r into two vertices $x_i^{r,0}$ and $x_i^{r,1}$. The adjacencies are as the following: $x_{2i}^{r,s} \sim x_{2i+1}^{r,t}$ and $x_{2i+1}^{r,s} \sim x_{2i+2}^{s,r}$, where $i \in \mathbb{Z}_n$ and $r, s, t \in \mathbb{Z}_2$ (see Fig. 2 for EX(5,2)).

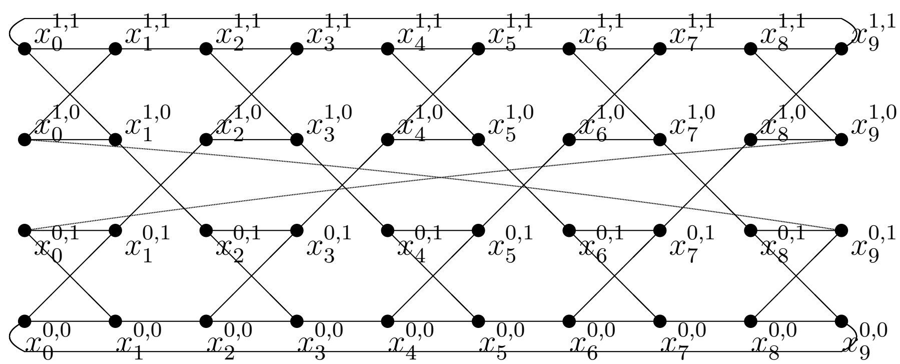

Figure 2: The graph $EX(5, 2)$

Note that $EX(n, 2)$ is vertex-transitive for each $n \geq 2$ (see [5, Proposition 3.3]). However, $EX(n, 2)$ is not necessarily a Cayley graph. Below, we shall give a sufficient condition for the graph $EX(n, 2)$ to be vertex-transitive non-Cayley. To do this, we need a lemma.

Lemma 8. *If $p > 7$ is a prime, then* $\mathrm{Aut}(C_p[2K_1])$ *has no subgroups of order* $8p$.

Proof. Set $A = \mathrm{Aut}(C_p[2K_1])$. It is easily known that $A \cong \mathbb{Z}_2^p \rtimes D_{2p}$. Recall that $C_p[2K_1]$ has vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{y_i, x_{i+1}\} \mid i \in \mathbb{Z}_p\}$. Let K be the maximal normal 2-subgroup of A. Then $K = \langle k_0 \rangle \times \langle k_1 \rangle \times \ldots \times \langle k_{p-1} \rangle$, where $k_i = (x_i\ y_i)$ for $i \in \mathbb{Z}_p$. Let $\alpha = (x_0\ x_1\ \ldots\ x_{p-1})(y_0\ y_1\ \ldots\ y_{p-1})$. It is easy to see that α is an automorphism of $C_p[2K_1]$ of order p. Set $P = \langle \alpha \rangle$.

Suppose to the contrary that A has a subgroup, say G, of order $8p$. By Sylow Theorem, one may assume that $P \leq G$. Since $p > 7$, Sylow Theorem gives $P \trianglelefteq G$. Noting that

$GK \leq A$, one has $|A : GK| = 1$ or 2. Consequently, $G \cap K$ is isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$ and is normal in G. It follows that $G \cap K \leq C_A(P)$. However, it is easy to see that $C_A(P) \cap K = \langle k_0 k_1 \dots k_{p-1} \rangle \cong \mathbb{Z}_2$, a contradiction. □

Theorem 9. *Let $p > 3$ be a prime. Then the graph $VNC_{8p}^2 := EX(p, 2)$ is a connected cubic VNC-graph of order $8p$.*

Proof. Let $X = VNC_{8p}^2 = EX(p, 2)$ and $A = \mathrm{Aut}(X)$. If $p = 5$ or 7 then by MAGMA [1], X is a connected cubic VNC-graph of order $8p$. In what follows, assume that $p > 7$. By [5, Proposition 3.3], X is vertex-transitive.

Clearly, for each $j \in \mathbb{Z}_p$, $C_j^0 = (x_{2j}^{0,0}, x_{2j+1}^{0,0}, x_{2j}^{0,1}, x_{2j+1}^{0,1})$ and $C_j^1 = (x_{2j}^{1,1}, x_{2j+1}^{1,1}, x_{2j}^{1,0}, x_{2j+1}^{1,0})$ are two 4-cycles. Set $\mathcal{F} = \{C_j^i \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_p\}$. From the construction of X, it is easy to see that in X passing each vertex there is exactly one 4-cycle, which belongs to $\mathcal{F}$. Clearly, any two distinct 4-cycles in $\mathcal{F}$ are vertex-disjoint. This implies that $\Delta = \{V(C_j^i) \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_p\}$ is an A-invariant partition of $V(X)$. Consider the quotient graph X_Δ, and let K be the kernel of A acting on Δ. Then $X_\Delta \cong C_p[2K_1]$, and hence $A/K \leq \mathrm{Aut}(C_p[2K_1]) \cong \mathbb{Z}_2^p \rtimes D_{2p}$. Noting that between any two adjacent vertices of X_Δ there is exactly one edge of X, K fixes each vertex of X and hence $K = 1$. If X is a Cayley graph, then $A = A/K$ has a regular subgroup of order $8p$, and hence $\mathrm{Aut}(C_p[2K_1])$ would contain a subgroup of order $8p$. However, this is impossible by Lemma 8. □

5 Classification

In this section, we classify all connected cubic VNC-graphs of order $8p$ for each prime p. Throughout this section, the notations FnA, FnB, etc. will refer to the corresponding graphs of order n in the Foster census of all cubic symmetric graphs [2, 4]. The following is the main result of this paper.

Theorem 10. *A connected cubic graph of order $8p$ for a prime p is a VNC-graph if and only if it is isomorphic to* F56B*,* F56C*, VNC_{8p}^1 or VNC_{8p}^2.*

Proof. By [4], F56B and F56C are cubic symmetric graphs. By MAGMA [1], Aut(F56B) and Aut(F56C) have no subgroups of order 56. It follows from Proposition 2 that F56B and F56C are non-Cayley graphs. By Theorems 5 and 9, the graphs VNC_{8p}^1 and VNC_{8p}^2 are connected cubic VNC-graphs of order $8p$.

To complete the proof, we only need to show necessity. Assume that X is a connected cubic VNC-graph of order $8p$. By McKay [18, pp.1114] and [21], all connected cubic vertex-transitive graphs of order 16 or 24 are Cayley graphs. If X is symmetric then by [7, Theorem 5.1], $X \cong$ F40A, F56B or F56C. By MAMGA [1], F40A $\cong VNC_{8 \cdot 5}^1$.

In what follows, assume that $p > 3$ and X is non-symmetric. Let $A = \mathrm{Aut}(X)$. Since X is non-Cayley, A has no subgroups acting regularly on $V(X)$ by Proposition 2. For each $v \in V(X)$, we have $|A_v| = 2^m$ and $|A| = 2^{m+3}p$ for some positive integer m. By Burnside's $p^a q^b$-theorem [24, 8.5.3], A is solvable. For notational convenience, in the remainder of the proof, we always use the following notations.

Assumption For each $q \in \{2, p\}$, use M_q to denote the maximal normal q-subgroup of A. Let X_{M_q} be the quotient graph of X relative to the orbit set of M_q, and let Ker_q be the kernel of A acting on $V(X_{M_q})$.

We first prove the following claims.

Claim 1 Suppose $M_2 > 1$. Then for any orbit B of M_2, we have $X[B]$ is the null graph.

Since $p > 2$, one has $|B| = 8, 4$ or 2, and hence $p \mid |X_{M_2}|$. This implies that $d(X_{M_2}) \geq 2$. If $X[B]$ is not a null graph, then it has valency 1. For each $v \in B$, one neighbor of v is in B, and the other two are in the two different orbits of M_2 adjacent to B, respectively. Because Ker_2 fixes each orbit of M_2, $(\mathrm{Ker}_2)_v$ fixes each neighbor of v. By the connectivity of X, $(\mathrm{Ker}_2)_v$ fixes all vertices of X, and hence $(\mathrm{Ker}_2)_v = 1$. This shows that $\mathrm{Ker}_2 = M_2$ is semiregular. Clearly, X_{M_2} must be a cycle of length $\ell = 8p/|B|$. The vertex-transitivity of A/M_2 on X_{M_2} implies that A/M_2 contains a subgroup, say G/M_2, acting regularly on $V(X_{M_2})$. As a result, G is regular on $V(X)$, a contradiction.

Claim 2 Suppose $M_p > 1$. Then $X_{M_p} \cong C_8$, $\mathrm{Ker}_p = M_p \rtimes A_v \cong D_{2p}$ and $A/\mathrm{Ker}_p \cong D_8$. Furthermore, for any two adjacent orbits, say B, B' of M_p, we have $X[B] \cong pK_1$ and $X[B \cup B'] \cong C_{2p}$ or pK_2.

Since $|A| = 2^{m+3}p$, M_p is a Sylow p-subgroup of A, and $|X_{M_p}| = 8$. So, $d(X_{M_p}) = 3$ or 2. Suppose $d(X_{M_p}) = 3$. Then the stabilizer $(\mathrm{Ker}_p)_v$ fixes the neighborhood of v in X pointwise because Ker_p fixes each orbit of M_p setwise. By the connectivity of X, $(\mathrm{Ker}_p)_v$ fixes each vertex in $V(X)$, forcing $(\mathrm{Ker}_p)_v = 1$. Hence, $\mathrm{Ker}_p = M_p$. By [21], X_{M_p} is a Cayley graph, and furthermore, either $X_{M_p} \cong Q_3$, the three dimensional hypercube, or $|\mathrm{Aut}(X_{M_p})| \leq 16$. Note that if $X_{M_p} \cong Q_3$ then $\mathrm{Aut}(X_{M_p}) \cong S_4 \times \mathbb{Z}_2$. Since $|A/M_p| = 2^{m+3} > 8$, A/M_p is always a Sylow 2-subgroup of $\mathrm{Aut}(X_{M_p})$. As X_{M_p} is a Cayley graph of order 8, $\mathrm{Aut}(X_{M_p})$ has a regular subgroup, say $\overline{G}$. By a Sylow Theorem, one may assume that $\overline{G} = G/M_p \leq A/M_p$. This forces that G is regular on $V(X)$, a contradiction.

Now we know that $d(X_{M_p}) = 2$, namely, $X_{M_p} \cong C_8$. Then $A/\mathrm{Ker}_p \leq \mathrm{Aut}(X_{M_p}) \cong D_{16}$. Let $V(X_{M_p}) = \{B_i \mid i \in \mathbb{Z}_8\}$ with $B_i \sim B_{i+1}$ for each $i \in \mathbb{Z}_8$. If some B_i contains an edge of X, then the connectivity of X_{M_p} implies that $d(X[B_i]) = 1$. This forces that $|B_i| = p$ is even, a contradiction. Thus, $X[B_i] \cong pK_1$ for every $i \in \mathbb{Z}_8$. Since X is cubic, for any two adjacent orbits B, B' of M_p, we have $X[B \cup B'] \cong C_{2p}$ or pK_2. Without loss of generality, assume that $X[B_0 \cup B_7] \cong pK_2$ and $X[B_0 \cup B_1] \cong C_{2p}$. Then A/Ker_p is not edge-transitive on X_{M_p}, and hence $A/\mathrm{Ker}_p \cong D_8$. Since $p > 3$, the subgroup Ker_p^* of Ker_p fixing B_0 pointwise also fixes B_1 and B_7 pointwise. The connectivity of X gives $\mathrm{Ker}_p^* = 1$, and consequently, $\mathrm{Ker}_p \leq \mathrm{Aut}(B_0 \cup B_1) \cong D_{4p}$. Since Ker_p fixes B_0, one has $\mathrm{Ker}_p \cong \mathbb{Z}_p$ or D_{2p}. Since $|A| > 8p$, it follows that $\mathrm{Ker}_p \cong D_{2p}$ and hence $|A| = 16p$. Since A/Ker_p is regular on $V(X_{M_p})$, one has $A_v = (\mathrm{Ker}_p)_v \cong \mathbb{Z}_2$ and $\mathrm{Ker}_p = M_p \rtimes A_v$.

Now we are ready to finish the proof. We distinguish two different cases.

Case 1 $M_p > 1$.

Since $|A| = 2^{m+3}p$, one has $M_p \cong \mathbb{Z}_p$. Let $C = C_A(M_p)$. Then $M_p \leq C$ and by Proposition 1, $A/C \leq \mathrm{Aut}(M_p) \cong \mathbb{Z}_{p-1}$. By Claim 2, $A/\mathrm{Ker}_p \cong D_8$ and $\mathrm{Ker}_p = M_p \rtimes A_v \cong$

D_{2p}. This means that $C_v = 1$, and hence C is semiregular on $V(X)$. So, $|C| = 2p$ or $4p$. If $|C| = 2p$, then $C/P \cong \mathbb{Z}_2$ is in the center of A/P. Since $(A/P)/(C/P) \cong A/P \leq \mathbb{Z}_{p-1}$, A/P is abelian. It follows that $A/\mathrm{Ker}_p \cong (A/P)/(\mathrm{Ker}_p/P)$ is abelian, a contradiction. So, the only possible is $|C| = 4p$.

Clearly, C has two orbits, say Δ and Δ' on $V(X)$, and the action of C on each of these two orbits is regular. It follows that $\Delta = \{u^h \mid h \in C\}$ and $\Delta' = \{v^h \mid h \in C\}$ for some fixed $u \in \Delta$ and $v \in \Delta'$, and furthermore, $u^{h_1} \neq u^{h_2}$ and $v^{h_1} \neq v^{h_2}$ for any two distinct $h_1, h_2 \in C$. Since Δ is an orbit of C, $X[\Delta]$ has valency $0, 1$ or 2.

First, suppose $d(X[\Delta]) = 0$. Then X is bipartite. Let the neighbors of u be v^{h_1}, v^{h_2} and v^{h_3} where $h_1, h_2, h_3 \in C$. Note that C is abelian. For any $h \in C$, the neighbors of u^h are v^{hh_1}, v^{hh_2} and v^{hh_3}, and furthermore, the neighbors of v^h are $u^{hh_1^{-1}}$, $u^{hh_2^{-1}}$ and $u^{hh_3^{-1}}$. Now it is easy to see that the map α defined by $v^h \mapsto u^{h^{-1}}, u^h \mapsto v^{h^{-1}}, \forall h \in C$, is an automorphism of X of order 2. Since $C \trianglelefteq A$, $\langle C, \alpha \rangle = C \rtimes \langle \alpha \rangle$ has order $8p$, implying that $\langle C, \alpha \rangle$ is regular on $V(X)$, a contradiction.

Next, suppose $d(X[\Delta]) = 1$. Let Q be a Sylow 2-subgroup of C. As C is abelian and normal in A, Q is characteristic in C, and hence it is normal in A. Clearly, every orbit of Q has cardinality 4 and is contained in Δ or Δ'. Let u^h be a neighbor of u, where $h \in C$. Clearly, $\{u, u^h\}^h = \{u^h, u^{h^2}\}$. Since $d(X[\Delta]) = 1$, one has $u^{h^2} = u$, implying that h is an involution. It follows that each orbit of Q of C consists of two pairs of adjacent vertices. This is impossible by Claim 1 because $Q \leq M_2$.

Now, suppose $d(X[\Delta]) = 2$. Since $M_p \leq C$, each orbit of M_p is contained in Δ or Δ'. By Claim 2, for any two adjacent orbits B, B' of M_p, $X[B] \cong pK_1$ and $X[B \cup B'] \cong pK_2$ or C_{2p}. Since $d(X[\Delta]) = 2$, we must have $X[\Delta] \cong X[\Delta'] \cong 2C_{2p}$. Let $\{x_0, x_1\}$ be an edge of $X[\Delta]$. Then there exists $a \in C$ such that $x_1 = x_0^a$. From Claim 1 we get a is not an involution. Let a have order s and let $x_i = x_0^{a^i}$ with $i \in \mathbb{Z}_s$. Then $C_1 = (x_0, x_1, \ldots, x_{s-1}, x_0)$ is an s-cycle. Since $X[\Delta] \cong 2C_{2p}$, one has $s = 2p$.

Suppose $C \cong \mathbb{Z}_{4p}$. Let $w \in \Delta'$ be adjacent to x_0, and let $\{w^b, w\} \in E(X[\Delta'])$ for some $b \in C$. Similar to an argument as above, we get that b has order $2p$, and $(w, w^b, w^{b^2}, \ldots, w^{b^{2p-1}}, w)$ is a $2p$-cycle. Since $C \cong \mathbb{Z}_{4p}$, one has $\langle a \rangle = \langle b \rangle$, and so $a = b^k$ for some $k \in \mathbb{Z}_{2p}^*$. This implies that for any $i \in \mathbb{Z}_{2p}$, $x_i = x_0^{a^i} \sim w^{a^i} = w^{b^{ik}}$. Consequently, the subgraph induced by $\{x_i, w^{b^i} \mid i \in \mathbb{Z}_{2p}\}$ has valency 3, contrary to the connectivity of X.

Now we know that $C \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, and hence there is an involution $c \in C \setminus \langle a \rangle$. Let $y_i = x_i^c$ with $i \in \mathbb{Z}_{2p}$. Since C is abelian, $C_2 = (y_0, y_1, \ldots, y_{s-1}, y_0)$ is also a $2p$-cycle. Clearly, C_1 and C_2 are vertex-disjoint, so $X[\Delta] = C_1 \cup C_2$. Note that the edges with one endpoint in Δ and the other endpoint in Δ' are independent. Assume that $\Delta' = \{u_i, v_i \mid i \in \mathbb{Z}_{2p}\}$ so that $u_i \sim x_i$ and $v_i \sim y_i$ for $i \in \mathbb{Z}_{2p}$. Since $X[\Delta'] \cong 2C_{2p}$, we may assume that $u_0 \sim u_\lambda$ or $u_0 \sim v_\lambda$ for some $\lambda \in \mathbb{Z}_{2p} - \{0\}$. If $u_0 \sim u_\lambda$, then the subgraph induced by $\{x_i, u_i \mid i \in \mathbb{Z}_{2p}\}$ has valency 3, contrary to the connectivity of X. Thus, $u_0 \sim v_\lambda$. Since $x_i^c = y_i$, one has $\{x_i, u_i\}^c = \{y_i, v_i\}$, and hence $u_i^c = v_i$. Since c is an involution, one has $\{u_0, v_\lambda\}^c = \{v_0, u_\lambda\}$. By Definition 3, $X \cong DP(2p, \lambda)$.

It is easy to see that $C \rtimes A_u$ is the kernel of A acting on $\{\Delta, \Delta'\}$, and $A/(C \rtimes A_u) \cong \mathbb{Z}_2$. Let $\beta \in A$ be a 2-element interchanging Δ and Δ'. Then $\beta^2 \in C \rtimes A_u$. If $\beta^2 \in C$ then $\langle C, \beta \rangle$ is regular on $V(X)$, a contradiction. Thus, $\beta^2 = gd$ where $g \in C$ and $A_v = \langle d \rangle$.

Recalling $\mathrm{Ker} = P \rtimes A_v \cong D_{2p}$, one has $\beta^{-2}a^2\beta^2 = a^{-2}$. It follows that $\beta^{-1}a^2\beta = a^{2t}$ for some $t \in \mathbb{Z}_p^*$ satisfying $t^2 \equiv -1 \pmod p$. Without loss of generality, assume $x_0^\beta = u_i$ for some $i \in \mathbb{Z}_{2p}$. Then $x_2^\beta = (x_0)^{a^2\beta} = u_i^{\beta^{-1}a^2\beta} = u_i^{a^{2t}} = u_{i+2t}$. Since the distance between x_0 and x_2 is 2, one has $u_{i+2t} = u_{i+2\lambda}$ or $u_{i-2\lambda}$. It follows that $2t \equiv \pm 2\lambda \pmod{2p}$, and hence $\lambda \equiv \pm t \pmod p$. This shows that $\lambda \in \mathbb{Z}_{2p}$ is a solution of Eq. (1). By Lemma 4 and Theorem 5, we have $X \cong VNC_{8p}^1$.

Case 2 $M_p = 1$

By the solvability of A, we have $M_2 > 1$. Let P be a Sylow p-subgroup of A. Then $P \ntrianglelefteq A$ but $PM_2/M_2 \trianglelefteq A/M_2$, namely, $PM_2 \trianglelefteq A$. If $P \trianglelefteq PM_2$, then P is characteristic in PM_2, and hence it is normal in A, a contradiction. Thus, P is not normal in PM_2. Let B be an orbit of M_2. Since $p > 2$, one has $|B| = 8, 4$ or 2, and hence $p \mid |X_{M_2}|$. This implies that X_{M_2} has valency greater than 1. If $d(X_{M_2}) = 3$, then $|B| = 2$ or 4, and it is easily seen that M_2 is semiregular, and so $|M_2| = |B|$. Since $p > 3$, Sylow Theorem implies that $P \trianglelefteq PM_2$, a contradiction. Thus, $d(X_{M_2}) = 2$. Also, since A/Ker_2 is transitive on $V(X_{M_2})$, Ker_2 is a 2-group. The maximality of M_2 gives $\mathrm{Ker}_2 = M_2$.

If $|B| = 8$, then $X_{M_2} \cong C_p$. By Claim 2, $X[B]$ is a null graph. So, the subgraph induced by any two adjacent orbits is of valency 1 or 2. This forces that $|X_{M_2}| = p$ is even, a contradiction. If $|B| = 2$, then $X_{M_2} \cong C_{4p}$, and hence $A/M_2 \leq \mathrm{Aut}(X_{M_2}) \cong D_{8p}$. Since A/M_2 is transitive on $V(X_{M_2})$, $A/M_2 \cong D_{4p}, \mathbb{Z}_{4p}$ or D_{8p}. This implies that A/M_2 always has a normal subgroup of order 2, contrary to the maximality of M_2.

It now only remains to deal with the case when $|B| = 4$. In this case, $X_{M_2} \cong C_{2p}$ and by Claim 2, $X[B] \cong 4K_1$. Let $V(X_{M_2}) = \{B_i \mid i \in \mathbb{Z}_{2p}\}$ with $B_i \sim B_{i+1}$. Since X is cubic, one may assume that $X[B_0 \cup B_1] \cong C_8$ or $2C_4$ and $X[B_0 \cup B_{2p-1}] \cong 4K_2$. Suppose $X[B_0 \cup B_1] \cong C_8$. The subgroup M_2^* of M_2 fixing B_0 pointwise also fixes B_1 and B_{2p-1} pointwise. The connectivity of X and the transitivity of A/M_2 on $V(X_{M_2})$ imply that $M_2^* = 1$, and consequently, $M_2 \leq \mathrm{Aut}(X[B_0 \cup B_1]) \cong D_{16}$. Hence, $\mathrm{Aut}(M_2)$ is a $\{2,3\}$-group. By Proposition 1, $PM_2/C_{PM_2}(M_2) \leq \mathrm{Aut}(M_2)$. Since $p \geq 5$, one has $P \leq C_{PM_2}(M_2)$, forcing $P \trianglelefteq PM_2$, a contradiction.

We now know that $X[B_0 \cup B_1]$ is a union of two 4-cycles, say $(x_0^{0,0}, x_1^{0,0}, x_0^{0,1}, x_1^{0,1})$ and $(x_0^{1,1}, x_1^{1,1}, x_0^{1,0}, x_1^{1,0})$, where $B_i = \{x_i^{0,0}, x_i^{0,1}, x_i^{1,0}, x_i^{1,1}\}$ with $i = 0$ or 1. Remember that $X_N = (B_0, B_1, \ldots, B_{2p-1})$ is a $2p$-cycle. Hence, A has an element, say σ, of order p such that $B_i^\sigma = B_{i+2}$ for each $i \in \mathbb{Z}_{2p}$. Without loss of generality, assume

$$\sigma = \prod_{(r,s)\in\mathbb{Z}_2\times\mathbb{Z}_2} (x_0^{r,s}\ x_2^{r,s}\ \ldots\ x_2^{r,s})(x_1^{r,s}\ x_3^{r,s}\ \ldots\ x_{2p-1}^{r,s}).$$

Then for each $i \in \mathbb{Z}_{2p}$, $B_i = \{x_i^{0,0}, x_i^{0,1}, x_i^{1,0}, x_i^{1,1}\}$, and $(x_{2j}^{0,0},\ x_{2j+1}^{0,0},\ x_{2j}^{0,1},\ x_{2j+1}^{0,1})$ and $(x_{2j}^{1,1},\ x_{2j+1}^{1,1},\ x_{2j}^{1,0},\ x_{2j+1}^{1,0})$ are the two 4-cycles of $X[B_{2j} \cup B_{2j+1}]$ for each $j \in \mathbb{Z}_p$.

Note that σ is an automorphism of X. Once the edges between B_{2j+1} and B_{2j+2} are given, the graph X will be determined. Let u, v be the neighbors of $x_{2i+1}^{0,0}$ and $x_{2i+1}^{0,1}$ in B_{2j+2}, respectively.

If u, v are in the same 4-cycle of $X[B_{2j+2} \cup B_{2j+3}]$, then by the connectivity of X, we

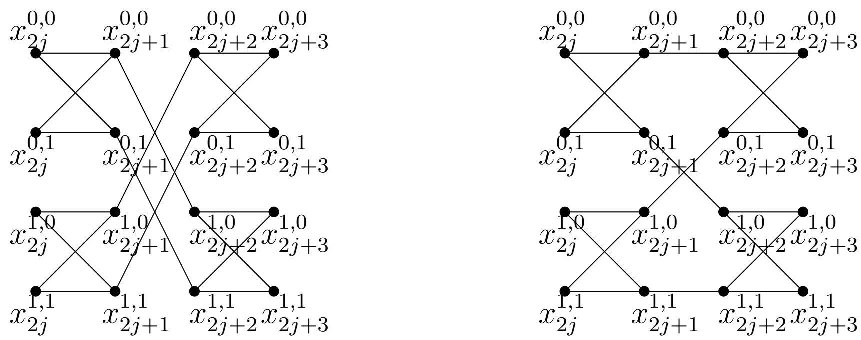

Figure 3: Two possible cases

get $\{u, v\} = \{x_{2i+2}^{1,0}, x_{2i+2}^{1,1}\}$. This gives rise to four graphs $X_i (0 \leq i \leq 4)$ such that

$$\begin{aligned} E(X_0) = & \ \{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{r+1,s}\} \mid i \in \mathbb{Z}_{2p}, r, s, t \in \mathbb{Z}_2\}; \\ E(X_1) = & \ \{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{r+1,s+1}\} \mid i \in \mathbb{Z}_{2p}, r, s, t \in \mathbb{Z}_2\}; \\ E(X_2) = & \ \{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{0,s}, x_{2i+2}^{1,s+1}\}, \{x_{2i+1}^{1,s}, x_{2i+2}^{0,s}\} \mid i \in \mathbb{Z}_{2p}, r, s, t \in \mathbb{Z}_2\}; \\ E(X_3) = & \ \{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{0,s}, x_{2i+2}^{1,s}\}, \{x_{2i+1}^{1,s}, x_{2i+2}^{0,s+1}\} \mid i \in \mathbb{Z}_{2p}, r, s, t \in \mathbb{Z}_2\}. \end{aligned}$$

Let $\delta = \prod_{i \in \mathbb{Z}_{2p}} (x_{2i+2}^{0,0}, x_{2i+2}^{0,1})(x_{2i+2}^{1,0}, x_{2i+2}^{1,1})$ and $\gamma = \prod_{i \in \mathbb{Z}_{2p}} (x_{2i+2}^{0,0}, x_{2i+2}^{0,1})$. It is easy to see that δ is an isomorphism from X_k to X_{k+1} with $k = 0, 2$, and γ is an isomorphism from X_0 to X_3. So, we may assume $X = X_0$. In this case, $X[B_{2j} \cup B_{2j+1} \cup B_{2j+2} \cup B_{2j+3}]$ is the first graph in Fig. 3. Since $p > 3$, it is easy to check that passing through each vertex of X there is one and only one 4-cycle. Set $\Omega = \{\{x_i^{0,0}, x_i^{0,1}\}, \{x_i^{1,0}, x_i^{1,1}\} \mid i \in \mathbb{Z}_{2p}\}$. Take an arbitrary $\Delta \in \Omega$. Without loss of generality, let $\Delta = \{x_i^{0,0}, x_i^{0,1}\}$ for some $i \in \mathbb{Z}_{4p}$. For any $g \in A$, $\Delta^g \subset B_i^g = B^j$ for some $j \in \mathbb{Z}_{4p}$. Since there is a 4-cycle in X passing through $(x_i^{0,0})^g$ and $(x_i^{0,1})^g$, one has $\Delta^g = \{x_j^{0,0}, x_j^{0,1}\}$ or $\{x_j^{1,0}, x_j^{1,1}\}$. It follows that $\Delta^g \in \Omega$. Clearly, any two distinct subsets in Ω are disjoint. Then Ω is an A-invariant partition of $V(X)$. From the structure of X we obtain that $X_\Omega \cong C_{4p}$ and $X[\Delta] \cong 2K_1$ for each $\Delta \in \Omega$. For notational convenience, let $V(X_\Omega) = \{\Delta_0, \Delta_1, \ldots, \Delta_{4p-1}\}$ such that $\Delta_i \in \Omega$ and $\Delta_i \sim \Delta_{i+1}$ for each $i \in \mathbb{Z}_{4p}$. Since X has valency 3, assume that $X[\Delta_0 \cup \Delta_1] \cong C_4$ and $X[\Delta_{4p-1} \cup \Delta_0] \cong 2K_2$. By the transitivity of A on $V(X)$, $X[\Delta_{2j} \cup \Delta_{2j+1}] \cong C_4$ and $X[\Delta_{2j-1} \cup \Delta_{2j}] \cong 2K_2$ for each $j \in \mathbb{Z}_{2p}$. Let $\Delta_i = \{x_i, y_i\}$ for each $i \in \mathbb{Z}_{4p}$. From the above analysis we may assume that $x_i \sim x_{i+1}, y_i \sim y_{i+1}, x_{2i} \sim y_{2i+1}$ and $y_{2i} \sim x_{2i+1}$ for each $i \in \mathbb{Z}_{4p}$. Let $\alpha : x_i \mapsto x_{i+2}, y_i \mapsto y_{i+2}$ $(i \in \mathbb{Z}_{4p})$, $\beta : x_i \mapsto y_i, y_i \mapsto x_i$ $(i \in \mathbb{Z}_{4p})$, and $\gamma : x_i \mapsto x_{4p+1-i}, y_i \mapsto y_{4p+1-i}$ $(i \in \mathbb{Z}_{4p})$ be the three permutations on $V(X)$. It is easy to check that α, β and γ are automorphisms of X. Furthermore, $\langle \alpha, \beta, \gamma \rangle \cong D_{4p} \times \mathbb{Z}_2$ is regular on $V(X)$, a contradiction.

Now suppose that u, v are in different 4-cycles of $X[B_{2j+2} \cup B_{2j+3}]$. By [5, Proposition 3.1], we may assume that $X[B_{2j} \cup B_{2j+1} \cup B_{2j+2} \cup B_{2j+3}]$ is the second graph in Fig. 3 In this case,

$$E(X) = \ \{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{s,r}\} \mid i \in \mathbb{Z}_{2p}, r, s, t \in \mathbb{Z}_2\}.$$

From Definition 7 and Theorem 9, we know that $X = VNC_{8p}^2$. □

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