A Dice Rolling Game on a Set of Tori

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Abstract

Some linear algebraic and combinatorial problems are widely studied in connection with σ-games. One particular issue is to characterize whether or not a given vector lies in the submodule generated by the rows of a given matrix over a commutative ring. In general, one can solve this problem easily and algorithmically using the linear algebra over commutative ring. However, if the matrix has some combinatorial structure, one may expect that some more can be asserted instead of merely giving an algorithm. A recent outstanding example appeared in this line of research is the paper by Florence and Meunier published in Journal of Algebraic Combinatorics in 2010. In the same spirit, we consider a matrix over $\mathbb{Z}_n$ to completely characterize the submodule generated by its rows and give a constructive proof. The main idea for the characterization is to find certain good basic elements in the row space and then express a given element as the linear combination of them as well as some additional term.

Keywords σ-game, Fiver, dice rolling game on a torus, dice rolling game on a circle, system of linear equations

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1 Introduction

We introduce a dice rolling game on a set of tori as a variant of Fiver. Fiver is a puzzle game in which you need to flip over all the counters so that your $n \times n$ board changes from being completely full of white pieces, and, instead, becomes entirely inhabited by black pieces. Clicking on any counter will not only flip that piece from being white to black (or vice versa), but its 4 neighboring pieces immediately above, below, to the left and right of the piece that you clicked will also reverse their allegiance, becoming black if they were white, or white if they were black. The game is named after Fiver the rabbit from the classic book and movie, Watership Down.

Fiver is a σ-game and was studied by Hunziker *et al.* [4] and Lee and Yang [5]. See [1–3,6,7] for some results on σ-games. Some linear algebraic and combinatorial problems are widely studied in connection with σ-games. One particular issue is to characterize whether or not a given vector lies in the submodule generated by the rows of a given matrix over a commutative ring. In general, one can solve this problem easily and algorithmically using the linear algebra over commutative ring. However, if the matrix has some combinatorial structure, one may expect that some more can be asserted instead of merely giving an algorithm. A recent outstanding example appeared in this line of research is the paper by Florence and Meunier [2]. In the same spirit, this paper considers a matrix over $\mathbb{Z}_n$ corresponding to a rectangular array on a torus and completely characterizes the submodule generated by its rows.

Given a positive integer n, an *n-dice* is a dice with n faces such that element i of $\mathbb{Z}_n$ is written on the ith face. Given positive integers α_1, α_2, we arbitrarily locate $\alpha_1\alpha_2$ n-dice in an $\alpha_1 \times \alpha_2$ rectangular array, and glue the lower and upper together and also the left and right edges. Then we have $\alpha_1\alpha_2$ n-dice on a torus (see Figure 1).

We denote by $\mathcal{D}((\alpha_1, \alpha_2), n)$ the set of tori on each of which $\alpha_1\alpha_2$ n-dice are located as described above. For positive integers β_1, β_2, $\beta_1 \leq \alpha_1$, $\beta_2 \leq \alpha_2$, we call the following action a "(β_1, β_2)-rolling procedure" (see Figure 1).

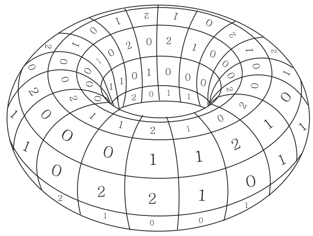

Figure 1: A torus belonging to $\mathcal{D}((19, 8), 3)$

> We roll the n-dice which form a $\beta_1 \times \beta_2$ rectangular array on the torus so that we increase the number on the top face of each of them by 1.

Then we may ask "Given a torus in $\mathcal{D}((\alpha_1, \alpha_2), n)$, is it possible to have 0 appear on the top face of each of $\alpha_1\alpha_2$ n-dice on the torus by repeatedly applying (β_1, β_2)-rolling proce-

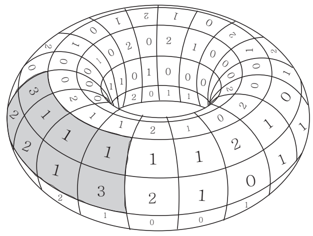

Figure 2: The torus resulting from going through a $(2,4)$-rolling procedure applied to the shaded array of the torus given in Figure 1

dures?" We call this game the *dice rolling game on* $\mathcal{D}((\alpha_1,\alpha_2),n)$ *with respect to* (β_1,β_2)-*rolling procedures* or the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-*DR game* for short. We say that a torus for which the answer to the above question is yes is a *solution of the* $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-*DR game*. Given positive integers, α_1, α_2, n, β_1, β_2, $\beta_1 \le \alpha_1$, $\beta_2 \le \alpha_2$, we will characterize the tori which are solutions of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game in the rest of this paper.

We define a module over $\mathbb{Z}_n$ as follows. We denote the set of $\alpha_1 \times \alpha_2$ matrices with elements in $\mathbb{Z}_n$ by $\mathcal{M}((\alpha_1,\alpha_2),n)$. For each element $A \in \mathcal{M}((\alpha_1,\alpha_2),n)$, we denote by $[A]_{i,j}$ the element in the (i,j)-entry. We define operations on $\mathcal{M}((\alpha_1,\alpha_2),n)$ in terms of addition and multiplication over $\mathbb{Z}_n$: Given two matrices A, $B \in \mathcal{M}((\alpha_1,\alpha_2),n)$,

$$[A+B]_{i,j} = [A]_{i,j} + [B]_{i,j} \quad \text{and} \quad [cA]_{i,j} = c[A]_{i,j} \tag{1.1}$$

for any $c \in \mathbb{Z}_n$, $0 \le i \le \alpha_1 - 1$, $0 \le j \le \alpha_2 - 1$. Throughout this paper, we assume that the 1st component and the 2nd component of every subscript are reduced to modulo α_1 and modulo α_2, respectively. Then we associate a torus in $\mathcal{D}((\alpha_1,\alpha_2),n)$ with a matrix in $\mathcal{M}((\alpha_1,\alpha_2),n)$ whose (i,j)-element equals the number on the top face of the n-dice in the (i,j) position of the torus for i, j, $0 \le i \le \alpha_1 - 1$, $0 \le j \le \alpha_2 - 1$. In this way, we can give an isomorphism between $\mathcal{D}((\alpha_1,\alpha_2),n)$ and $\mathcal{M}((\alpha_1,\alpha_2),n)$. Thus, in order to characterize the solutions of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game, it is sufficient to characterize the matrices whose corresponding tori are its solutions. Let $E_{i,j}$ denote $\alpha_1 \times \alpha_2$ matrix with 1 in the (i,j)-entry and 0 elsewhere. We define a matrix $J^{m_1 \times m_2}_{k_1,k_2}$ by

$$J^{m_1 \times m_2}_{k_1,k_2} = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} E_{k_1+i,k_2+j} \tag{1.2}$$

for integers m_i, k_i such that $1 \le m_i \le \alpha_i$ for $i = 1, 2$. For example, let $\alpha_1 = 6$, $\alpha_2 = 8$, $\beta_1 = 2$, $\beta_2 = 2$, and $n = 5$. Then

$$J^{2\times 2}_{1,2} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&0&1&1&0&0&0&0\\\hline 0&0&1&1&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array}, \qquad J^{3\times 4}_{5,5} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&0&0&0&0&1&1&1\\\hline 1&0&0&0&0&1&1&1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 1&0&0&0&0&1&1&1\\\hline\end{array}.$$

In particular, we denote $J_{k_1,k_2}^{\beta_1\times\beta_2}$ by J_{k_1,k_2}^*. The torus corresponding to A in $\mathcal{M}((\alpha_1,\alpha_2),n)$ is a solution of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game if the zero matrix O can be obtained by adding a linear combination of $J_{\cdot,\cdot}^*$ to A. That is, A is a solution of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game if and only if there exist $c_{i,j}\in\mathbb{Z}$ satisfying the system of linear equations

$$A+\sum_{j=0}^{\alpha_2-1}\sum_{i=0}^{\alpha_1-1}c_{i,j}J_{i,j}^*=O.$$

In this aspect, we call A a *solution matrix of the* $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$*-DR game.* We call matrix $(c_{i,j})$ a *solving coefficient matrix* of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game corresponding to A. We characterize the matrices in $\mathcal{M}((\alpha_1\times\alpha_2),n)$ which are solutions of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game in the following section. The main idea is to find certain good basic elements in the row space and then express a given element as the linear combination of them as well as some additional term.

2 Solution matrices of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game

In this section, we characterize the solution matrices of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game. Throughout this paper, for $i=1,2$, let g_i,r_i,s_i denote the integers such that

$$g_i=\gcd(\alpha_i,\beta_i),\qquad \alpha_i=r_ig_i,\qquad \beta_i=s_ig_i.$$

For each integer j, by the division algorithm, there exist integers u_j,w_j satisfying

$$j\equiv u_j\pmod{g_1}\quad 0\le u_j\le g_1-1,\qquad j\equiv w_j\pmod{g_2}\quad 0\le w_j\le g_2-1.$$

First note that for integers i and j,

$$\sum_{a=0}^{\beta_1-1}E_{u_a,j}=s_1\sum_{a=0}^{g_1-1}E_{a,j}\quad\text{and}\quad\sum_{b=0}^{\beta_2-1}E_{i,w_b}=s_2\sum_{b=0}^{g_2-1}E_{i,b}.\tag{2.1}$$

For integers i,j,k_1,k_2 such that $1\le k_1\le\alpha_1$ and $1\le k_2\le\alpha_2$, we define matrices

$$\begin{aligned}C_{i,j}^{k_1\times 1}&=J_{i,w_j}^{k_1\times 1}-J_{i,j}^{k_1\times 1},&(2.2)\\ R_{i,j}^{1\times k_2}&=J_{u_i,j}^{1\times k_2}-J_{i,j}^{1\times k_2},&(2.3)\\ Q_{i,j}&=E_{u_i,w_j}-E_{i,w_j}-E_{u_i,j}+E_{i,j}.&(2.4)\end{aligned}$$

Note that, by definitions,

$$\begin{aligned}C_{i,j}^{k_1\times 1}=O&\qquad\text{if } 0\le j\le g_2-1,&(2.5)\\ R_{i,j}^{1\times k_2}=O&\qquad\text{if } 0\le i\le g_1-1,&(2.6)\\ Q_{i,j}=O&\qquad\text{if } 0\le i\le g_1-1\text{ and }0\le j\le g_2-1.&(2.7)\end{aligned}$$

For example, consider the $((6,8);(2,2);5)$-DR game. Then $g_1 = \gcd(6,2) = 2$ and $g_2 = \gcd(8,2) = 2$. We take $i = 3$ and $j = 7$. Then $u_3 = 1$, $w_7 = 1$. Consider $C^{2\times 1}_{1,7}$, that is, by definition,

$$C^{2\times 1}_{1,7} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&1&0&0&0&0&0&-1\\\hline 0&1&0&0&0&0&0&-1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array}.$$

By the definition of $J^*_{i,j}$,

$$J^*_{1,1} - J^*_{1,2} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&1&0&-1&0&0&0&0\\\hline 0&1&0&-1&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array},$$

$$J^*_{1,3} - J^*_{1,4} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&1&0&-1&0&0\\\hline 0&0&0&1&0&-1&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array},$$

$$J^*_{1,5} - J^*_{1,6} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&1&0&-1\\\hline 0&0&0&0&0&1&0&-1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array}.$$

It can be checked that

$$C^{2\times 1}_{1,7} = (J^*_{1,1} - J^*_{1,2}) + (J^*_{1,3} - J^*_{1,4}) + (J^*_{1,5} - J^*_{1,6})$$

or

$$C^{2\times 1}_{1,7} + (J^*_{1,2} - J^*_{1,1}) + (J^*_{1,4} - J^*_{1,3}) + (J^*_{1,6} - J^*_{1,5}) = O.$$

In addition, it is true that

$$C^{2\times 1}_{2,7} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline 0&1&0&0&0&0&0&-1\\\hline 0&1&0&0&0&0&0&-1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array} = (J^*_{2,1} - J^*_{2,2}) + (J^*_{2,3} - J^*_{2,4}) + (J^*_{2,5} - J^*_{2,6}).$$

Therefore, $C_{1,7}^{2\times 1}$ and $C_{2,7}^{2\times 1}$ are solutions matrices of the $((6,8);(2,2);5)$-DR game. In addition, the matrix $Q_{3,7}$ can be represented by using $C_{1,7}^{2\times 1}$ and $C_{2,7}^{2\times 1}$, as

$$Q_{3,7} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0&0\\\hline 0&1&0&0&0&0&0&-1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&-1&0&0&0&0&0&1\\\hline 0&0&0&0&0&0&0&0\\\hline 0&0&0&0&0&0&0&0\\\hline\end{array} = C_{1,7}^{2\times 1} - C_{2,7}^{2\times 1}.$$

Since $C_{1,7}^{2\times 1}$ and $C_{2,7}^{2\times 1}$ are solution matrices, $Q_{3,7}$ is a solution matrix of the $((6,8);(2,2);5)$-DR game. In fact, the following lemma holds.

Lemma 2.1. *For any integers i and j, matrices $C_{i,j}^{\beta_1\times 1}$, $R_{i,j}^{1\times\beta_2}$, and $Q_{i,j}$ are solution matrices of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game.*

Proof. Since each of linear congruence equations $j-u_j \equiv x\beta_1 \pmod{\alpha_1}$ and $j-w_j \equiv x\beta_2 \pmod{\alpha_2}$ has a solution, there exist positive integers ζ_j and η_j satisfying that

$$j-u_j \equiv \zeta_j\beta_1 \pmod{\alpha_1} \quad \text{and} \quad j-w_j \equiv \eta_j\beta_2 \pmod{\alpha_2}. \tag{2.8}$$

To show that $C_{i,j}^{\beta_1\times 1}$ are $R_{i,j}^{1\times\beta_2}$ are solution matrices, it is sufficient to show

$$C_{i,j}^{\beta_1\times 1} = \sum_{m=0}^{\eta_j-1}\left(J^*_{i,w_j+m\beta_2} - J^*_{i,w_j+m\beta_2+1}\right), \tag{2.9}$$

$$R_{i,j}^{1\times\beta_2} = \sum_{m=0}^{\zeta_i-1}\left(J^*_{u_i+m\beta_1,j} - J^*_{u_i+m\beta_1+1,j}\right). \tag{2.10}$$

From the definition given in (1.2), it immediately follows that $J^*_{i,j} = \sum_{b=0}^{\beta_2-1} J^{\beta_1\times 1}_{i,j+b}$. Then (2.9) holds as

$$\begin{aligned}
\sum_{m=0}^{\eta_j-1}\left(J^*_{i,w_j+m\beta_2} - J^*_{i,w_j+m\beta_2+1}\right) &= \sum_{m=0}^{\eta_j-1}\left[\left(\sum_{b=0}^{\beta_2-1} J^{\beta_1\times 1}_{i,w_j+m\beta_2+b}\right) - \left(\sum_{b=0}^{\beta_2-1} J^{\beta_1\times 1}_{i,w_j+m\beta_2+b+1}\right)\right] \\
&= \sum_{m=0}^{\eta_j-1}\sum_{b=0}^{\beta_2-1}\left(J^{\beta_1\times 1}_{i,w_j+m\beta_2+b} - J^{\beta_1\times 1}_{i,w_j+m\beta_2+b+1}\right) \\
&= \sum_{m=0}^{\eta_j-1}(J^{\beta_1\times 1}_{i,w_j+m\beta_2} - J^{\beta_1\times 1}_{i,w_j+m\beta_2+\beta_2}) \\
&= \sum_{m=0}^{\eta_j-1}(J^{\beta_1\times 1}_{i,w_j+m\beta_2} - J^{\beta_1\times 1}_{i,w_j+(m+1)\beta_2}) \\
&= J^{\beta_1\times 1}_{i,w_j} - J^{\beta_1\times 1}_{i,w_j+\eta_j\beta_2} \\
&= J^{\beta_1\times 1}_{i,w_j} - J^{\beta_1\times 1}_{i,j} && \text{(by (2.8).)} \\
&= C^{\beta_1\times 1}_{i,j}.
\end{aligned}$$

Similarly, we can show that (2.10) is true.

To prove that $Q_{i,j}$ is a solution matrix, it is sufficient to show that $Q_{i,j}$ is a linear combination of $C_{i,j}^{\beta_1\times 1}$ which are solution matrices. Actually,

$$\begin{aligned}
&\sum_{m=0}^{\zeta_i-1}\left(C_{u_i+m\beta_1,j}^{\beta_1\times 1}-C_{u_i+m\beta_1+1,j}^{\beta_1\times 1}\right)\\
&=\sum_{m=0}^{\zeta_i-1}\left[\left(J_{u_i+m\beta_1,w_j}^{\beta_1\times 1}-J_{u_i+m\beta_1,j}^{\beta_1\times 1}\right)-\left(J_{u_i+m\beta_1+1,w_j}^{\beta_1\times 1}-J_{u_i+m\beta_1+1,j}^{\beta_1\times 1}\right)\right]\\
&=\sum_{m=0}^{\zeta_i-1}\left[\left(J_{u_i+m\beta_1,w_j}^{\beta_1\times 1}-J_{u_i+m\beta_1+1,w_j}^{\beta_1\times 1}\right)+\left(-J_{u_i+m\beta_1,j}^{\beta_1\times 1}+J_{u_i+m\beta_1+1,j}^{\beta_1\times 1}\right)\right]\\
&=\sum_{m=0}^{\zeta_i-1}\left[\left(E_{u_i+m\beta_1,w_j}-E_{u_i+(m+1)\beta_1,w_j}\right)+\left(-E_{u_i+m\beta_1,j}+E_{u_i+(m+1)\beta_1,j}\right)\right]\\
&=\sum_{m=0}^{\zeta_i-1}\left(E_{u_i+m\beta_1,w_j}-E_{u_i+(m+1)\beta_1,w_j}\right)+\sum_{m=0}^{\zeta_i-1}\left(-E_{u_i+m\beta_1,j}+E_{u_i+(m+1)\beta_1,j}\right)\\
&=\left(E_{u_i,w_j}-E_{u_i+\zeta_i\beta_i,w_j}\right)+\left(-E_{u_i,j}+E_{u_i+\zeta_i\beta_i,j}\right)\\
&=E_{u_i,w_j}-E_{i,w_j}-E_{u_i,j}+E_{i,j}.
\end{aligned}$$

The last equality holds by (2.8). $\square$

Now we define a function $\mathcal{T}:\mathcal{M}((\alpha_1\times\alpha_2),n)\to\mathcal{M}((\alpha_1\times\alpha_2),n)$ by

$$\mathcal{T}(A)=A-\sum_{j=0}^{\alpha_2-1}\sum_{i=0}^{\alpha_1-1}[A]_{i,j}Q_{i,j}. \tag{2.11}$$

By the definition, for a, b, $0\le a\le\alpha_1-1$, $0\le b\le\alpha_2-1$,

$$\mathcal{T}(E_{a,b})=E_{u_a,b}+E_{a,w_b}-E_{u_a,w_b}. \tag{2.12}$$

By (1.1), the following lemma immediately holds:

Lemma 2.2. *The function $\mathcal{T}$ on $\mathcal{M}((\alpha_1,\alpha_2),n)$ defined by (2.11) is a module homomorphism.*

From Lemma 2.1, we know that $Q_{i,j}$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game. Thus, the following lemma is true.

Lemma 2.3. *Matrix $A\in\mathcal{M}((\alpha_1,\alpha_2),n)$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game if and only if $\mathcal{T}(A)$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game.*

We define a function $\mathcal{S}$ from $\mathcal{M}((\alpha_1,\alpha_2),n)$ to $\mathcal{M}((\alpha_1,\alpha_2),n)$ by

$$\mathcal{S}(A)=\mathcal{T}(A)+\sum_{i=0}^{\alpha_1-1}[\mathcal{T}(A)]_{i,0}\,R_{i,0}^{1\times g_2}+\sum_{j=0}^{\alpha_2-1}[\mathcal{T}(A)]_{0,j}\,C_{0,j}^{g_1\times 1}. \tag{2.13}$$

Then, by (1.1) and Lemma 2.2, the following lemma holds:

Lemma 2.4. *The function $\mathcal{S}$ defined by (2.13) is a module homomorphism.*

The following theorem is our main result which gives a characterization of the solution matrices.

Theorem 2.5. *A matrix A in $\mathcal{M}((\alpha_1,\alpha_2),n)$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game if and only if*

$$\mathcal{T}(A)=s_1\sum_{j=0}^{\alpha_2-1}d_jJ_{0,j}^{g_1\times 1}+s_2\sum_{i=0}^{\alpha_1-1}e_iJ_{i,0}^{1\times g_2}-s_1s_2tJ_{0,0}^{g_1\times g_2}$$

for some d_j, e_i, $t\in\mathbb{Z}_n$ and, specifically,

$$\mathcal{S}(A)=s_1s_2tJ_{0,0}^{g_1\times g_2}.$$

Theorem 2.5 tells us that we can determine whether or not a given matrix A is a solution matrix by computing $\mathcal{T}(A)$ and $\mathcal{S}(A)$. Furthermore we have an $O(\alpha_1\alpha_2)$-algorithm to determine if a matrix is a solution matrix. By Theorem 2.5, we need to compute $\mathcal{T}$ and $\mathcal{S}$ given in (2.11) and check two equalities given in the theorem. The algorithm for computing the function $\mathcal{T}$ given in (2.11) iterates $\alpha_1\alpha_2$ times as each $Q_{i,j}$ has at most 4 nonzero components. Thus the time complexity for computing $\mathcal{T}$ is $O(\alpha_1\alpha_2)$. Similarly, the algorithm for computing the function $\mathcal{S}$ iterates α_1 times for each $R_{i,0}^{1\times g_2}$ as it has at most $2\alpha_2$ components, and iterates α_2 times for each $C_{0,j}^{g_1\times 1}$ as it has at most $2\alpha_1$ components. Therefore the complexity for computing $\mathcal{S}$ is $O(\alpha_1\alpha_2)$. Now it remains to check two equalities hold. This involves scanning the nonzero components and solving congruence equations in the form of $pqt=r \pmod n$ or $pq=r \pmod n$ where at least one of p, q is s_1 or s_2. This also takes $O(\alpha_1\alpha_2)$. Hence the time complexity to determine if the matrix is a solution matrix is $O(\alpha_1\alpha_2)$.

In the rest of section, we devote ourselves to prove Theorem 2.5. To do so, we need several lemmas.

Lemma 2.6. *For any integers i and j, matrices $s_2C_{0,j}^{g_1\times 1}$ and $s_1R_{i,0}^{1\times g_2}$ are solution matrices of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game.*

Proof. By (2.2), (2.12), and Lemma 2.2,

$$\begin{aligned}
\mathcal{T}(C^{\beta_1\times 1}_{0,j}) &= \mathcal{T}\left(J^{\beta_1\times 1}_{0,w_j} - J^{\beta_1\times 1}_{0,j}\right) = \mathcal{T}\left(\sum_{a=0}^{\beta_1-1}\left(E_{a,w_j}-E_{a,j}\right)\right) = \sum_{a=0}^{\beta_1-1}\left(\mathcal{T}(E_{a,w_j})-\mathcal{T}(E_{a,j})\right)\\
&= \sum_{a=0}^{\beta_1-1}\left[(E_{u_a,w_j}+E_{a,w_j}-E_{u_a,w_j})-(E_{u_a,j}+E_{a,w_j}-E_{u_a,w_j})\right]\\
&= \sum_{a=0}^{\beta_1-1}(E_{u_a,w_j}-E_{u_a,j}) \underset{\text{by (2.1)}}{=} s_1\sum_{a=0}^{g_1-1}(E_{a,w_j}-E_{a,j}) = s_1(J^{g_1\times 1}_{0,w_j}-J^{g_1\times 1}_{0,j}) = s_1 C^{g_1\times 1}_{0,j}.
\end{aligned}$$

Similarly, we can show that

$$\mathcal{T}(R^{1\times\beta_2}_{i,0}) = s_2 R^{1\times g_2}_{i,0}.$$

As $\mathcal{T}(C^{\beta_1\times 1}_{0,j})$ and $\mathcal{T}(R^{1\times\beta_2}_{i,0})$ are solution matrices by Lemmas 2.1 and 2.3, the lemma follows. □

Lemma 2.7. *For any two integers i, j,*

$$\mathcal{T}(J^*_{i,j}) \;=\; s_1 J^{g_1\times\beta_2}_{0,j} + s_2 J^{\beta_1\times g_2}_{i,0} - s_1 s_2 J^{g_1\times g_2}_{0,0}.$$

Proof. By Lemma 2.2,

$$\begin{aligned}
\mathcal{T}(J^*_{i,j}) &= \mathcal{T}\left(\sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1} E_{a,b}\right) = \sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1}\mathcal{T}(E_{a,b}) && \text{(by Lemma 2.2)}\\
&= \sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1}(E_{u_a,b}+E_{a,w_b}-E_{u_a,w_b}) && \text{(by (2.12))}\\
&= \sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1}E_{u_a,b} + \sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1}E_{a,w_b} - \sum_{b=j}^{j+\beta_2-1}\sum_{a=i}^{i+\beta_1-1}E_{u_a,w_b}\\
&= \sum_{b=j}^{j+\beta_2-1}\left(s_1\sum_{a=0}^{g_1-1}E_{a,b}\right) + \sum_{a=i}^{i+\beta_1-1}\left(s_2\sum_{b=0}^{g_2-1}E_{a,b}\right) - s_1 s_2\sum_{b=0}^{g_2-1}\sum_{a=0}^{g_1-1}E_{a,b} && \text{(by (2.1))}\\
&= s_1\sum_{b=j}^{j+\beta_2-1}\sum_{a=0}^{g_1-1}E_{a,b} + s_2\sum_{a=i}^{i+\beta_1-1}\sum_{b=0}^{g_2-1}E_{a,b} - s_1 s_2\sum_{b=0}^{g_2-1}\sum_{a=0}^{g_1-1}E_{a,b}\\
&= s_1 J^{g_1\times\beta_2}_{0,j} + s_2 J^{\beta_1\times g_2}_{i,0} - s_1 s_2 J^{g_1\times g_2}_{0,0}.
\end{aligned}$$

Therefore, the lemma holds. □

Lemma 2.8. *For any integers i, j, $\mathcal{S}(J^*_{i,j})$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game and*

$$\mathcal{S}(J^*_{i,j}) \;=\; s_1 s_2 J^{g_1\times g_2}_{0,0}.$$

Proof. First, we will show that $\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j}))\right]_{a,0} R^{1\times g_2}_{a,0}$ is a solution matrix and that

$$\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j}))\right]_{a,0} R^{1\times g_2}_{a,0} = s_2\left(s_1 J^{g_1\times g_2}_{0,0} - J^{\beta_1\times g_2}_{i,0}\right). \tag{2.14}$$

Consider the case where $\alpha_1 = \beta_1$. Then $g_1 = \gcd(\alpha_1, \beta_1) = \alpha_1$ and $s_1 = 1$. Then by (2.6), it holds that

$$\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j})\right]_{a,0} R^{1\times g_2}_{a,0} = O$$

and so $\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j}))\right]_{a,0} R^{1\times g_2}_{a,0}$ is a trivial solution matrix. On the other hand, since $s_1 = 1$ and $g_1 = \beta_1$,

$$s_2\left(s_1 J^{g_1\times g_2}_{0,0} - J^{\beta_1\times g_2}_{i,0}\right) = s_2\left(J^{g_1\times g_2}_{0,0} - J^{g_1\times g_2}_{0,0}\right)$$

and so the equality (2.14) holds.

Now consider the case where $\alpha_1 > \beta_1$. Then $g_1 < \alpha_1$. Let $X = \{0, 1, 2, \dots, g_1 - 1\}$, $Y = \{g_1, \dots, \alpha_1 - 1\}$, and $Z = \{i, i+1 \dots, i+\beta_1-1\}$. By Lemma 2.7, for a, $0 \le a \le \alpha_1 - 1$,

$$[\mathcal{T}(J^*_{i,j})]_{a,0} = s_1\left[J^{g_1\times\beta_2}_{0,j}\right]_{a,0} + s_2\left[J^{\beta_1\times g_2}_{i,0}\right]_{a,0} - s_1 s_2\left[J^{g_1\times g_2}_{0,0}\right]_{a,0}.$$

For $a \in Y$, $s_1\left[J^{g_1\times\beta_2}_{0,j}\right]_{a,0} = 0$ and $\left[J^{g_1\times g_2}_{0,0}\right]_{a,0} = 0$. Thus

$$[\mathcal{T}(J^*_{i,j})]_{a,0} = \begin{cases} s_2 & \text{if } a \in Y \cap Z \\ 0 & \text{if } Y \setminus Z \end{cases}. \tag{2.15}$$

Then

$$\begin{aligned}\sum_{a\in X}[\mathcal{T}(J^*_{i,j})]_{a,0} R^{1\times g_2}_{a,0} &\underset{\text{by (2.6)}}{=} \sum_{a\in Y}[\mathcal{T}(J^*_{i,j})]_{a,0} R^{1\times g_2}_{a,0} \underset{\text{by (2.15)}}{=} \sum_{a\in Y\cap Z} s_2 R^{1\times g_2}_{a,0} \\ &\underset{\text{by (2.6)}}{=} \sum_{a\in Y\cap Z} s_2 R^{1\times g_2}_{a,0} + \sum_{a\in X\cap Z} s_2 R^{1\times g_2}_{a,0} = \sum_{a\in Z} s_2 R^{1\times g_2}_{a,0}.\end{aligned}$$

Therefore,

$$\sum_{a\in X}[\mathcal{T}(J^*_{i,j})]_{a,0} R^{1\times g_2}_{a,0} = \sum_{a=0}^{\alpha_1-1}[\mathcal{T}(J^*_{i,j})]_{a,0} R^{1\times g_2}_{a,0} = \sum_{a\in Z} s_2 R^{1\times g_2}_{a,0} = \sum_{a=i}^{i+\beta_1-1} s_2 R^{1\times g_2}_{a,0}$$

and so $\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j}))\right]_{a,0} R^{1\times g_2}_{a,0}$ is a solution matrix by Lemma 2.6. In addition,

$$\begin{aligned}\sum_{a=i}^{i+\beta_1-1} s_2 R^{1\times g_2}_{a,0} &= s_2 \sum_{a=i}^{i+\beta_1-1} (J^{1\times g_2}_{u_a,0} - J^{1\times g_2}_{a,0}) = s_2 \sum_{a=i}^{i+\beta_1-1}\left(\sum_{b=0}^{g_2-1}(E_{u_a,b}-E_{a,b})\right)\\ &= s_2 \sum_{b=0}^{g_2-1}\sum_{a=i}^{i+\beta_1-1}(E_{u_a,b}-E_{a,b}) = s_2\left(\sum_{b=0}^{g_2-1}\sum_{a=i}^{i+\beta_1-1} E_{u_a,b} - \sum_{b=0}^{g_2-1}\sum_{a=i}^{i+\beta_1-1} E_{a,b}\right)\\ &= s_2\left(s_1\sum_{b=0}^{g_2-1}\sum_{a=0}^{g_1-1}E_{a,b} - J^{\beta_1\times g_2}_{i,0}\right) = s_2\left(s_1 J^{g_1\times g_2}_{0,0} - J^{\beta_1\times g_2}_{i,0}\right).\end{aligned}$$

Therefore, (2.14) holds. Similarly, we can show that $\sum_{b=0}^{\alpha_2-1}\left[\mathcal{T}(J^*_{i,j})\right]_{0,b} C^{g_1\times 1}_{0,b}$ is a solution matrix and that

$$\sum_{b=0}^{\alpha_2-1}\left[\mathcal{T}(J^*_{i,j})\right]_{0,b} C^{g_1\times 1}_{0,b} = s_1\left(s_2 J^{g_1\times g_2}_{0,0} - J^{g_1\times \beta_2}_{0,j}\right). \tag{2.16}$$

By the definition of $\mathcal{S}$ given in (2.13),

$$\mathcal{S}(J^*_{i,j}) = \mathcal{T}(J^*_{i,j}) + \sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j})\right]_{a,0} R^{1\times g_2}_{a,0} + \sum_{b=0}^{\alpha_2-1}\left[\mathcal{T}(J^*_{i,j})\right]_{0,b} C^{g_1\times 1}_{0,b}.$$

Note that both $\sum_{a=0}^{\alpha_1-1}\left[\mathcal{T}(J^*_{i,j})\right]_{a,0} R^{1\times g_2}_{a,0}$ and $\sum_{b=0}^{\alpha_2-1}\left[\mathcal{T}(J^*_{i,j})\right]_{0,b} C^{g_1\times 1}_{0,b}$ are solution matrices. Furthermore, by Lemma 2.3, $\mathcal{T}(J^*_{i,j})$ is a solution matrix. Therefore, $\mathcal{S}(J^*_{i,j})$ is a solution matrix. On the other hand, from (2.14), (2.16) and Lemma 2.7, it follows that

$$\begin{aligned}\mathcal{S}(J^*_{i,j}) &= s_1 J^{g_1\times\beta_2}_{0,j} + s_2 J^{\beta_1\times g_2}_{i,0} - s_1 s_2 J^{g_1\times g_2}_{0,0} + s_2\big(s_1 J^{g_1\times g_2}_{0,0} - J^{\beta_1\times g_2}_{i,0}\big) + s_1\big(s_2 J^{g_1\times g_2}_{0,0} - J^{g_1\times\beta_2}_{0,j}\big)\\ &= s_1 s_2 J^{g_1\times g_2}_{0,0}.\end{aligned}$$

Hence the lemma holds. □

Now we are ready to prove the main result:

Proof of Theorem 2.5. Suppose that a matrix A in $\mathcal{M}((\alpha_1,\alpha_2),n)$ is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game. By definition, there exists a solving coefficient matrix $(c_{i,j})$ of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game corresponding to A. That is,

$$A = -\sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1} c_{a,b} J^*_{a,b}.$$

Then

$$\begin{aligned}
\mathcal{T}(A) &= \sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})\mathcal{T}(J^*_{a,b}) && \text{(by Lemma 2.2)}\\
&= \sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})\left(s_1J_{0,b}^{g_1\times\beta_2}+s_2J_{a,0}^{\beta_1\times g_2}-s_1s_2J_{0,0}^{g_1\times g_2}\right) && \text{(by Lemma 2.7)}\\
&= \sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})\left(s_1\sum_{j=b}^{b+\beta_2-1}J_{0,j}^{g_1\times 1}+s_2\sum_{i=a}^{a+\beta_1-1}J_{i,0}^{1\times g_2}-s_1s_2J_{0,0}^{g_1\times g_2}\right)\\
&= s_1\sum_{b=0}^{\alpha_2-1}\sum_{j=b}^{b+\beta_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})J_{0,j}^{g_1\times 1}+s_2\sum_{a=0}^{\alpha_1-1}\sum_{i=a}^{a+\beta_1-1}\sum_{b=0}^{\alpha_2-1}(-c_{a,b})J_{i,0}^{1\times g_2}\\
&\quad - s_1s_2\sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})J_{0,0}^{g_1\times g_2}.
\end{aligned}$$

It is not difficult to check that

$$\begin{aligned}
s_1\sum_{b=0}^{\alpha_2-1}\sum_{j=b}^{b+\beta_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})J_{0,j}^{g_1\times 1} &= s_1\sum_{j=0}^{\alpha_2-1}\sum_{b=j-\beta_2+1}^{j}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})J_{0,j}^{g_1\times 1} = s_1\sum_{j=0}^{\alpha_2-1}d_jJ_{0,j}^{g_1\times 1}\\
s_2\sum_{a=0}^{\alpha_1-1}\sum_{i=a}^{a+\beta_1-1}\sum_{b=0}^{\alpha_2-1}(-c_{a,b})J_{i,0}^{1\times g_2} &= s_2\sum_{i=0}^{\alpha_1-1}\sum_{a=i-\beta_1+1}^{i}\sum_{b=0}^{\alpha_2-1}(-c_{a,b})J_{i,0}^{1\times g_2} = s_2\sum_{a=0}^{\alpha_1-1}e_iJ_{i,0}^{1\times g_2},
\end{aligned}$$

where $d_j=\sum_{b=j-\beta_2+1}^{j}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})$ and $e_i=\sum_{a=i-\beta_1+1}^{i}\sum_{b=0}^{\alpha_2-1}(-c_{a,b})$. Thus we obtain

$$\mathcal{T}(A)=s_1\sum_{j=0}^{\alpha_2-1}d_jJ_{0,j}^{g_1\times 1}+s_2\sum_{a=0}^{\alpha_1-1}e_iJ_{i,0}^{1\times g_2}-s_1s_2tJ_{0,0}^{g_1\times g_2}$$

where $t=\sum_{b=0}^{\alpha_2-1}\sum_{a=0}^{\alpha_1-1}(-c_{a,b})$. On the other hand, by Lemmas 2.4 and 2.8,

$$\begin{aligned}
\mathcal{S}(A) &= \sum_{j=0}^{\alpha_2-1}\sum_{i=0}^{\alpha_1-1}(-c_{i,j})\mathcal{S}(J^*_{i,j}) = \sum_{j=0}^{\alpha_2-1}\sum_{i=0}^{\alpha_1-1}(-c_{i,j})s_1s_2J_{0,0}^{g_1\times g_2}\\
&= \left[\sum_{j=0}^{\alpha_2-1}\sum_{i=0}^{\alpha_1-1}(-c_{i,j})\right]s_1s_2J_{0,0}^{g_1\times g_2} = s_1s_2tJ_{0,0}^{g_1\times g_2}.
\end{aligned}$$

Thus the 'only if' part is true. To show the converse, assume that

$$\mathcal{T}(A)=s_1\sum_{j=0}^{\alpha_2-1}d_jJ_{0,j}^{g_1\times 1}+s_2\sum_{i=0}^{\alpha_1-1}e_iJ_{i,0}^{1\times g_2}-s_1s_2tJ_{0,0}^{g_1\times g_2} \tag{2.17}$$

and $\mathcal{S}(A) = s_1 s_2 t J_{0,0}^{g_1\times g_2}$ for some $d_j, e_i, t \in \mathbb{Z}_n$. Then, by the definition of $\mathcal{S}$ given in (2.13),

$$s_1 s_2 t J_{0,0}^{g_1\times g_2} = \mathcal{T}(A) + \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} R_{i,0}^{1\times g_2} + \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} C_{0,j}^{g_1\times 1},$$

or

$$\mathcal{T}(A) = s_1 s_2 t J_{0,0}^{g_1\times g_2} - \sum_{i=0}^{\alpha_1-1} [\mathcal{T}(A)]_{i,0} R_{i,0}^{1\times g_2} - \sum_{j=0}^{\alpha_2-1} [\mathcal{T}(A)]_{0,j} C_{0,j}^{g_1\times 1}.$$

The first term $s_1 s_2 t J_{0,0}^{g_1\times g_2}$ of the right hand side of above inequality is equal to $t\mathcal{S}(J_{i,j}^*)$ for some integers i, j and so it is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game by Lemma 2.8. By (2.5), (2.6) and (2.17), the matrix $[\mathcal{T}(A)]_{i,0} R_{i,0}^{1\times g_2}$ is equal to either O or $s_2 e_i R_{i,0}^{1\times g_2}$, and the matrix $[\mathcal{T}(A)]_{0,j} C_{0,j}^{g_1\times 1}$ is equal to either O or $s_1 d_j C_{0,j}^{g_1\times 1}$, depending on whether or not $0 \le i \le g_1 - 1$, $0 \le j \le g_2 - 1$. Therefore, by Lemma 2.6, both $[\mathcal{T}(A)]_{i,0} R_{i,0}^{1\times g_2}$ and $[\mathcal{T}(A)]_{0,j} C_{0,j}^{g_1\times 1}$ are solution matrices. Thus we can conclude that $\mathcal{T}(A)$ is a solution matrix. By Lemma 2.3, A is a solution matrix of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game. □

3 Closing remarks

In this paper, we give a necessary and sufficient condition for a torus to be a solution of the $((\alpha_1,\alpha_2);(\beta_1,\beta_2);n)$-DR game for positive integers α_1, α_2, n, β_1, β_2 such that $\beta_1 \le \alpha_1$ and $\beta_2 \le \alpha_2$.

When $\alpha_1 = \beta_1 = 1$, a matrix in $\mathcal{M}((\alpha_1,\alpha_2),n)$ becomes an α_2-tuple, so $\mathcal{D}((1,\alpha_2),n)$ is the set of circles on each of which α_2 n-dice are located. By using the results obtained in the previous section, we can characterize a circle which is a solution of dice rolling game on the set $\mathcal{D}((1,\alpha_2),n)$ with respect to $(1,\beta_2)$-rolling procedures.

Suppose that $\alpha_1 = \beta_1$. Then $g_1 = \gcd(\alpha_1,\beta_1) = \alpha_1$ and so $s_1 = 1$. Then $0 \le i \le \alpha_1 - 1$ if an only if $0 \le i \le g_1 - 1$. By (2.7) and the definition of $\mathcal{T}$ given in (2.11), if $0 \le i \le g_1 - 1$ or $0 \le j \le g_2 - 1$, then $\mathcal{T}$ is the identity function, that is, $\mathcal{T}(A) = A$. Since $\alpha_1 = g_1$, $R_{i,0}^{1\times g_2} = O$ by (2.6) and so

$$\mathcal{S}(A) = A + \sum_{j=0}^{\alpha_2-1} [A]_{0,j} C_{0,j}^{\alpha_1\times 1}. \tag{3.1}$$

Hence a characterization of a solution matrix in $\mathcal{M}((1,\alpha_2),n)$ immediately follows from Theorem 2.5:

A matrix A is a solution matrix of the $((\alpha_1,\alpha_2);(\alpha_1,\beta_2);n)$-DR game if and only if

$$A = \sum_{j=0}^{\alpha_2-1} d_j J_{0,j}^{\alpha_1\times 1} + s_2 \sum_{i=0}^{\alpha_1-1} e_i J_{i,0}^{1\times g_2} - s_2 t J_{0,0}^{\alpha_1\times g_2} \tag{3.2}$$

for some d_j, e_i, $t \in \mathbb{Z}_n$ and $\mathcal{S}(A) = u s_2 J_{0,0}^{\alpha_1\times g_2}$ for some $u \in \mathbb{Z}_n$.

However, if $\mathcal{S}(A) = us_2 J_{0,0}^{\alpha_1 \times g_2}$ for some $u \in \mathbb{Z}_n$, then it holds that

$$\begin{aligned}
A &= -\mathcal{S}(A) + \sum_{j=0}^{\alpha_2-1} [A]_{0,j} C_{0,j}^{\alpha_1 \times 1} && \text{(by (3.1))} \\
&= -us_2 J_{0,0}^{\alpha_1 \times g_2} + \sum_{j=0}^{\alpha_2-1} [A]_{0,j} C_{0,j}^{\alpha_1 \times 1} \\
&= -us_2 \sum_{j=0}^{g_2-1} J_{0,j}^{\alpha_1 \times 1} + \sum_{j=0}^{\alpha_2-1} [A]_{0,j} (J_{0,w_j}^{g_1 \times 1} - J_{0,j}^{\alpha_1 \times 1}) && \text{(by (2.2))} \\
&= \sum_{j=0}^{g_2-1} (-us_2) J_{0,j}^{\alpha_1 \times 1} + \sum_{j=0}^{\alpha_2-1} \Big([A]_{0,j} J_{0,w_j}^{\alpha_1 \times 1} - [A]_{0,j} J_{0,j}^{\alpha_1 \times 1} \Big) \\
&= \sum_{j=0}^{\alpha_2-1} f_j J_{0,j}^{\alpha_1 \times 1} = s_2 \sum_{i=0}^{\alpha_1-1} 0 \cdot J_{i,0}^{1 \times g_2} + \sum_{j=0}^{\alpha_2-1} f_j J_{0,j}^{\alpha_1 \times 1} - s_2 \cdot 0 \cdot J_{0,0}^{\alpha_1 \times g_2}
\end{aligned}$$

for some $f_j \in \mathbb{Z}_n$. Thus (3.2) is true if $\mathcal{S}(A) = us_2 J_{0,0}^{\alpha_1 \times g_2}$ for some $u \in \mathbb{Z}_n$. Hence the characterization of a solution matrix in $\mathcal{M}((1, \alpha_2), n)$ given above can be simplified as follows:

Corollary 3.1. *A matrix A is a solution matrix of the $((\alpha_1, \alpha_2); (\alpha_1, \beta_2); n)$-DR game if and only if*

$$\mathcal{S}(A) = us_2 J_{0,0}^{\alpha_1 \times g_2}$$

for some $u \in \mathbb{Z}_n$.

When $\alpha_1 = \beta_1 = 1$, the above proposition can be stated as:

> *An ordered α_2-tuple $\mathbf{v}$ is a solution matrix of the $((1, \alpha_2); (1, \beta_2); n)$-DR game if and only if $\mathcal{S}(\mathbf{v}) = (\underbrace{us_2, \ldots, us_2}_{g_2}, 0, \ldots, 0)$ for some $u \in \mathbb{Z}_n$.*

which characterizes the solution set of the dice rolling game on a set of circles.

For the case where $\alpha_2 = \beta_2$, we can give a similar argument. We believe that our characterization for the 2-dimensional case can be generalized to the t-dimensional case for $t \geq 1$ if we can find a way to manipulate notations more effectively.

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