A Combinatorial Proof of the Recurrence for Rook Paths

Emma Yu Jin* and Markus E. Nebel†

Department of Computer Science
University of Kaiserslautern, Germany

{jin,nebel}@cs.uni-kl.de

Submitted: Feb 23, 2012; Accepted: Mar 5, 2012; Published: Mar 19, 2012
Mathematics Subject Classifications: 05A19, 68R05

Abstract

Let a_n count the number of 2-dimensional rook paths $\mathcal{R}_{n,n}$ from $(0,0)$ to $(2n,0)$. Rook paths $\mathcal{R}_{m,n}$ are the lattice paths from $(0,0)$ to $(m+n,m-n)$ with allowed steps (x,x) and $(y,-y)$ where $x,y \in \mathbb{N}^+$. In answer to the open question proposed by M. Erickson *et al.* (2010), we shall present a combinatorial proof for the recurrence of a_n, i.e., $(n+1)a_{n+1} + 9(n-1)a_{n-1} = 2(5n+2)a_n$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Furthermore, our proof can be extended to show the recurrence for the number of multiple Dyck paths d_n, i.e., $(n+2)d_{n+1} + 9(n-1)d_{n-1} = 5(2n+1)d_n$ with $d_0 = 1$ and $d_1 = 1$, where $d_n = \mathcal{N}_n(4)$ and $\mathcal{N}_n(x)$ is Narayana polynomial.

1 Introduction

The set of rook paths $\mathcal{R}_{m,n}$ is given by all 2-dimensional lattice paths from $(0,0)$ to $(m+n,m-n)$ with allowed steps (x,x) and $(y,-y)$ where $x,y \in \mathbb{N}^+$. They owe their name to the following view: consider a rook placed on the lower left corner $(0,0)$ of a chess board of size $m \times n$. Then $\mathcal{R}_{m,n}$ counts the number of ways the rook can reach the opposite corner if in each single step it can move an arbitrary number of squares in the horizontal or vertical direction. Generalizations to d-dimensional chess boards are available [3]. There the number of corresponding paths can also be interpreted as the number of ways a debitor can repay all of its d creditors if the extension of the chess board in dimension i equals the specific amount owing creditor i.

*The work of this author has been supported by the Alexander von Humboldt Foundation by a postdoctoral research fellowship.

†Author to whom correspondence should be addressed.

We set $|\mathcal{R}_{m,n}| = a_{m,n}$ and $\mathcal{R}_n = \mathcal{R}_{n,n}$ for simplicity. M. Erickson *et al.* [1] gave the following generating function $f(s,t)$ for the number of rook paths:

$$f(s,t) = \sum_{m\geq 0, n\geq 0} a_{m,n} s^m t^n = \frac{(1-s)(1-t)}{1-2(s+t)+3st}. \tag{1}$$

For the special case $a_n = a_{n,n}$ the generating function $f(x) = \sum_{n\geq 0} a_n x^n$ can be derived from eq. (1) by first substituting $t = \frac{x}{s}$ and then finding the coefficient of s^0 using partial fractions and Laurent series [1, 6]. Explicitly, the double generating function $f(s,t)$ is rational and it converges for sufficiently small $|s|$ and $|t|$. As a result, $f(x) = [s^0]F(s, \frac{x}{s})$ converges for $|x|$ small. We shall fix a small x; then the series $f(s,t)$, as a function of s, converges in some circle $|s| = r > 0$. According to the Cauchy integral theorem and the Residue theorem, we have

$$f(x) = [s^0] f\left(s, \frac{x}{s}\right) = \frac{1}{2\pi i}\int_{|s|=r} f\left(s, \frac{x}{s}\right)\frac{ds}{s} = \sum_{S_0} \mathrm{Res}\left[f\left(s, \frac{x}{s}\right)\cdot\frac{1}{s}\right]$$

where S_0 is the set of all the poles of $\frac{1}{s}f(s, \frac{x}{s})$ inside the circle $|s| = r$ and for any pole $s^* \in S_0$, $\lim_{x\to 0} s^* = 0$. Therefore we arrive at

$$\begin{aligned} f(x) &= \sum_{S_0} \mathrm{Res}\left[f\left(s, \frac{x}{s}\right)\cdot\frac{1}{s}\right] \\ &= \frac{1}{2\pi i}\int_{|s|=r} \frac{(1-s)(1-\frac{x}{s})}{1-2(s+\frac{x}{s})+3x}\cdot\frac{ds}{s}. \end{aligned}$$

The only pole of $\frac{1}{s}f(s, \frac{x}{s})$ approaching 0 as $x \to 0$ is at $s_0 = \frac{3}{4}x + \frac{1}{4} - \frac{1}{4}\sqrt{9x^2 - 10x + 1}$. Consequently,

$$f(x) = \mathrm{Res}_{s=s_0}\left[\frac{(1-s)(1-\frac{x}{s})}{1-2(s+\frac{x}{s})+3x}\cdot\frac{1}{s}\right] + \frac{1}{2} = \frac{1}{2}\left(1 + \sqrt{\frac{1-x}{1-9x}}\right) \tag{2}$$

by taking the initial condition $f(0) = 1$ into account. The recurrence relation for $a_n = [x^n]f(x)$ can be derived from eq. (2)[1], i.e.,

$$\begin{aligned} &a_0 = 1, a_1 = 2, \\ &a_n = \frac{n+1}{2(5n+2)}a_{n+1} + \frac{9(n-1)}{2(5n+2)}a_{n-1}. \end{aligned} \tag{3}$$

The set of multiple Dyck paths $\mathcal{MD}_n$ is given by the 2-dimensional lattice paths from $(0,0)$ to $(2n,0)$ with allowed steps (x,x) and $(y,-y)$ where $x, y \in \mathbb{N}^+$ that never go below x-axis. $|\mathcal{MD}_n| = d_n$. Similarly the generating function of d_n is given by

$$g(x) = \sum_{n\geq 0} d_n x^n = \frac{1}{8}\left(3 + \frac{1}{x} - \frac{1}{x}\sqrt{(1-x)(1-9x)}\right), \tag{4}$$

and the recurrence for d_n is

$$d_0 = 1, d_1 = 1,$$
$$d_n = \frac{n+2}{5(2n+1)} d_{n+1} + \frac{9(n-1)}{5(2n+1)} d_{n-1}. \tag{5}$$

In the next section we shall present the combinatorial proof of eq. (3). Eq. (5) follows analogously. Part of the proof employs the bijective proof of recurrence for Dyck paths [2, 5, 7].

Furthermore, we refer the readers to the related combinatorial proof of some famous numbers, like the recurrence for the Delannoy numbers was proved bijectively by P. Peart *et al.*[4], the recurrence for the large Schröder numbers was shown by Foata *et al.*[2], and R.A. Sulanke [7] independently.

2 The Main Proof

We can represent each rook path by the combination of U (i.e., up-step) and D (i.e., down-step) with multiplicities. Accordingly, let $U^i = \underbrace{\underline{U \cdots U}}_{i\text{-times}}$ (resp. $D^i = \underbrace{\underline{D \cdots D}}_{i\text{-times}}$) be a single up-step (resp. down-step) of length i (the underline is used to represent the step's *atomicity*). For our proof we will count the occurrence of UU, UD, DU and DD ignoring the underlines.

Let $S_1S_2 \uparrow$ ($S_1S_2 \downarrow$) denote the steps S_1S_2 above x-axis (resp. below x-axis) where $S_i \in \{U, D\}$ for $i = 1, 2$ subject to the condition that the y-the coordinates of both, the start point and the end point of S_1, are non-negative (resp. non-positive). Let m_{11} count the occurrence of $UU \uparrow$ and $DD \downarrow$, m_{22} count the occurrence of $DD \uparrow$ and $UU \downarrow$, m_{12} count the occurrence of $UD \uparrow$ and $DU \downarrow$, and m_{21} count the occurrence of $DU \uparrow$ and $UD \downarrow$. For example, $m_{11}, m_{12}, m_{21}, m_{22}$ of the rook path $\underline{UU}D\underline{DD}U\underline{UUU}D\underline{DD}$ is $4, 3, 0, 4$. $\sum_{i,j=1,2} m_{ij} = 2n - 1$ holds for each path $\mathcal{M} \in \mathcal{R}_n$.

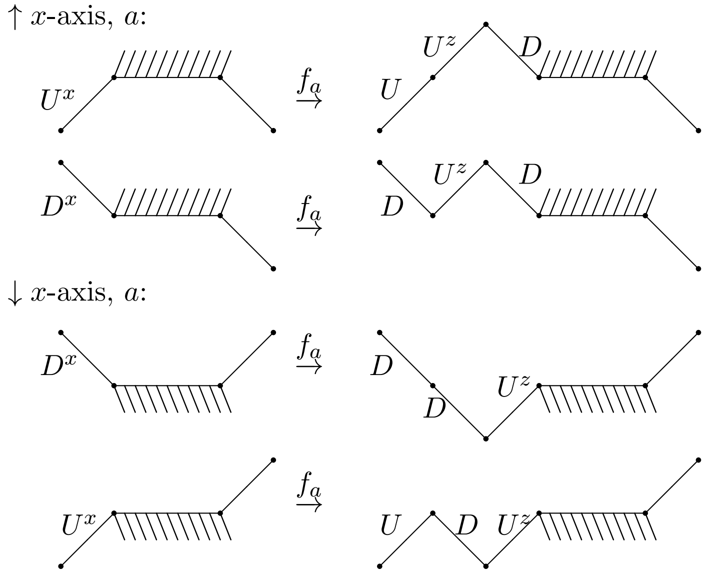
Here is the sketch of the idea of the combinatorial proof. $(n+1)a_{n+1}$ counts the family of rook paths $\mathcal{R}_{n+1}$ having one unit U-step labeled with z, denoted by $(U^z, \mathcal{R}_{n+1})$. In Lemma 1 we focus on the map from $(U^z, \mathcal{R}_{n+1})$ to $(s^x \cup \varnothing, \mathcal{R}_n, \{a, b\})$ with multiplicity on each patten U^xU, $\underline{U^xU}$ and D^xD, $\underline{D^xD}$, and U^xD, D^xU. In Lemma 2, $3(n-1)a_{n-1}$ counts the family of $(U^z, \mathcal{R}_{n-1}, [i, j])$ where $[i, j] = [1, 2], [2, 1], [2, 2]$ and having one unit U-step labeled with z, accordingly we present map from $(U^z, \mathcal{R}_{n-1}, [i, j])$ where $[i, j] = [1, 2], [2, 1], [2, 2]$ to $(U^x, \mathcal{R}_n, \{s, t\})$ with multiplicity on each pattern s^xU and s^xD for $s = U$ or D.

Lemma 1. *For each $\mathcal{M} \in \mathcal{R}_n$, let m_{11} count the occurrence of $UU \uparrow$ and $DD \downarrow$, m_{22} count the occurrence of $DD \uparrow$ and $UU \downarrow$, m_{12} count the occurrence of $UD \uparrow$ and $DU \downarrow$, and m_{21} count the occurrence of $DU \uparrow$ and $UD \downarrow$. Then for $a_{n+1} = |\mathcal{R}_{n+1}|$ we have*

$$(n+1)a_{n+1} = \sum_{\mathcal{M} \in \mathcal{R}_n} 5(m_{11} + m_{21}) + 8m_{12} + 2m_{22} + 6 \tag{6}$$

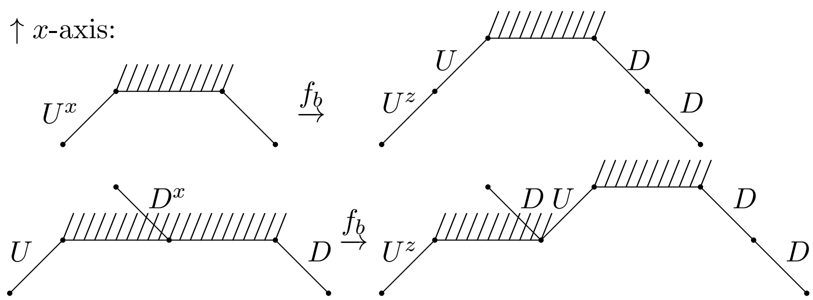
Proof. Let $\mathcal{F}_n$ be the family of free Dyck paths, i.e., the lattice paths from $(0,0)$ to $(2n,0)$ with allowed steps $(1,1)$ and $(1,-1)$, and pair $(U^z, \mathcal{F}_{n+1})$ be the set of paths $\mathcal{F}_{n+1}$ having one of the unit up-steps U labeled with z. Tuple $(s^x \cup \varnothing, \mathcal{F}_n, \{a,b\})$ represents the set of paths $\mathcal{F}_n$ having one of the unit steps s (U or D) labeled with x, or having nothing labeled. We set $f_i(\mathcal{F}_n) = f(s^x \cup \varnothing, \mathcal{F}_n, \{i\})$ for $i = a, b$. In the sequel we will construct a bijection $f : (s^x \cup \varnothing, \mathcal{F}_n, \{a,b\}) \to (U^z, \mathcal{F}_{n+1})$.

Case 1: $(s^x \cup \varnothing, \mathcal{F}_n, \{a\})$: For the paths having labeling as $P_1 = \cdots s^x \cdots \in \mathcal{F}_n$, we set $f_a(P_1) = \cdots sU^zD \cdots \in \mathcal{F}_{n+1}$ if s is above x-axis, otherwise $f_a(P_1) = \cdots sDU^z \cdots \in \mathcal{F}_{n+1}$. For the paths without labeling as P_2 we have $f_a(P_2) = U^zDP_2$.



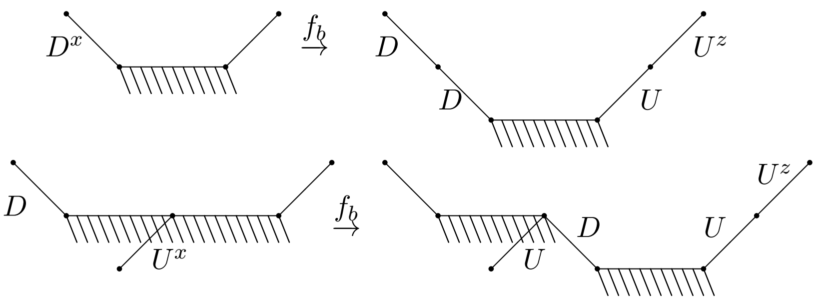
For step s, we define the level of the step s, denoted by $\ell^1(s)$ as the y-th coordinates of its final point.

Case 2: $(s^x, \mathcal{F}_n, \{b\})$ and $\ell^1(s^x) \neq 0$: For an arbitrary path $P_1 = \cdots s^x \cdots \in (s^x, \mathcal{F}_n)$ and $\ell^1(s^x) > 0$, let U_s be the first U-step preceding s for which $\ell^1(U_s) = \ell^1(s)$ and let D_s be the first D-step that follows s such that $\ell^1(D_s) = \ell^1(s) - 1$. Note that $U_s = s$ if s is a U-step. As a result, we can write $P_1 = \cdots U_s \cdots s^x R D_s \cdots$ where R is the Dyck path between s^x and D_s. Then $f_b(P_1) = \cdots U_s^z \cdots sURDD_s \cdots \in \mathcal{F}_{n+1}$.



Similarly, for the paths $P_2 = \cdots s^x \cdots \in \mathcal{F}_n$ and $\ell^1(s) < 0$, let U_s be the first U-step after s for which $\ell^1(U_s) = \ell^1(s) + 1$ and let D_s be the first D-step preceding s such that $\ell^1(D_s) = \ell^1(s)$. Note that $D_s = s$ if s is a D-step. As a result, we can write $P_2 = \cdots D_s \cdots s^x R U_s \cdots$ where R is the Dyck path between s^x and U_s. Then $f_b(P_2) = \cdots D_s \cdots s D R U U_s^z \cdots \in \mathcal{F}_{n+1}$.

$\downarrow$ x-axis:



Case 3: $(\varnothing, \mathcal{F}_n, \{b\})$ or $(s^x, \mathcal{F}_n, \{b\})$ and $\ell^1(s) = 0$: In the former case, let $P_1 \in \mathcal{F}_n$, $f_b(P_1) = DU^z P_1$. In the later case, if $P_2 = \cdots D^x$ then $f_b(P_2) = \cdots DDU^z$ holds, if $P_3 = \cdots U^x$ then $f_b(P_3) = \cdots UU^z D$. We omit the proof that the map f is bijective since that proof follows similarly to that for Dyck paths [2, 5, 7]. The next step we shall extend f^{-1} to the map $g : (U^z, \mathcal{R}_{n+1}) \to (s^x \cup \varnothing, \mathcal{R}_n, \{a, b\})$ by allowing the atomic up step and down step to be of arbitrary length, according to which g is surjective but not injective. We start from studying the number of labeled paths in $(U^z, \mathcal{R}_{n+1})$ induced by $f(s^x \cup \varnothing, \mathcal{R}_n, \{a, b\})$ via integrating the newly added 2-steps to the paths in $\mathcal{R}_n$.

Case 1: $(\varnothing, \mathcal{R}_n, \{a, b\})$: W.l.o.g. we assume path $p_1 = U^m p_0 \in \mathcal{R}_n$ where the first step of p_1 is U-step of length m ($m \geq 1$). Then $f_a(p_1) = U^z D U^m p_0$ and $f_b(p_1) = DU^z U^m p_0$. $D\underline{U^z U^m} p_0 \in (U^z, \mathcal{R}_{n+1})$ is obtained by gluing U^z and U^m ($m \geq 1$), therefore we set

$$\begin{aligned} g(U^z, U^z D U^m p_0) &= (\varnothing, p_1, \{a\}), \\ g(U^z, DU^z U^m p_0) &= g(U^z, D\underline{U^z U^m} p_0) = (\varnothing, p_1, \{b\}). \end{aligned}$$

For each image $(\varnothing, p_1) \in (\varnothing, \mathcal{R}_n)$ by map g, there are three preimages in $(U^z, \mathcal{R}_{n+1})$ such that the first unit U-step is labeled with z, which has cardinality $3a_n$.

Case 2: $(s^x, \mathcal{R}_n, \{a, b\})$ and s is the last unit step: W.l.o.g. we assume path $p_2 = \cdots D^x$. Then $f_a(p_2) = \cdots DU^z D$ and $f_b(p_2) = \cdots DDU^z$. $\cdots \underline{DD} U^z \in (U^z, \mathcal{R}_{n+1})$ is obtained by gluing two D-steps, therefore we set

$$\begin{aligned} g(U^z, \cdots DU^z D) &= (D^x, p_2, \{a\}), \\ g(U^z, \cdots DDU^z) &= g(U^z, \cdots \underline{DD} U^z) = (D^x, p_2, \{b\}). \end{aligned}$$

For each image $(s^x, p_2) \in (s^x, \mathcal{R}_n)$ by map g, there are three preimages in $(U^z, \mathcal{R}_{n+1})$ such that the last unit U-step is labeled with z, which has cardinality $3a_n$.

In what follows we consider all the possible scenarios for $s^x Y \subseteq P$ where Y is the step immediately following s^x for an arbitrary but fixed path $p \in \mathcal{R}_n$.

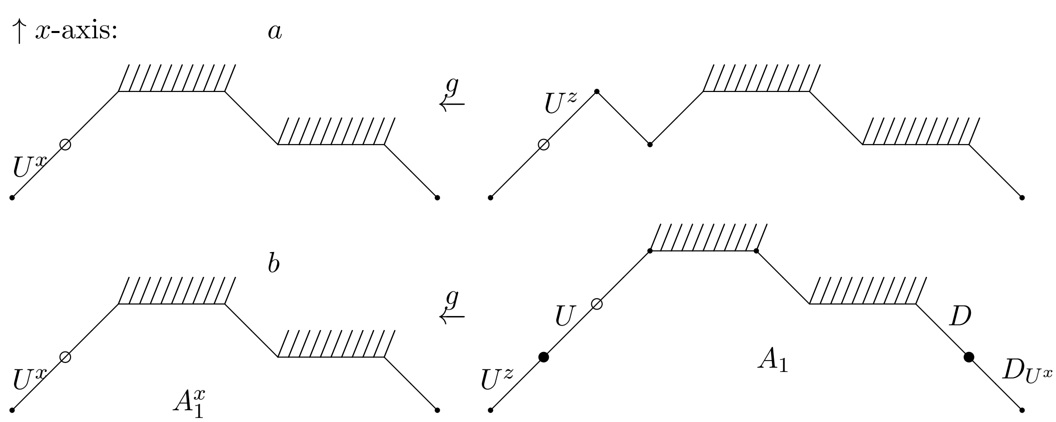
Let $\underline{U}\,\underline{U}$ represent $\underline{UU}$ xor UU, i.e., the two exclusive cases of either an atomic up-step of length two or of two up-steps which for sure belong to two different atomic up-steps. For a multiple Dyck path A that has a t-step of length m_1 as its first atomic step and s-step of length m_2 as its last atomic step, i.e., $A = t^{m_1}A's^{m_2}$ where $t = U, s = D$ or $t = D, s = U$ and $m_1, m_2 \geq 1$, we set

$$\begin{aligned}
\underline{tA}s &= t^{m_1+1}A's^{m_2}s\\
\underline{tAs} &= t^{m_1+1}A's^{m_2+1}\\
t\underline{As} &= tt^{m_1}A's^{m_2+1}\\
tAs &= tt^{m_1}A's^{m_2}s\\
\underline{tA}\;\underline{s} &= \underline{tA}s \text{ xor } \underline{tAs}\\
\underline{t}\;\underline{As} &= t\underline{As} \text{ xor } \underline{tAs}
\end{aligned}$$

Case 3: $\underline{U^x}\,\underline{U}\uparrow$: Let $p_3 = \cdots\underline{U^x}\,\underline{U}\cdots$, then $f_a(p_3) = \cdots\underline{U}\,\underline{U^z}DU\cdots$ and recall that D_{U^x} is the step that after U^x for which $\ell^1(U^x) = \ell^1(D_{U^x}) + 1$. Let A_1^x (resp. A_1) be the sub multiple Dyck paths from U^x to D_{U^x} with (resp. without) labeling x, then we can assume $p_3 = \cdots A_1^x\cdots$, and therefore $f_b(p_3) = \cdots U^zA_1D\cdots$. By setting

$$\begin{aligned}
&g(U^z, \cdots\underline{UU^z}DU\cdots) = (\underline{U^xU}\,, p_3, \{a\})\ xor\\
&g(U^z, \cdots UU^zDU\cdots) = (U^xU\,, p_3, \{a\}),\\
&g(U^z, \cdots\underline{U^zA_1}D\cdots) = g(U^z, \cdots\underline{U^zA_1D}\cdots) = (\underline{U^x}\,\underline{U}\,, p_3, \{b\}),\\
&g(U^z, \cdots U^z\underline{A_1D}\cdots) = g(U^z, \cdots U^zA_1D\cdots) = (\underline{U^x}\,\underline{U}\,, p_3, \{b\}),
\end{aligned}$$

all the five preimages in $(U^z, \mathcal{R}_{n+1})$ are mapped by g to $(\underline{U^x}\,\underline{U}\,, \mathcal{R}_n, \{a, b\})$. We name the "active point" to be the point that connects two S-steps ($S = U$ or D) as $\underline{SS}$ or SS, Furthermore, the hollow circled dot represents the point that connects one of the patterns from $\{\underline{SS}, SS\}$ exclusively. The figure below shows the images $(\underline{U^x}\,\underline{U}\,, p_3, \{a, b\})$ representing the active points by big black dots.

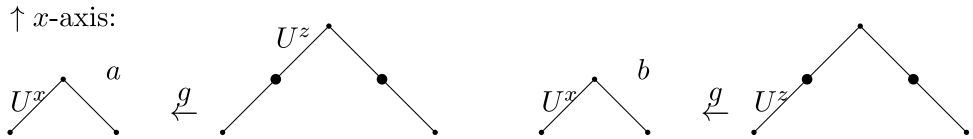


Case 4: $U^xD\uparrow$: Let $p_4 = \cdots U^xD\cdots$, then $f_a(p_4) = \cdots UU^zDD\cdots$ and $f_b(p_4) =$

$\cdots U^zUDD\cdots$ holds. Accordingly we set

$$\begin{aligned}
g(U^z,\cdots \underline{UU^z}\,DD\cdots) &= g(U^z,\cdots \underline{UU^z}\,\underline{DD}\cdots) = (U^xD,p_4,\{a\}),\\
g(U^z,\cdots UU^z\,\underline{DD}\cdots) &= g(U^z,\cdots UU^zDD\cdots) = (U^xD,p_4,\{a\}),\\
g(U^z,\cdots \underline{U^zU}\,DD\cdots) &= g(U^z,\cdots \underline{U^zU}\,\underline{DD}\cdots) = (U^xD,p_4,\{b\}),\\
g(U^z,\cdots U^zU\,\underline{DD}\cdots) &= g(U^z,\cdots U^zUDD\cdots) = (U^xD,p_4,\{b\}).
\end{aligned}$$

Every eight preimages in $(U^z,\mathcal{R}_{n+1})$ are mapped by g to one image $(U^xD,p_4,\{a,b\})$.



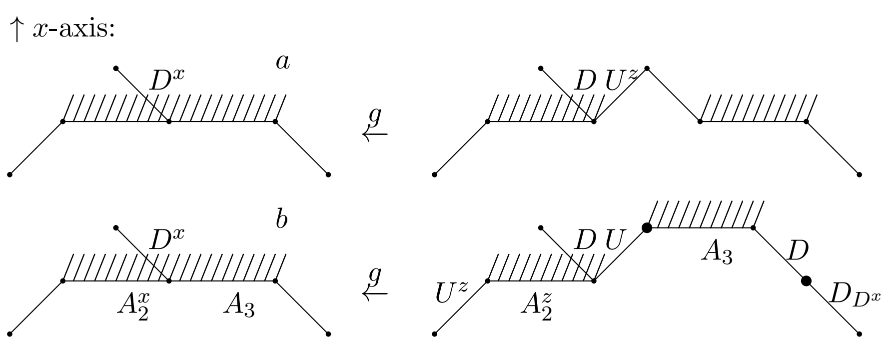
Case 5: D^xU ↑: First we discuss the subcase in which D^x touches the x-axis, i.e., $\ell^1(D^x) = 0$, and is followed by U-step. Let $p_5 = \cdots D^xU\cdots$, then $f_a(p_5) = \cdots DU^zDU\cdots$ and $f_b(p_5) = \cdots DDU^zU\cdots$ hold. Thus, we set

$$\begin{aligned}
g(U^z,\cdots DU^zDU\cdots) &= (D^xU,p_5,\{a\}),\\
g(U^z,\cdots \underline{DD}\,U^zU\cdots) &= g(U^z,\cdots \underline{DD}\,\underline{U^zU}\cdots) = (D^xU,p_5,\{b\}),\\
g(U^z,\cdots DD\,\underline{U^zU}\cdots) &= g(U^z,\cdots DDU^zU\cdots) = (D^xU,p_5,\{b\}).
\end{aligned}$$

Next we study the subcase in which D^x is above x-axis. Let $p_5 = \cdots D^xU\cdots$, then $f_a(p_5) = \cdots DU^zDU\cdots$ holds. Let D_{D^x} be the first step after D^x such that $\ell^1(D^x) = \ell^1(D_{D^x})+1$. Let U_{D^x} be the first step preceding D^x such that $\ell^1(U_{D^x}) = \ell^1(D^x)$. Then we assume A_2^x (resp. A_2^z) is the path from U_{D^x} to D^x with labeling D^x (resp. $U^z_{D^x} = U^z$) and A_3 is the path from the end of D^x to D_{D^x}. Therefore we can interpret p_5 as $p_5 = \cdots A_2^xA_3\cdots$, and $f_b(p_5) = \cdots A_2^zUA_3D\cdots$. Thus, we set

$$\begin{aligned}
g(U^z,\cdots DU^zDU\cdots) &= (D^xU,p_5,\{a\}),\\
g(U^z,\cdots A_2^z\underline{UA_3D}\cdots) &= g(U^z,\cdots A_2^z\underline{UA_3}D\cdots) = (D^xU,p_5,\{b\}),\\
g(U^z,\cdots A_2^zU\underline{A_3D}\cdots) &= g(U^z,\cdots A_2^zUA_3D\cdots) = (D^xU,p_5,\{b\}).
\end{aligned}$$

In both cases, every five preimages in $(U^z,\mathcal{R}_{n+1})$ are mapped by g to one image in $(D^xU,\mathcal{R}_n,\{a,b\})$.



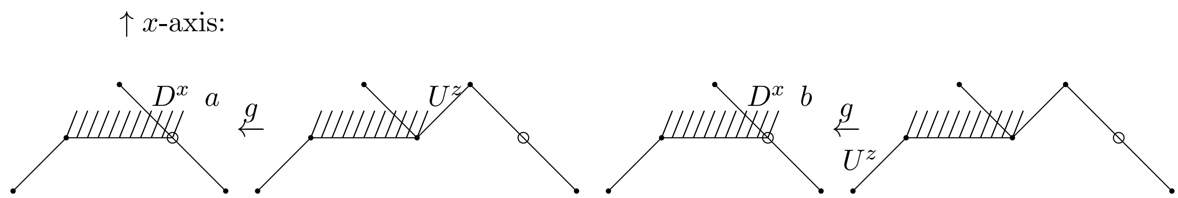
Case 6: $\underline{D^x}\ \underline{D} \uparrow$: First we discuss the subcase in which D^x touches the x-axis, i.e., $\ell^1(D^x) = 0$, and is followed by a D-step. Let $p_5 = \cdots \underline{D^x}\ \underline{D}$, then $f_a(p_5) = \cdots DU^zDD\cdots$ and $f_b(p_5) = \cdots DDU^zD\cdots$. We set

$$\begin{aligned}
g(U^z, \cdots DU^zDD\cdots) &= (D^xD, p_5, \{a\}) \quad \text{xor} \\
g(U^z, \cdots DU^z\underline{DD}\cdots) &= (\underline{D^xD}, p_5, \{a\}), \\
g(U^z, \cdots DDU^zD\cdots) &= (D^xD, p_5, \{b\}) \quad \text{xor} \\
g(U^z, \cdots \underline{DD}U^zD\cdots) &= (\underline{D^xD}, p_5, \{b\}).
\end{aligned}$$

Next we study the subcase that D^x is above x-axis. For $p_6 = \cdots \underline{D^x}\ \underline{D}\cdots$, $f_a(p_6) = \cdots DU^z\underline{D}\ \underline{D}\cdots$ holds. Let U_{D^x} be the first step preceding D^x such that $\ell^1(U_{D^x}) = \ell^1(D^x)$. Then we follow the assumption of A_2^x and A_2^z in Case 5, and represent p_6 as $p_6 = \cdots \underline{A_2^x}\ \underline{D}\cdots$, therefore $f_b(p_6) = \cdots A_2^zU\underline{D}\ \underline{D}\cdots$. Consequently, by setting

$$\begin{aligned}
g(U^z, \cdots DU^z\underline{DD}\cdots) &= (\underline{D^xD}, p_6, \{a\}) \quad \text{xor} \\
g(U^z, \cdots DU^zDD\cdots) &= (D^xD, p_6, \{a\}), \\
g(U^z, \cdots A_2^zU\underline{DD}\cdots) &= (\underline{D^xD}, p_6, \{b\}) \quad \text{xor} \\
g(U^z, \cdots A_2^zUDD\cdots) &= (D^xD, p_6, \{b\}),
\end{aligned}$$

every two preimages in $(U^z, \mathcal{R}_{n+1})$ are mapped by g to one image in $(D^xD, \mathcal{R}_n, \{a, b\})$.



For those labeled step $s^xY \downarrow$, the case discussion follows similarly. To sum up, for each path $p \in \mathcal{R}_n$, the number of paths in $(U^z, \mathcal{R}_{n+1})$ resulted from p is $5(m_{11}+m_{21}) + 8m_{12} + 2m_{22} + 6$. Hence eq. (6) follows and the proof is complete. □

Lemma 2. *Under the assumption of Lemma 1 we have*

$$\sum_{\mathcal{M}\in\mathcal{R}_n} [(m_{11} + m_{21}) + 2m_{12}] + 3(n-1)a_{n-1} = 2na_n \tag{7}$$

Proof. We shall construct the map

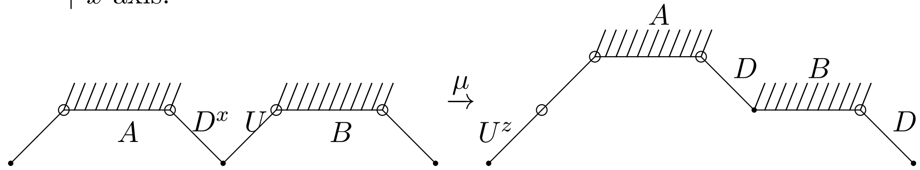
$$\begin{aligned}
\mu : &(U^xU \cup D^xU, \mathcal{R}_n \uparrow)\dot\cup(U^xD, \mathcal{R}_n \uparrow, \{a, b\})\dot\cup(D^xD \cup U^xD, \mathcal{R}_n \downarrow)\dot\cup(D^xU, \mathcal{R}_n \downarrow, \{a, b\}) \\
&\dot\cup(U^x, \mathcal{R}_{n-1}, \{[1, 2], [2, 1], [2, 2]\}) \to (U^z, \mathcal{R}_n, \{s, t\}).
\end{aligned}$$

Note that $(D^xD \cup U^xD, \mathcal{R}_n \downarrow)$ and $(D^xU, \mathcal{R}_n \downarrow, \{a, b\})$ are the mirror images of $(U^xU \cup D^xU, \mathcal{R}_n \uparrow)$ and $(U^xD, \mathcal{R}_n \uparrow, \{a, b\})$ under reflection $\mathcal{K} : y = 0$, i.e., for $a \in (U^xU \cup D^xU, \mathcal{R}_n \uparrow)$, then $\mathcal{K}(a) \in (D^xD \cup U^xD, \mathcal{R}_n \downarrow)$ and $g(\mathcal{K}(a)) = \mathcal{K}(g(a))$, for which we omit the case discussion for $(D^xD \cup U^xD, \mathcal{R}_n \downarrow)$ and $(D^xU, \mathcal{R}_n \downarrow, \{a, b\})$.

Case 1: For $w_1 \in (U^xU, \mathcal{R}_n \uparrow)$ or $w_1 \in (U^xD, \mathcal{R}_n \uparrow, \{a,b\})$, μ simply replaces the labeling x with z, i.e., $\mu(U^xU, w_1) = (U^zU, w_1, s)$ and $\mu(U^xD, w_1, a) = (U^zD, w_1, s)$, $\mu(U^xD, w_1, b) = (U^zD, w_1, t)$.

Case 2: $w_2 \in (D^xU, \mathcal{R}_n \uparrow)$: Let U_x be the first U-step preceding D^x such that $\ell^1(U_x) = \ell^1(D^x) + 1$. Similarly, let D_U be the first D-step that follows U and for which $\ell^1(U) = \ell^1(D_U) + 1$ holds. We assume A_4^x (resp. A_4) is the path from U_x to D^x with (resp. without) labeling x, and A_5 is the path from the end of U to D_U. Then we consider path $w_2 = \cdots A_4^x\underline{U}\ \underline{A_5}\cdots$, $\mu(w_2) = (U^z, \cdots \underline{U^z}\ \underline{A_4}A_5\cdots, t)$. We observe that in the case $A_5 \neq D$, μ is a one-to-one correspondence. In the case $A_5 = D$, three patterns are missing from $(U^zU, \mathcal{R}_n, t)$, i.e., $(\cdots \underline{U^zA_4}\ A_5\cdots, t)$, $(\cdots \underline{U^zA_4A_5}\cdots, t)$ and $(\cdots U^z\ \underline{A_4A_5}\cdots, t)$. Case 3 will make up for this.

$\uparrow$ x-axis:



Case 3: $w_3 \in (U^x, \mathcal{R}_{n-1}, \{[1,2],[2,1],[2,2]\})$: If $U^x \in \mathcal{R}_{n-1}$ (thus $w_3 = \cdots \underline{U^x}\ \underline{A}\ \underline{D}\cdots$ in which A is a multiple Dyck path (can be $\varnothing$)), then we set

$$\begin{aligned}
\mu(w_3, [2,1]) &= (U^z, \cdots \underline{U^zU}\ \underline{A}\ \underline{D}D\cdots, t) \\
\mu(w_3, [1,2]) &= (U^z, \cdots U^z\ \underline{U}\ \underline{A}\ \underline{DD}\cdots, t) \\
\mu(w_3, [2,2]) &= (U^z, \cdots \underline{U^zU}\ \underline{A}\ \underline{DD}\cdots, t),
\end{aligned}$$

which make up three other possible patterns in Case 2.

It remains to prove the map μ is bijective. Note the union of the image set for these 3 cases is exactly the set $(U^z, \mathcal{R}_n, \{s,t\})$ and μ is injective by construction. Consequently, eq. (7) holds.

□

Theorem 3. *Let a_n denote the number of 2-dimensional rook paths from $(0,0)$ to $(2n,0)$. Then the recurrence of a_n is given by*

$$\begin{aligned}
&a_0 = 1, a_1 = 2, \\
&a_n = \frac{n+1}{2(5n+2)}a_{n+1} + \frac{9(n-1)}{2(5n+2)}a_{n-1}.
\end{aligned} \tag{8}$$

Proof. In combination of Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
&(n+1)a_{n+1} + 9(n-1)a_{n-1} \\
&= 2(2n-1) + 6 + 3\sum_{\mathcal{M}\in\mathcal{R}_n}[(m_{11}+m_{21}) + 2m_{12}] + 9(n-1)a_{n-1} \\
&= 2(5n+2)a_n,
\end{aligned}$$

with initial condition $a_0 = 1$ and $a_1 = 2$. □

The combinatorial proof for the recurrence of rook paths can also be extended to the combinatorial proof for the recurrence of multiple Dyck paths. Eq. (5) follows according to the lemmas below.

Lemma 4. *For each $\mathcal{M} \in \mathcal{D}_n$, let $n_{11}, n_{12}, n_{21}, n_{22}$ count the occurrence of UU, UD, DU and DD. Then for $d_{n+1} = |\mathcal{D}_{n+1}|$ we have*

$$(n+2)d_{n+1} = \sum_{\mathcal{M} \in \mathcal{M}\mathcal{D}_n} 5(n_{11} + n_{21}) + 8n_{12} + 2n_{22}. \tag{9}$$

Lemma 5. *Under the assumptions of Lemma 4 we have*

$$\sum_{\mathcal{M} \in \mathcal{M}\mathcal{D}_n} 2n_{12} + 3(n-1)d_{n-1} = (n+1)d_n. \tag{10}$$

References

[1] M. Erickson, S. Fernando and K. Tran. Enumerating rook and Queen paths. *Bull. Inst. Combin. Appl.*, 60:37-48, 2010.

[2] D. Foata and D. Zeilberger. A classic proof of a recurrence for a very classical sequence. *J. Comb. Theor., Ser. A.*, 80:380-384, 1997.

[3] M. Kauers and D. Zeilberger. The computational challenge of enumerating high-dimensional rook walks, arXiv:1011.4671v1 [math.CO] 21 Nov 2010.

[4] P. Peart and W. J. Woan. A bijective proof of the delannoy recurrence. *Proceeding of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL)*, 158:29-33. Congressus Numerantium, 2002.

[5] J-L. Rémy. Un procédé itératif de dénombrement d'arbres binaires et son application à leur génération aléatpoire. *RAIRO Inform. Théor.*, 19:179-195, 1985.

[6] R.P. Stanley. Enumerative Combinatorics (Vol.2). Cambridge University Press, New York, 1999.

[7] R.A. Sulanke. Bijective Recurrences concerning Schröder paths. *Electron. J. Combin.*, 5:#R47, 1998.