The full automorphism group of Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$

Edward Dobson*

Department of Mathematics and Statistics
Mississippi State University
Mississippi State, MS 39762 U.S.A.

dobson@math.msstate.edu

and

University of Primorska
IAM
Muzejski trg 2
6000 Koper, Slovenia

Submitted: Oct 15, 2010; Accepted: Mar 5, 2012; Published: Mar 19, 2012
Mathematics Subject Classifications: 05C25, 20B25

Abstract

Let $p \geq 5$ be prime. We determine the full automorphism groups of Cayley digraphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$.

1 Introduction

Determining the full automorphism group of a Cayley digraph of a group G is perhaps one of the most fundamental questions that one can ask about a Cayley digraph, and seems to be a very difficult question to answer. In recent years, progress towards solving this problem has begun, usually focusing on Cayley digraphs of specific groups G or Cayley digraphs that have particular properties, such Cayley digraphs that 1/2-transitive, or edge-transitive. The groups G for which the full automorphism groups of Cayley digraphs of G have been explicitly determined are $G = \mathbb{Z}_p$ [1], $\mathbb{Z}_p^2$ [15], $\mathbb{Z}_{p^2}$ [18] (see [15] for a later proof), $\mathbb{Z}_{pq}$ [18] (see [10] for a later proof), the nonabelian groups of order pq [10], and $\mathbb{Z}_p^3$ [12], where p and q are distinct primes. Additionally, strong constraints on the structure of the full automorphism group of Cayley digraphs of $\mathbb{Z}_n$ have been obtained (see [20]), and independently for n square-free [13]. Using these constraints, Ponomarenko [22] has

*Project sponsored by the National Security Agency under Grant Number H98230-11-1-0179

found a polynomial time algorithm to compute the full automorphism group of circulant digraphs. In this paper, we determine the full automorphism groups of Cayley digraphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $p \geq 5$. Our approach basically follows the approach used to determine the full automorphism groups of Cayley digraphs of $\mathbb{Z}_p^3$ given in [12]. We use the implicit determination of all Sylow p-subgroups of Cayley digraphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ given in [8], and then either determine the overgroups of these p-subgroups or use known results giving the overgroups of these p-groups.

For permutation group terms not defined here, see [6]. We begin with some definitions, and then state some of the many results in the literature that we will have need of.

Definition 1 *Let G be a group and $S \subset G$ such that $1_G \notin S$. Define a digraph $D = D(G,S)$ by $V(D) = G$ and $E(D) = \{(u,v) : v^{-1}u \in S\}$. Such a digraph is a Cayley digraph of G with connection set S. A Cayley graph of G is defined analogously although we insist that $S = S^{-1} = \{s^{-1} : s \in S\}$. A circulant (di)graph of order n is simply a Cayley (di)graph of $\mathbb{Z}_n$.*

It is straightforward to verify that for $g \in G$, the map $g_L : G \to G$ by $g_L(x) = gx$ is an automorphism of a Cayley digraph D of a group G. Thus $G_L = \{g_L : g \in G\}$, the *left regular representation of* G, is a subgroup of the automorphism group of D, $\mathrm{Aut}(D)$. Sabidussi has shown [23] that a digraph D is isomorphic to a Cayley digraph of D if and only if $\mathrm{Aut}(D)$ contains a regular subgroup isomorphic to G.

Definition 2 *Let G be a transitive permutation group with complete block system $\mathcal{B}$. We say that $\mathcal{B}$ is genuine if $\mathcal{B}$ is formed by the orbits of some normal subgroup of G. By $G/\mathcal{B}$, we mean the subgroup of $S_\mathcal{B}$ induced by the action of G on $\mathcal{B}$, and by $\mathrm{fix}_G(\mathcal{B})$ the kernel of this action. Thus $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. By $\mathrm{Stab}_G(\mathcal{B})$, we mean the set-wise stabilizer in G of the block $B \in \mathcal{B}$. Hence $\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}$, and $\mathrm{fix}_G(\mathcal{B}) = \cap_{B\in\mathcal{B}}\mathrm{Stab}_G(B)$. If $\mathcal{C}$ is a complete block system of G such that every block of $\mathcal{C}$ is a union of blocks of $\mathcal{B}$, we write $\mathcal{B} \preceq \mathcal{C}$, and denote the complete block system of $G/\mathcal{B}$ induced by $\mathcal{C}$ by $\mathcal{C}/\mathcal{B}$. Thus $C/\mathcal{B} \in \mathcal{C}/\mathcal{B}$ consists of those blocks of $\mathcal{B}$ whose union is $C \in \mathcal{C}$.*

In most situations, we will be determining all 2-closed groups, which are a slightly larger class of groups than automorphism groups of digraphs, and are defined below.

Definition 3 *Let Ω be a set and $G \leq S_\Omega$. Let G act on $\Omega\times\Omega$ by $g(\omega_1,\omega_2) = (g(\omega_1), g(\omega_2))$ for every $g \in G$ and $\omega_1,\omega_2 \in \Omega$. We define the 2-closure of G, denoted $G^{(2)}$, to be the largest subgroup of S_Ω whose orbits on $\Omega\times\Omega$ are the same as G's. Let $\mathcal{O}_1,\ldots,\mathcal{O}_r$ be the orbits of G acting on $\Omega\times\Omega$. Define digraphs $\Gamma_1,\ldots,\Gamma_r$ by $V(\Gamma_i) = \Omega$ and $E(\Gamma_i) = \mathcal{O}_i$. Each Γ_i, $1 \leq i \leq r$, is an orbital digraph of G, and it is straightforward to show that $G^{(2)} = \cap_{i=1}^r \mathrm{Aut}(\Gamma_i)$. Equivalently, $G^{(2)}$ is the automorphism group of a color digraph.*

Definition 4 *For a positive integer n, define $N(n) = \{x \to ax + b : a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n\}$. Thus $N(n)$ is the normalizer of the left regular representation $(\mathbb{Z}_n)_L$ of $\mathbb{Z}_n$ in S_n. We remark that for p a prime, $N(p)$ is usually denoted $\mathrm{AGL}(1,p)$.*

The following classical result of Burnside [2] is quite useful for analyzing transitive groups of prime degree, especially now that, as a consequence of the Classification of Finite Simple Groups, all doubly transitive groups are known [3]. We remark that the following versions of this result also makes use another of Burnside's results, namely [6, Theorem 4.1B].

Theorem 5 *Let G be a transitive group of prime degree. Then either G is doubly transitive with nonabelian simple socle, or G contains a normal Sylow p-subgroup.*

Equivalently (see [6, Exercise 3.5.1]), we have

Theorem 6 *Let G be a transitive group of prime degree p. Then we may relabel the set upon which G acts so that $G \leq \mathrm{AGL}(1,p)$, or G is doubly transitive with nonabelian simple socle.*

The following result is [9, Theorem 33], and is an extension of the previous result to prime-powers.

Theorem 7 *Let $p \geq 3$ be prime, and $G \leq S_{p^m}$, $m \geq 1$, be transitive such that every minimal transitive subgroup of G is cyclic. Then either G contains a transitive normal Sylow p-subgroup, or G is doubly-transitive and*

1. *$G = A_{p^m}$ or S_{p^m}, and $m = 1$,*
2. *$\mathrm{PSL}(n,k) \leq G \leq \mathrm{P\Gamma L}(n,k)$, for some prime power k and $n \geq 2$ with $p^m = (k^n - 1)/(k-1)$,*
3. *$\mathrm{PSL}(2,11)$ or M_{11} and $p^m = 11$,*
4. *M_{23} and $p^m = 23$.*

The following is [9, Lemma 17], and gives more information concerning some of the doubly-transitive groups of the preceding result.

Lemma 8 *Let $\mathrm{PSL}(n,k) \leq G \leq \mathrm{P\Gamma L}(n,k)$ be primitive of degree $(k^n-1)/(k-1) = p^m$, where k is a prime power, $n \geq 2$, p an odd prime, and $m \geq 1$. If $(n,k) \neq (2,8)$, then a Sylow p-subgroup of $\mathrm{P\Gamma L}(n,k)$ is regular and cyclic, and so a Sylow p-subgroup of G is regular and cyclic. Consequently, if $N(p^m) \leq G$, $m \geq 2$, then $p^m = 9$ and $\mathrm{PSL}(2,2^3) < G \leq \mathrm{P\Gamma L}(2,2^3)$.*

The following definition and result are very useful in determining Sylow p-subgroups of automorphism groups of Cayley digraphs of prime-power order (among other things).

Definition 9 *Let G be a transitive permutation group that admits a complete block system $\mathcal{B}$ of m blocks of size p, p a prime, and $\mathcal{B}$ is formed by the orbits of some normal subgroup $N \triangleleft G$. Then for each $B \in \mathcal{B}$ there exists $\alpha_B \in N$ such that $\alpha_B|_B$ is a p-cycle. Define an equivalence relation $\equiv$ on the blocks of $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\alpha \in N$ then $\alpha|_B$ is a p-cycle if and only if $\alpha|_{B'}$ is also a p-cycle. Denote the equivalence classes of $\equiv$ by $C_0, \ldots, C_a$ and let $E_i = \cup_{B \in C_i} B$.*

The following result is [7, Lemma 3].

Lemma 10 *Let G be as in Definition 9, and $\alpha \in N$ be such that $|\alpha| = p$. Then for each $0 \leq i \leq a$ there exists $\alpha_i \in G^{(2)}$ such that $\alpha_i|_{E_i} = \alpha|_{E_i}$ and $\alpha_i|_{E_j} = 1$ for every $i \neq j$. Furthermore, each E_i is a block of G.*

We remark that the statement of Lemma 10 is more general than in [9], but it is straightforward to show this more general version holds using the fact that the 2-closure of G is the intersection of the automorphism groups of the orbital digraphs of G.

We shall have need of the following result of Kalužnin and Klin [17] (this result is also contained in the more easily accessible [4, Theorem 5.1]).

Lemma 11 *Let $G \leq S_X$ and $H \leq S_Y$ be transitive groups. Then in their coordinate-wise action on $X \times Y$, we have*

$$(G \times H)^{(2)} = G^{(2)} \times H^{(2)}, \text{ and } (G \wr H)^{(2)} = G^{(2)} \wr H^{(2)}.$$

For the remainder of this paper, we define $\tau_1, \tau_2 : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ by $\tau_1(i,j) = (i+1,j)$ and $\tau_2(i,j) = (i,j+1)$. Then $\langle \tau_1, \tau_2 \rangle = (\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L$ and so $\langle \tau_1, \tau_2 \rangle \leq \mathrm{Aut}(\Gamma)$ for every Cayley digraph Γ of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$.

The following result can be extracted from [8, Theorem 10], together with the previous result, and gives the Sylow p-subgroups of the automorphism groups of Cayley digraphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Let $\mathcal{B}_{1,1}$ be the complete block system of $\langle \tau_1, \tau_2 \rangle$ formed by the orbits of $\langle \tau_1, \tau_2^p \rangle$, and $\mathcal{B}_2$ the complete block system of $\langle \tau_1, \tau_2 \rangle$ formed by the orbits of $\langle \tau_2 \rangle$. In the following result, by $\gamma|_B$ we mean the permutation of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ defined by $\gamma|_B(x) = \gamma(x)$ if $x \in B$, and $\gamma|_B(x) = x$ if $x \notin B$.

Theorem 12 *Let $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ be 2-closed with Sylow p-subgroup P which contains $(\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L$. Then one of the following is true for some $\alpha_1 \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L$:*

(i) $H = S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$,

(ii) $P = (\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L$,

(iii) $P = \alpha_1^{-1} \langle \tau_1, \tau_2, \tau_2^p|_B : B \in \mathcal{B}_{1,1} \rangle \alpha_1 \cong \mathbb{Z}_p \times (\mathbb{Z}_p \wr \mathbb{Z}_p)$,

(iv) $P = \alpha_1^{-1} \langle \tau_1, \tau_2, \tau_2^p|_B : B \in \mathcal{B}_2 \rangle \alpha_1$,

(v) $P = \alpha_1^{-1} \langle \tau_1, \tau_2, \tau_1|_B : B \in \mathcal{B}_{1,1} \rangle \alpha_1$,

(vi) *if* $\gamma : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \to \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ *by* $\gamma(i,j) = (i + [aj \pmod p], j + ibp)$, $a, b \in \mathbb{Z}_p^*$, *then* $P = \langle \tau_1, \tau_2, \gamma \rangle$, *and* $|P| = p^4$,

(vii) $\alpha_1^{-1} P \alpha_1 = P_1 \wr P_2$, *where P_1 is 2-closed p-group of degree p^2 and contains a regular subgroup isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$, and $P_2 \leq S_p$ is cyclic of order p,*

(viii) $\alpha_1^{-1} P \alpha_1 = P_2 \wr P_1$, *where $P_2 \leq S_p$ is cyclic of order p, and $P_1 \leq S_{p^2}$ is 2-closed p-subgroup of degree p^2 and contains a regular subgroup isomorphic to $\mathbb{Z}_{p^2}$.*

We remark that in [8, Theorem 10], there is an additional case, namely when P admits a complete block system $\mathcal{B}'$ consisting of p^2 blocks of size p formed by the orbits of $\langle\tau_1\rangle$, $\mathrm{fix}_P(\mathcal{B}') = \langle\tau_1\rangle$, P admits $\mathcal{B}_{1,1}$ as a complete block system, $\mathrm{fix}_P(\mathcal{B}_{1,1})|_B \not\leq \langle\tau_1,\tau_2^p\rangle|_B$ for some $B\in\mathcal{B}_{1,1}$, and $\langle\tau_1,\tau_2\rangle \triangleleft P$. This case is superfluous, as if $\langle\tau_1,\tau_2\rangle \triangleleft P$, then P admits a complete block system formed by the orbits of $\langle\tau_2^p\rangle$. This follows as if $\langle\tau_1,\tau_2\rangle\triangleleft P$, then $\gamma^{-1}\langle\tau_2\rangle\gamma = \langle\tau_2\tau_1^a\rangle$ for some $a\in\mathbb{Z}_p$, and so $\gamma^{-1}\langle\tau_2^p\rangle\gamma = \langle\tau_2^p\rangle$. Thus $\langle\tau_2^p\rangle\triangleleft P$ and its orbits form a complete block system (which is a case considered separately in [8, Theorem 10].

We shall also have need of the following result [11, Corollary 7.3], which gives the 2-closed groups which contain a regular abelian Sylow p-subgroup that is of rank 2 (i.e. is a direct product of two cyclic groups).

Theorem 13 *Let $G \leq S_{p^k}$ be transitive and 2-closed with Sylow p-subgroup P that is abelian of rank two. Then one of the following is true:*

1. *G has a normal Sylow p-subgroup,*
2. *G is primitive, $k=2$, and G is permutation isomorphic to $S_2 \wr S_p$,*
3. *$k=2$, and G is permutation isomorphic to $S_p\times S_p$, or*
4. *G is permutation isomorphic to $S_p\times A$, where $A\leq N(p^{k-1})$ has order dividing $(p-1)p^{k-1}$.*

The following result deals with the case where a Sylow p-subgroup P of a 2-closed group is permutation isomorphic to $\mathbb{Z}_p\times(\mathbb{Z}_p\wr\mathbb{Z}_p)$ and is extracted from [12, Proposition 5.12]. We remark that $\mathbb{Z}_p\times(\mathbb{Z}_p\wr\mathbb{Z}_p)$ contains regular subgroups isomorphic to both $\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ and to $\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$ as $\mathbb{Z}_p\wr\mathbb{Z}_p$ contains regular subgroups isomorphic to $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p\times\mathbb{Z}_p$ [15, Lemma 4], and so any Cayley digraph of $\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ that contains a Sylow p-subgroup permutation isomorphic to $\mathbb{Z}_p\times(\mathbb{Z}_p\wr\mathbb{Z}_p)$ is also isomorphic to a Cayley digraph of $\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$.

Lemma 14 *Let $H\leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ be such that $\mathbb{Z}_p\times(\mathbb{Z}_p\wr\mathbb{Z}_p)$. Then H is permutation isomorphic to $X\times S_{p^2}$ or $C((X\wr Y)\times Z)$, where $X,Y,Z\leq S_p$ are 2-closed and $C\leq \mathrm{Aut}(\mathbb{Z}_p^3)$.*

The following fact is quite well-known, and will be used implicitly throughout this paper. It is stated and proved here for convenience and conpleteness.

Lemma 15 *Let G be a regular abelian group of order n and $H\leq S_n$ such that $G\leq H$. Then every complete block system of H is genuine and is also formed by the orbits of some subgroup of G.*

Proof. Let $\mathcal{B}$ be a complete block system of H consisting of, say, m blocks of size k. We will show that $\mathrm{fix}_H(\mathcal{B})$ contains a subgroup of G of order k. Indeed, $G/\mathcal{B}$ is transitive and abelian, and as a transitive abelian group is regular [24, Proposition 4.3], $|G/\mathcal{B}| = m$, and so $\mathrm{fix}_G(\mathcal{B})$ has order k. Then $\mathcal{B}$ is formed by the orbits of $\mathrm{fix}_G(\mathcal{B})$, and so of $\mathrm{fix}_H(\mathcal{B})\triangleleft H$.

□

The following is essentially a variant of [12, Lemma 2.8].

Lemma 16 *Let G be a group and $H \leq S_G$ such that $G_L \leq H$. Suppose that $G_L \leq K \triangleleft H$ and any two regular subgroups of K isomorphic to G are conjugate in K. Then $H \leq \mathrm{Aut}(G) \cdot K$.*

Proof. Let $h \in H$, so that $h^{-1}Kh = K$ and $h^{-1}G_Lh \leq K$ is a regular subgroup isomorphic to G. By hypothesis, there exists $k \in K$ such that $k^{-1}h^{-1}G_Lhk = G_L$. By [6, Corollary 4.2B], we then have that $hk \in \mathrm{Aut}(G)\cdot G_L$, so that $hk = \alpha g_L$, $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. As $k \in K$, we have that $h = \alpha(g_Lk^{-1})$ and $g_Lk^{-1} \in K$, so the result follows. □

2 Overgroups of some 2-closed p-subgroups

Our goal in this section is to find the overgroups of the 2-closed p-groups given in Theorem 12 parts (iv), (v), and (vi), and when the overgroups do not normalize some natural p-subgroup, determine all such 2-closed overgroups.

The overgroups for Theorem 12 (iv) are given in Lemma 21, overgroups for Theorem 12 (v) are given in Lemma 25, while such 2-closed overgroups are given in Lemma 26. It is then shown that every group with Sylow p-subgroup as in Theorem 12 (vi) normalizes $\langle \tau_1, \tau_2\rangle = (\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L$ in Lemma 30. We begin with a technical lemma.

Lemma 17 *Let $H \leq S_{n\cdot m}$, $m \geq 4$, admit a complete block system $\mathcal{D}$ with blocks of size m. If $\mathrm{fix}_H(\mathcal{D})|_D \geq A_m$ and $\mathrm{fix}_H(\mathcal{D})$ acts faithfully on $D \in \mathcal{D}$, then $H \leq S_n \times S_m$ or $m = 6$.*

Proof. As $\mathrm{fix}_H(\mathcal{D})|_D \geq A_m$ is primitive, $\mathrm{Stab}_{\mathrm{fix}_H(\mathcal{D})|_D}(d)$, $d \in D$, fixes exactly one point. If $m = 4$, then A_4 has a unique subgroup of order 4, and so by the comments following [6, Lemma 1.6B], A_4 has a unique transitive representation of degree 4. Thus $\mathrm{fix}_H(\mathcal{D})|_D$ is equivalent to $\mathrm{fix}_H(\mathcal{D})|_{D'}$ for all $D, D' \in \mathcal{D}$. If $m \geq 5$, $m \neq 6$, then by [3, Table], we have that $\mathrm{fix}_H(\mathcal{D})|_D$ is equivalent to $\mathrm{fix}_H(\mathcal{D})|_{D'}$ for every $D, D' \in \mathcal{D}$. The result then follows by [11, Lemma 4.1]. □

For the following result, let $P_1 = \langle \tau_1, \tau_2, \tau_1|_B : B \in \mathcal{B}_{1,1}\rangle$ and $P_2 = \langle \tau_1, \tau_2, \tau_2^p|_B : B \in \mathcal{B}_2\rangle$. Also, we define a *minimal complete block system* $\mathcal{B}$ of a transitive group H to be a complete block system of H such that there is no nontrivial complete block system $\mathcal{C} \prec \mathcal{B}$.

Lemma 18 *Let $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$, $p \geq 3$, have Sylow p-subgroup $P = P_1$ or P_2. Then H is imprimitive, and if $\mathcal{D}$ is a minimal complete block system of H, then $\mathcal{D}$ has blocks of prime size.*

Proof. By [24, Theorem 25.5], H is imprimitive or doubly-transitive. The primitive groups that contain a transitive abelian subgroup are given in [19, Theorem 1.1], while the doubly-transitive groups are given in [3, Table]. We conclude that H is imprimitive. Let $\mathcal{D}$ be a minimal complete block system of H. We assume that $\mathcal{D}$ consists of p blocks of size p^2. By [6, Exercise 1.5.10], we have that $\mathrm{Stab}_H(D)|_D$ is primitive for $D \in \mathcal{D}$. As

$H/\mathcal{D} \leq S_p$, we have that a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{D})$ has order p^{p+1} as P has order p^{p+2}.

Suppose that $\mathrm{fix}_H(\mathcal{D})$ acts faithfully on $D \in \mathcal{D}$. Then a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{D})|_D$ has order p^{p+1} and degree p^2, and so must be $\mathbb{Z}_p \wr \mathbb{Z}_p$. Perusing the list of primitive groups that contain a transitive abelian subgroup in [19, Theorem 1.1], we see that $\mathrm{Stab}_H(\mathcal{D})|_D \geq A_{p^2}$. As $p \geq 3$, A_{p^2} is simple, and so $\mathrm{fix}_H(\mathcal{D})|_D \geq A_{p^2}$. It then follows by Lemma 17 that H is canonically isomorphic to a subgroup of $S_p \times S_{p^2}$, and so has Sylow p-subgroup isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}_p \wr \mathbb{Z}_p)$. However, P is not isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}_p \wr \mathbb{Z}_p)$, a contradiction. We henceforth assume that $\mathrm{fix}_H(\mathcal{D})$ acts unfaithfully on $D \in \mathcal{D}$.

As $\mathcal{D}$ is genuine, $\mathcal{D}$ is formed by the orbits of some subgroup of $\langle \tau_1, \tau_2 \rangle$ of order p^2, and so $\mathcal{D}$ is formed by the orbits of $\langle \tau_1^a \tau_2 \rangle$ or $\langle \tau_1, \tau_2^p \rangle$ for some $a \in \mathbb{Z}_p$. Then $\langle \tau_2^p \rangle$ is contained in every subgroup of $\langle \tau_1, \tau_2 \rangle$ of order p^2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two distinct complete block systems of $\langle \tau_1, \tau_2 \rangle$ with blocks of size p^2. Considering intersections of blocks chosen from $\mathcal{D}_1$ and $\mathcal{D}_2$, we see that each such intersection has at most p elements contained in it, every element of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ is contained in such an intersection, and there are at most p^2 such intersections. We conclude that each such intersection has order exactly p (and as $\langle \tau_2^p \rangle \leq \mathrm{fix}_H(\mathcal{D}_1) \cap \mathrm{fix}_H(\mathcal{D}_2)$ each such intersection is an orbit of $\langle \tau_2^p \rangle$).

Let $D \in \mathcal{D}$. Then for some $D' \neq D$, $D' \in \mathcal{D}$, there exists $\gamma \in \mathrm{fix}_H(\mathcal{D})$ such that $\gamma|_D = 1$ but $\gamma|_{D'} \neq 1$. Then $\langle \gamma \rangle^{\mathrm{fix}_H(\mathcal{D})}$, the normal closure of $\langle \gamma \rangle$ in $\mathrm{fix}_H(\mathcal{D})$, is normal in $\mathrm{fix}_H(\mathcal{D})$, so the orbits of $\langle \gamma \rangle^{\mathrm{fix}_H(\mathcal{D})}|_{D'}$ form a complete block system of $\mathrm{fix}_H(\mathcal{D})|_{D'}$, and so have order p or p^2. We conclude that some element δ of $\mathrm{fix}_H(\mathcal{D})$ that is the identity on D has order p on D'. After an appropriate conjugation by an element of $\mathrm{fix}_H(\mathcal{D})$, if necessary, we may assume that this element is in P. If $P = P_1$ then the only nontrivial elements that fix a point are contained in $\langle \tau_1|_B : B \in \mathcal{B}_{1,1} \rangle$, while if $P = P_2$ then the only nontrivial elements that fix a point are contained in $\langle \tau_2^p|_B : B \in \mathcal{B}_2 \rangle$. In the former case, let $B \in \mathcal{B}_{1,1}$ such that $\delta|_B$ is semiregular of order p while in the latter case let $B \in \mathcal{B}_2$ such that $\delta|_B$ is semiregular of order p. If $P = P_1$ and $\mathcal{D} \neq \mathcal{B}_{1,1}$ or $P = P_2$ and $\mathcal{D} \neq \mathcal{B}_2$, then by arguments in the preceding paragraph $|D \cap B| = p$ for every $B \in \mathcal{B}_{1,1}$ if $P = P_1$ or $B \in \mathcal{B}_2$ if $P = P_2$. We conclude that δ is not the identity on any block of $\mathcal{D}$, a contradiction. Thus if $P = P_1$ then $\mathcal{D} = \mathcal{B}_{1,1}$ while if $P = P_2$ then $\mathcal{D} = \mathcal{B}_2$.

As $P = P_1$ and $\mathcal{D} = \mathcal{B}_{1,1}$ or $P = P_2$ and $\mathcal{D} = \mathcal{B}_2$, for some element $\tau \in \langle \tau_1, \tau_2 \rangle$ we have that $\tau|_{D'} \in P$ ($\tau = \tau_1$ if $P = P_1$ while $\tau = \tau_2^p$ if $P = P_2$) for some $D' \in \mathcal{D}$. Then $\tau|_D \in \mathrm{fix}_H(\mathcal{D})|_{D'}$ for every $D \in \mathcal{D}$. As a normal subgroup of a primitive group is transitive [24, Theorem 8.8], the normal closure of $\langle \tau|_D \rangle$ in $\mathrm{fix}_H(\mathcal{D})$ is transitive on D and the identity on any block of $\mathcal{D}$ not D. We conclude that the order of $\mathrm{fix}_H(\mathcal{D})$ is divisible by $(p^2)^p$, and so $2p = p + 1$ or $p = 1$, a contradiction. □

Definition 19 *For a permutation group $H \leq S_\Omega$, we define the support of H, denoted* $\mathrm{supp}(H)$*, to be the set of all $x \in \Omega$ such that there exists $h \in H$ such that $h(x) \neq x$.*

Lemma 20 *Let $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ have Sylow p-subgroup $P = \langle \tau_1, \tau_2, \tau_2^p|_B : B \in \mathcal{B}_2 \rangle$. Then $\beta H \beta^{-1} \leq \{(i,j) \mapsto (\omega(i), \alpha j + a + pb_i) : \omega \in S_p, \alpha \in \mathbb{Z}_{p^2}^*, a, b_i \in \mathbb{Z}_p\}$ for some $\beta \in$* $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

Proof. By Lemma 18 we have that H admits a genuine complete block system $\mathcal{D}$ with blocks of prime size formed by the orbits of some subgroup K of $\langle\tau_1,\tau_2\rangle$. Thus $K = \langle\tau_2^p\rangle$ or $\langle\tau_1\tau_2^{ap}\rangle$, where $a \in \mathbb{Z}_p$. As $\tau_2|_B \in P$ for every $B \in \mathcal{B}_2$, it cannot be the case that $K = \langle\tau_1\tau_2^{ap}\rangle$ (as this would imply that $\mathrm{fix}_P(\mathcal{D}) = \langle\tau_1\tau_2^{ap}\rangle \triangleleft P$, which is not true), so that $\mathcal{D}$ is formed by the orbits of $\langle\tau_2^p\rangle$. Observe that $\langle\tau_2^p|_B : B \in \mathcal{B}_2\rangle$ is a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{D})$, so if $h \in H$ there exists $g \in \mathrm{fix}_H(\mathcal{D})$ such that $(hg)^{-1}\langle\tau_2^p|_B : B \in \mathcal{B}_2\rangle(hg) = \langle\tau_2^p|_B : B \in \mathcal{B}_2\rangle$. As g fixes each block of $\mathcal{D}$ and $\mathcal{D} \prec \mathcal{B}_2$, we have that g fixes each block of $\mathcal{B}_2$. Now, if $B \in \mathcal{B}_2$ and $h \in H$, then $(hg)^{-1}\tau_2^p|_B(hg) = \tau_2^{xp}|_{B'}$ for some $B' \in \mathcal{B}_2$ and $x \in \mathbb{Z}_p^*$. We conclude that hg maps the support of $\tau_2^p|_B$ to the support of $\tau_2^p|_{B'}$, and so $hg(B) = B'$. As $g(B) = B$, $h(B) = B'$. Thus $\mathcal{B}_2$ is a complete block system of H as well.

Now, a Sylow p-subgroup of $\mathrm{Stab}_H(\mathcal{B}_2)|_B$, $B \in \mathcal{B}_2$, must be cyclic of order p^2 as a Sylow p-subgroup of $\mathrm{fix}_P(\mathcal{B}_2)|_B = \langle\tau_2\rangle|_B$, $B \in \mathcal{B}_2$, and the fact that $\mathrm{Stab}_P(B) = \mathrm{fix}_P(\mathcal{B}_2)$ as $P/\mathcal{B}_2$ has order p and so is regular. As the blocks of $\mathcal{D}$ contained within a block $B \in \mathcal{B}_2$ form a complete block system of $\mathrm{Stab}_H(\mathcal{B}_2)|_B$, $\mathrm{Stab}_H(\mathcal{B}_2)|_B$ is imprimitive. By Theorem 7 we have that $\mathrm{Stab}_H(\mathcal{B}_2)|_B$ has a unique Sylow p-subgroup, which is $\langle\tau_2\rangle|_B$, and so $\mathrm{Stab}_H(\mathcal{B}_2) \le \{x \mapsto ax+b : a \in \mathbb{Z}_{p^2}^*, b \in \mathbb{Z}_{p^2}\}$ as this latter group is the normalizer of a regular cyclic subgroup in S_{p^2} by [6, Corollary 4.2B]. Additionally, as a Sylow p-subgroup of $\mathrm{Stab}_H(\mathcal{B}_2)|_B$ has order p^2, if $x \mapsto ax+b$ is contained in $\mathrm{Stab}_H(\mathcal{B}_2)|_B$, then $\gcd(|a|,p) = 1$. By the Embedding Theorem [21, Theorem 1.2.6], we have that $h(i,j) = (\omega(i), \alpha_i j + c_i)$, $\omega \in S_p$, $\alpha_i \in \mathbb{Z}_{p^2}^*$ of order relatively prime to p, and $c_i \in \mathbb{Z}_{p^2}$. As $\mathrm{fix}_P(\mathcal{D})$ has order p^p and $|P| = p^{p+2}$, we have that a Sylow p-subgroup of $H/\mathcal{D}$ has order p^2 and, as $\mathcal{B}_2$ is a complete block system of H, $H/\mathcal{D}$ is imprimitive. By [15, Theorem 4 and Lemma 1], we have that $H/\mathcal{D}$ is conjugate to a subgroup of $S_p \times S_p$.

Assume for the moment that $H/\mathcal{D} \le S_p \times S_p$. Then $\alpha_i \equiv \alpha_{i'} \pmod p$ and $c_i \equiv c_{i'} \pmod p$ for every $i, i' \in \mathbb{Z}_p$. We may thus assume without loss of generality that $c_i \equiv 0 \pmod p$ for every $i \in \mathbb{Z}_p$. As $\tau_2^p|_B \in H$ for all $B \in \mathcal{B}_2$, we may assume without loss of generality that $c_i = 0$ for all $i \in \mathbb{Z}_p$. Equivalently, $c_i = a + pb_i$ for some $a, b_i \in \mathbb{Z}_p$.

If $\alpha_k = c + bp$ for some $b, c \in \mathbb{Z}_p$ with $b \ne 0$, then $\alpha_k^p \equiv c^p \pmod{p^2}$, and so p divides $|\alpha_k|$, which is not possible. Hence $\alpha_k = c$ for all $k \in \mathbb{Z}_p$. The result will then follow provided there exists $\beta \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ such that $\beta H \beta^{-1}/\mathcal{D} \le S_p \times S_p$.

Now, as $H/\mathcal{D}$ is conjugate in S_{p^2} to a subgroup of $S_p \times S_p$, there exists $\delta \in S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ such that $\delta H \delta^{-1} \le \{(i,j) \mapsto (\omega(i), \alpha j + a + pb_i : \omega \in S_p, a, b_i \in \mathbb{Z}_p\} = L$. By [8, Lemma 8] (we remark that the statement of [8, Lemma 8] holds for a Cayley graph Γ, but the proof only really depends on a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ being P) there exists $\gamma \in L$ such that $\gamma\delta\langle\tau_1,\tau_2\rangle\delta^{-1}\gamma^{-1} = \langle\tau_1,\tau_2\rangle$, and, of course, $\gamma\delta H\delta^{-1}\gamma^{-1} \le L$. As $\gamma\delta\langle\tau_1,\tau_2\rangle\delta^{-1}\gamma^{-1} = \langle\tau_1,\tau_2\rangle$, by [6, Corollary 4.2B], $\gamma\delta \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) \cdot \langle\tau_1,\tau_2\rangle$, so we may assume that $\gamma\delta = \beta \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$. Then $\beta H \beta^{-1}/\mathcal{D} \le S_p \times S_p$ and the result follows. $\square$

Lemma 21 *Let $H \le S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ have Sylow p-subgroup $P = \langle\tau_1,\tau_2,\tau_2^p|_B : B \in \mathcal{B}_2\rangle$. Also suppose that $H \le \{(i,j) \mapsto (\omega(i), \alpha j + a + pb_i) : \omega \in \mathrm{AGL}(1,p), \alpha \in \mathbb{Z}_{p^2}^*, a, b_i \in \mathbb{Z}_p\}$. Then $H = A \cdot P$ for some $A \le \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.*

PROOF. Straightforward computations will show that if $h \in H$ then h normalizes $\langle\tau_2\rangle$, and as each $\omega \in \mathrm{AGL}(1,p)$, we see that $h^{-1}\tau_1 h \in \langle\tau_1, \tau_2^p|_B : B \in \mathcal{B}\rangle$. We conclude that H normalizes P. By [8, Lemma 8], we have that any two transitive subgroups of P isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ are conjugate in P. The result follows then follows from Lemma 16. □

This completes our consideration of overgroups of p-groups given in Theorem 12 (iv), and we now begin to consider the overgroups of those p-groups given in Theorem 12 (v).

Lemma 22 *Let $H \leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ be transitive with Sylow p-subgroup $P = \langle\tau_1, \tau_2, \tau_1|_B : B \in \mathcal{B}_{1,1}\rangle$. Assume that H admits a complete block system $\mathcal{D}$ with blocks of prime size and $H/\mathcal{D}$ is imprimitive. Then H admits a complete block system with blocks formed by the orbits of $\langle\tau_1\rangle$.*

PROOF. Suppose that $\mathcal{D}$ is formed by the orbits of $K \neq \langle\tau_1\rangle$, where $K \leq \langle\tau_1,\tau_2\rangle$. Then a Sylow p-subgroup of $H/\mathcal{D}$ is isomorphic to $\mathbb{Z}_p \wr \mathbb{Z}_p$, and so admits a unique complete block system $\mathcal{F}$ of p blocks of size p. As a Sylow p-subgroup of H is P, $\mathcal{F}$ must be formed by the orbits of $\langle\tau_1\rangle/\mathcal{D}$. Then H admits a complete block system $\mathcal{E}$ consisting of p blocks of size p^2 induced by $\mathcal{F}$, so that $\mathcal{E}$ is formed by the orbits of $\langle\tau_1, \tau_2^p\rangle$. Hence $\mathcal{E} = \mathcal{B}_{1,1}$. Then $\tau_1|_B \in \mathrm{fix}_H(\mathcal{B}_{1,1})$ for every $B \in \mathcal{B}_{1,1}$. Let $B \in \mathcal{B}_{1,1}$, and $L = \langle\tau_1|_B\rangle^{\mathrm{Stab}_H(B)}$. Note that every element of order p of L fixes all points outside B. The only elements of P with this property are elements of $\langle\tau_1|_B : B \in \mathcal{B}_{1,1}\rangle$, as P admits a complete block system $\mathcal{L}$ formed by the orbits of $\langle\tau_1\rangle$ and $P/\mathcal{L} \cong \mathbb{Z}_{p^2}$. If $L|_B$ is transitive, then $L|_B$ contains a transitive Sylow p-subgroup, say P_B. We conclude that $1_{S_p} \wr P_B \leq H$, and so a Sylow p-subgroup of H has order at least $p \cdot (p^2)^p$. However, P has order $p^2 \cdot p^p$ and so $p = 1$, a contradiction. Thus $L|_B$ is intransitive, and so the orbits of $L|_B$ form a complete block system of $\mathrm{Stab}_H(B)|_B$, which is necessarily formed by the orbits of $\langle\tau_1|_B\rangle$. By [6, Exercise 1.5.10], the set of all blocks conjugate to an orbit of $\langle\tau_1|_B\rangle$ forms a complete block system of H, and so H admits a complete block system formed by the orbits of $\langle\tau_1\rangle$. □

Lemma 23 *Let $p \geq 5$ and $H \leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ be transitive with Sylow p-subgroup $P = \langle\tau_1, \tau_2, \tau_1|_B : B \in \mathcal{B}_{1,1}\rangle$. Assume that H admits a complete block system $\mathcal{D}$ with blocks of prime size and $H/\mathcal{D}$ is primitive. Then $\mathcal{D}$ is formed by the orbits of $\langle\tau_1\rangle$.*

PROOF. Assume $\mathcal{D}$ is not formed by the orbits of $\langle\tau_1\rangle$, so that a Sylow p-subgroup of $H/\mathcal{D}$ is $\mathbb{Z}_p\wr\mathbb{Z}_p$. As $H/\mathcal{D}$ is primitive and $\mathbb{Z}_p\wr\mathbb{Z}_p$ contains a regular subgroup isomorphic to $\mathbb{Z}_{p^2}$, we have that $H/\mathcal{D}$ is doubly-transitive as $\mathbb{Z}_{p^2}$ is a Burnside group [6, Theorem 3.5A]. Perusing the list of doubly-transitive groups of degree p^2 [19, Theorem 1.1] (or [16, Theorem 3]) that contain a regular cyclic subgroup and observing that no doubly-transitive group of prime-squared degree contained in any $\mathrm{P\Gamma L}(n,k)$ contains a Sylow p-subgroup isomorphic to $\mathbb{Z}_p\wr\mathbb{Z}_p$ by Lemma 8, we have that $A_{p^2} \leq H/\mathcal{D}$. Thus $A_p \wr A_p \leq H/\mathcal{D}$ and A_p is nonsolvable as $p \geq 5$. Let $L \leq H$ be maximal such that $L/\mathcal{D} = A_p \wr A_p$. Then $L/\mathcal{D}$ is imprimitive, and so by Lemma 22, L admits a complete block system $\mathcal{I}$ formed by the orbits of $\langle\tau_1\rangle$. Then a Sylow p-subgroup of $L/\mathcal{I}$ is cyclic of order p^2 as $P/\mathcal{I}$ is cyclic. As $L/\mathcal{D} = A_p \wr A_p$,

$L/\mathcal{D}$ admits a complete block system $\mathcal{E}$ necessarily formed by the orbits of $\langle\tau_1\rangle/\mathcal{D}$, and so L admits a complete block system $\mathcal{F}$ consisting of p blocks of size p^2 formed by the orbits of $\langle\tau_1,\tau_2^p\rangle$. Note that $\mathcal{I}\prec\mathcal{F}$. As $A_p=(L/\mathcal{D})/\mathcal{E}=L/\mathcal{F}$, we have that $L/\mathcal{I}$ is nonsolvable as $p\geq 5$. As a Sylow p-subgroup of $L/\mathcal{I}$, is cyclic and imprimitive, by Theorem 7 we have that $L/\mathcal{I}$ contains a normal transitive cyclic subgroup, a contradiction as the normalizer of a p^2-cycle in S_{p^2} is isomorphic to $\{x\mapsto ax+b : a\in\mathbb{Z}_{p^2}^*, b\in\mathbb{Z}_{p^2}\}$ by [6, Corollary 4.2B].

□

Definition 24 *View each element of $\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ uniquely as $(i,j+kp)$, where $i,j,k\in\mathbb{Z}_p$. Define $\alpha_2:\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ by $\alpha_2(i,j+kp)=(i+k,j+kp)$. Note that $\alpha_2^{-1}(i,j+kp)=(i-k,j+kp)$. Straightforward computations will then show that $\alpha_2^{-1}\tau_2\alpha_2(i,j+kp)=(i,j+1+kp)$ if $j\neq p-1$ while $\alpha_2^{-1}\tau_2\alpha_2(i,p-1+kp)=(i-1,(k+1)p)$. Then $\mathcal{B}_{1,1}=\{B_j : j\in\mathbb{Z}_p\}$, where $B_j=\{(i,j+kp) : i,k\in\mathbb{Z}_p\}$. We then have that $\alpha_2^{-1}\tau_2\alpha_2=\tau_2(\tau_1^{-1}|_{B_{p-1}})$. It is not hard to see that α_2 commutes with $\tau_1|_B$ for every $B\in\mathcal{B}_{1,1}$ and $\alpha_2^{-1}\tau_2^p\alpha_2=\tau_1^{-1}\tau_2^p$. Then α_2 normalizes $\langle\tau_1|_B,\tau_2 : B\in\mathcal{B}_{1,1}\rangle$. We now view $\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ in the usual fashion. For $a\in\mathbb{Z}_{p^*}$, define $\bar{a}:\mathbb{Z}_p\times\mathbb{Z}_{p^2}\mapsto\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ by $\bar{a}(i,j)=(a^{-1}i,j)$. Note that $\bar{a}^{-1}\tau_1\bar{a}=\tau_1^a$ and $\bar{a}$ commutes with τ_2.*

Lemma 25 *Let $H\leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ be transitive with Sylow p-subgroup $P=\langle\tau_1,\tau_2,\tau_1|_B : B\in\mathcal{B}_{1,1}\rangle$. If $p\geq 5$, then for $c=0$ or 1, and $a\in\mathbb{Z}_p^*$, $\alpha_2^{-c}\bar{a}^{-1}H\bar{a}\alpha_2^c\leq\{(i,j)\mapsto(\beta_{j \pmod p}(i),\alpha j+b) : \beta_{j \pmod p}\in S_p, \alpha\in\mathbb{Z}_{p^2}^*, b\in\mathbb{Z}_{p^2}\}$.*

Proof. By Lemma 18, we have that H is imprimitive and a minimal complete block system $\mathcal{D}$ of H has blocks of size p formed by the orbits of some subgroup $K\leq G$. As $H/\mathcal{D}$ is imprimitive or primitive, by Lemma 22 or 23, respectively, we have that H admits a complete block system formed by the orbits of $\langle\tau_1\rangle$. We thus assume without loss of generality that $\mathcal{D}$ is formed by the orbits of $\langle\tau_1\rangle$.

As $\mathcal{D}$ is formed by the orbits of $\langle\tau_1\rangle$, and a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{D})$ is $\langle\tau_1|_B : B\in\mathcal{B}_{1,1}\rangle$, by Lemma 10 $\mathcal{B}_{1,1}$ is a complete block system of H. Thus $H/\mathcal{D}$ is imprimitive. Also, a Sylow p-subgroup of $H/\mathcal{D}$ is regular and cyclic as a Sylow p-subgroup of $P/\mathcal{D}$ is regular and cyclic. By Theorem 7 we conclude that $H/\mathcal{D}$ contains a normal regular cyclic subgroup, and so if $h\in H$, then $h(i,j)=(\beta_j(i),\alpha j+b)$, $\beta_j\in S_p$, $\alpha\in\mathbb{Z}_{p^2}^*$ of order not divisible by p, and $b\in\mathbb{Z}_{p^2}$.

As $\mathcal{B}_{1,1}$ is a complete block system of H and as P is a Sylow p-subgroup of H, we have that a Sylow p-subgroup of $\mathrm{Stab}_H(B)|_B$ is elementary abelian and $\mathrm{Stab}_H(B)|_B$ is imprimitive, $B\in\mathcal{B}_{1,1}$. By [15, Theorem 4], we have that $\mathrm{Stab}_H(B)|_B$ is permutation isomorphic to a subgroup of $S_p\times S_p$, and so $\mathrm{Stab}_H(B)|_B$ admits a complete block system $\mathcal{E}$ consisting of p blocks of size p and no block of $\mathcal{E}$ is contained in $\mathcal{D}$. By [6, Exercise 1.5.10], H admits a complete block system $\mathcal{F}$ whose blocks consist of those blocks conjugate in H to a block of $\mathcal{E}$. Then $\mathcal{F}$ is genuine, being formed by the orbits of $\langle\tau_1^c\tau_2^{ap}\rangle$, $c=0,1$, $a\in\mathbb{Z}_p^*$, and if $c=0$, then we may and do take $a=1$. If $c=1$, then $\alpha_2^{-c}\bar{a}^{-1}\tau_1\bar{a}\alpha_2^c=\tau_1^a$ and $\alpha_2^{-c}\tau_2^{ap}\alpha_2^c=\tau_1^{-a}\tau_2^{ap}$. Thus $\alpha_2^{-1}\bar{a}^{-1}\tau_1\tau_2^{ap}\bar{a}\alpha_2=\tau_2^{ap}$. Also, α_2 normalizes P as does $\bar{a}$.

We may then assume without loss of generality that $\mathcal{F}$ is formed by the orbits of $\langle \tau_2^p \rangle$ by replacing H with $\alpha_2^{-c}\bar{a}^{-1}H\bar{a}\alpha_2^c$ for $c = 0, 1$. Then $\mathrm{Stab}_H(B)|_B \leq S_p \times S_p$.

Now let $h \in H$. Then $h(i,j) = (\beta_j(i), \alpha j + b)$, where $\beta_j \in S_p$, $\alpha \in \mathbb{Z}_{p^2}^*$ has order relatively prime to p, and $b \in \mathbb{Z}_{p^2}$. Also, $h^{-1}\langle \tau_1, \tau_2 \rangle h \leq H$ and is a regular subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Setting $K = (\langle \tau_1, \tau_2, h^{-1}\langle \tau_1, \tau_2 \rangle h \rangle)^{(2)}$, we see that a Sylow p-subgroup of K is either $\langle \tau_1, \tau_2 \rangle$ or P by Theorem 12. In the latter case, by [8, Lemma 9], and in the former case by a Sylow Theorem, there exists $k \in K$ such that $k^{-1}h^{-1}\langle \tau_1, \tau_2 \rangle hk = \langle \tau_1, \tau_2 \rangle$. Then kh normalizes $\langle \tau_1, \tau_2 \rangle$. Observe now that $\langle \tau_1, \tau_2, h^{-1}\langle \tau_1, \tau_2 \rangle h \rangle \leq \mathbb{Z}_{p^2} \wr S_p$ as $H/\mathcal{D}$ has a normal cyclic Sylow p-subgroup of order p^2, in which case $K \leq \mathbb{Z}_{p^2} \wr S_p$ as $\mathbb{Z}_{p^2} \wr S_p$ is 2-closed. Thus $K/\mathcal{D} = \langle \tau_2 \rangle/\mathcal{D}$, and so by replacing k with $\tau_2^d k$ for appropriate $d \in \mathbb{Z}$, we may assume without loss of generality that $k \in \mathrm{fix}_K(\mathcal{D})$. Let $B' \in \mathcal{B}_{1,1}$. Then there exists $e_{B'} \in \mathbb{Z}$ such that $kh\tau_2^{e_{B'}}(B') = B'$, and there exists $d_{B'} \in \mathbb{Z}_p$ such that $\tau_2^{-d_{B'}}(kh\tau_2^{e_{B'}})\tau_2^{d_{B'}}$ stabilizes B. Then $\tau_2^{-d_{B'}}(kh\tau_2^{e_{B'}})\tau_2^{d_{B'}}|_B \leq S_p \times S_p$. As $\tau_2^{-d_{B'}}k\tau_2^{d_{B'}}|_B \leq S_p \times S_p$ as $\tau_2^{-d_{B'}}k\tau_2^{d_{B'}} \in \mathrm{Stab}_H(B)$, we see that $\tau_2^{-d_{B'}}h\tau_2^{e'_B}\tau_2^{d_{B'}}|_B \leq S_p \times S_p$ for every $B \in \mathcal{B}_{1,1}$. As $\tau_2(i,j) = (i, j+1)$, we conclude that $h\tau_2^{e_B}|_{B'} \leq S_p \times S_p$. Hence if $i \equiv i' \pmod{p}$, then $\beta_i = \beta_{i'}$, and the result follows. □

Lemma 26 *Let $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ be transitive and 2-closed with Sylow p-subgroup $P = \langle \tau_1, \tau_2, \tau_1|_B : B \in \mathcal{B}_{1,1} \rangle$. If $H \leq \{(i,j) \mapsto (\beta_{j \pmod{p}}(i), \alpha j + b) : \beta_{j \pmod{p}} \in \mathrm{AGL}(1,p), \alpha \in \mathbb{Z}_{p^2}^*, b \in \mathbb{Z}_{p^2}\}$, then there exists $D \leq \mathbb{Z}_p^*$ and $A \leq \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ such that $H = A \cdot \{(i,j) \mapsto (d_{j \pmod{p}} i + c_{j \pmod{p}}, j + b) : d_{j \pmod{p}} \in D, c_{j \pmod{p}} \in \mathbb{Z}_p,$ and $b \in \mathbb{Z}_{p^2}\}$.*

Proof. First observe that H admits a complete block system $\mathcal{B}$ of p^2 blocks of size p formed by the orbits of $\langle \tau_1 \rangle$. As $H \leq \{(i,j) \mapsto (\beta_{j \pmod{p}}(i), \alpha j + b) : \beta_{j \pmod{p}} \in \mathrm{AGL}(1,p), \alpha \in \mathbb{Z}_{p^2}^*, b \in \mathbb{Z}_{p^2}\}$ we have that $\mathrm{fix}_H(\mathcal{B})|_B \leq \mathrm{AGL}(1,p)$. Let $\mathrm{fix}_H(\mathcal{B})|_B = D \cdot (\mathbb{Z}_p)_L$, where $D \leq \mathbb{Z}_p^*$. We observe that such a D exists as $\mathrm{fix}_H(\mathcal{B})|_B$ has a unique Sylow p-subgroup $(\mathbb{Z}_p)_L$, and [6, Corollary 4.2B]. Define an equivalence relation $\equiv$ on the blocks of $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $h \in \mathrm{fix}_H(\mathcal{B})$ then $h|_B$ is a p-cycle if and only if $h|_{B'}$ is also a p-cycle. Clearly the equivalence classes of $\equiv$ form $\mathcal{B}_{1,1}$ as $\langle \tau_1|_{B_{1,1}} : B_{1,1} \in \mathcal{B}_{1,1} \rangle$ is a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$. By Lemma 10 we have that if $h \in \mathrm{fix}_H(\mathcal{B})$, then $h|_{B_{1,1}} \in \mathrm{fix}_H(\mathcal{B})$ for every $B_{1,1} \in \mathcal{B}_{1,1}$, where as usual $h|_{B_{1,1}}$ is the permutation equal to h on $B_{1,1}$ and the identity on every other block of $\mathcal{B}_{1,1}$. Thus $\mathrm{fix}_H(\mathcal{B}) = \{(i,j) \mapsto (d_{j \pmod{p}} i + c_{j \pmod{p}}, j) : d_{j \pmod{p}} \in D, c_{j \pmod{p}} \in \mathbb{Z}_p\}$. Furthermore, as $\langle \tau_2 \rangle/\mathcal{B} \triangleleft H/\mathcal{B}$ and $\mathrm{fix}_H(\mathcal{B}) \triangleleft H$, we have that $K = \{(i,j) \mapsto (d_{j \pmod{p}} i + c_{j \pmod{p}}, j + b) : d_{j \pmod{p}} \in D, c_{j \pmod{p}} \in \mathbb{Z}_p,$ and $b \in \mathbb{Z}_{p^2}\} \triangleleft H$.

Now let $h \in H$. Then $h^{-1}\langle \tau_1, \tau_2 \rangle h \leq K$ and is contained in a Sylow p-subgroup of K. Hence there exists $k_1 \in K$ such that $k_1^{-1}h^{-1}\langle \tau_1, \tau_2 \rangle hk_1 \leq P$. By [8, Lemma 9], there exists $k_2 \in P$ such that $k_2^{-1}k_1^{-1}h^{-1}\langle \tau_1, \tau_2 \rangle hk_1k_2 = \langle \tau_1, \tau_2 \rangle$. The result then follows by Lemma 16. □

This completes our consideration of overgroups of p-groups given in Theorem 12 (v), and we now begin to consider the overgroups of those p-groups given in Theorem 12 (vi).

Lemma 27 *Let $\gamma : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ by $\gamma(i,j) = (i + [aj \pmod p], j + ibp)$, $a, b \in \mathbb{Z}_p^*$, and $P = \langle \tau_1, \tau_2, \gamma\rangle$ be of order p^4. Then the only nontrivial complete block systems of P are formed by the orbits of $\langle \tau_2^p\rangle$ and $\langle \tau_1, \tau_2^p\rangle$.*

Proof. Straightforward computations will show that $\gamma^{-1}(i,j) = (i - [aj \pmod p], j - [i - (aj \pmod p)]bp)$. Then $\gamma^{-1}\tau_1\gamma = \tau_1\tau_2^{-bp}$, while $\gamma^{-1}\tau_2\gamma = \tau_1^{-a}\tau_2^{1+abp}$. Then P admits a complete block system $\mathcal{B}$ formed by the orbits of $\langle \tau_2^p\rangle$ as $\mathrm{Z}(P) = \langle \tau_2^p\rangle$ where $\mathrm{Z}(P)$ is the center of P. Also, $K = \langle \tau_1, \tau_2^p, \gamma\rangle \triangleleft P$ as it is a subgroup of index p (or just using direct computation) in P, and so P admits $\mathcal{B}_{1,1}$ as a complete block system. Note that any subgroup L of P of order p^3 different from $\langle \tau_1, \tau_2\rangle$ must contain a subgroup of $\langle \tau_1, \tau_2\rangle$ of order p^2, so L contains either $\langle \tau_1, \tau_2^p\rangle$ or $\langle \tau_1^c\tau_2\rangle$. In the latter case, as $L \triangleleft P$ as L is of index p in P, $\gamma^{-1}\tau_1^c\tau_2\gamma = \tau_1^{c-a}\tau_2^{1+bp(a-c)}$ is contained in L. If $c \neq a$, then $L \geq \langle \tau_1^c\tau_2, \tau_1^{c-a}\tau_2^{1+bp(a-c)}\rangle = \langle \tau_1, \tau_2\rangle$, a contradiction. If $c = a$, then L contains τ_2 as well, and L contains all of $\langle \tau_1, \tau_2\rangle$, again a contradiction. Thus L contains $\langle \tau_1, \tau_2^p\rangle$. As if $\mathcal{D}$ is a complete block system of P with blocks of size p^2, then $P/\mathcal{D}$ has order p, we must have that $\mathcal{D}$ is formed by the orbits of a subgroup of P of order p^3 that is intransitive, and contains $\langle \tau_1, \tau_2^p\rangle$, and so must be $\mathcal{B}_{1,1}$. By Lemma 15, any complete block system $\mathcal{D}$ of P with blocks of prime size is formed by the orbits of $\langle \tau_2^p\rangle$ or $\langle \tau_1\tau_2^{cp}\rangle$ for some $c \in \mathbb{Z}_p$. If the latter case occurs, then note that the orbits of γ are not contained in the orbits of $\langle \tau_1\tau_2^{cp}\rangle$ for any $c \in \mathbb{Z}_p$ (the orbit of $\langle\gamma\rangle$ that contains $(1,1)$ also contains $(1+a, 1+bp)$ and $(1+2a, 1+2bp+abp)$, for example), and so $\mathrm{fix}_P(\mathcal{D})$ has order p. Thus if $\mathcal{D}$ is formed by the orbits of $\langle \tau_1\tau_2^{cp}\rangle$, then $\langle \tau_1\tau_2^{cp}\rangle \triangleleft P$. However, $\gamma^{-1}\tau_1\tau_2^{cp}\gamma = \tau_1\tau_2^{cp-bp}$. Thus $\mathcal{D}$ is not formed by the orbits of $\langle \tau_1\tau_2^{cp}\rangle$ and so $\mathcal{D}$ is formed by $\langle \tau_2^p\rangle$. The result then follows. $\square$

Lemma 28 *Let $\gamma : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ by $\gamma(i,j) = (i + [aj \pmod p], j + ibp)$, $a, b \in \mathbb{Z}_p^*$, and $P = \langle \tau_1, \tau_2, \gamma\rangle$ be of order p^4. Let $H \leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ have Sylow p-subgroup P and admit $\mathcal{B}_{1,1}$ as a complete block system. Then there exists $K \leq H$ with Sylow p-subgroup P such that $K/\mathcal{B}_{1,1} = H/\mathcal{B}_{1,1}$ and K also admits a complete block system formed by the orbits of $\langle \tau_2^p\rangle$.*

Proof. Let $L = \langle \tau_1, \tau_2^p, \gamma\rangle$, so that L is a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B}_{1,1})$. Let $h \in H$, so that $h^{-1}Lh \leq \mathrm{fix}_H(\mathcal{B}_{1,1})$ is also a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B}_{1,1})$. Hence there exists $\beta_h \in \mathrm{fix}_H(\mathcal{B}_{1,1})$ such that $\beta_h^{-1}h^{-1}Lh\beta_h = L$, so that $h\beta_h$ normalizes L and $h\beta_h/\mathcal{B}_{1,1} = h/\mathcal{B}_{1,1}$. Let $K = \langle h\beta_h : h \in H\rangle$, so that $K/\mathcal{B}_{1,1} = H/\mathcal{B}_{1,1}$ and $L \triangleleft K$. Note that the center $\mathrm{Z}(L)$ of L is $\langle \tau_2^p\rangle$ and, as the center of a group is characteristic, $\langle \tau_2^p\rangle \triangleleft K$ so that K admits the required complete block system. $\square$

Lemma 29 *Let $p \geq 3$, $\gamma : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ by $\gamma(i,j) = (i + [aj \pmod p], j + ibp)$, $a, b \in \mathbb{Z}_p^*$, and $P = \langle \tau_1, \tau_2, \gamma\rangle$ be of order p^4. If $H \leq S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$ with Sylow p-subgroup P, then H admits a complete block system formed by the orbits of $\langle \tau_2^p\rangle$ or $\langle \tau_1, \tau_2\rangle \triangleleft H$.*

Proof. Examining the list of primitive groups that contain a regular abelian subgroup given by [19, Theorem 1.1], we see that H is not primitive. As any complete block system of H is also a complete block system of P, by Lemma 27 we have that either the result follows or $\mathcal{B}_{1,1}$ is the only nontrivial complete block system of H. By [6, Exercise 1.5.10] we have that $\mathrm{Stab}_H(B)|_B$ is primitive for every $B \in \mathcal{B}_{1,1}$. Additionally, $\mathrm{Stab}_H(B)|_B$ contains a Sylow p-subgroup of order p^3 with a normal elementary abelian subgroup of order p^2 (and does not contain a regular cyclic subgroup as $p \geq 3$ - see [15, Lemma 4]). By [19, Theorem 1.1], we have that $\mathrm{Stab}_H(B)|_B \leq \mathrm{AGL}(2,p)$ for every $B \in \mathcal{B}_{1,1}$. By the Embedding Theorem [21, Theorem 1.2.6], we have that H is permutation isomorphic to a subgroup of $H/\mathcal{B}_{1,1} \wr \mathrm{AGL}(2,p)$.

By Lemma 28, there exists $P \leq K \leq H$ such that $K/\mathcal{B}_{1,1} = H/\mathcal{B}_{1,1}$ and the orbits of $\langle \tau_2^p \rangle$ form a complete block system $\mathcal{E}$ of K. Then $K/\mathcal{E}$ has a Sylow p-subgroup of order p^3 that contains a regular elementary abelian subgroup and is imprimitive. By [15, Theorem 4] and [15, Lemma 6], we have that $K/\mathcal{B}_{1,1}$ is permutation isomorphic to a subgroup of $\mathrm{AGL}(1,p)$. Hence H is permutation isomorphic to a subgroup of $\mathrm{AGL}(1,p) \wr \mathrm{AGL}(2,p)$.

As $H \leq \mathrm{AGL}(1,p) \wr \mathrm{AGL}(2,p)$, we have that H normalizes $L = \langle \tau_1|_B, \tau_2^p|_B : B \in \mathcal{B}_{1,1}\rangle \cap H$. Also, $\langle L, P\rangle \leq \mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$, a Sylow p-subgroup of S_{p^3}, so L is contained in P, and so L is contained in $\mathrm{fix}_P(\mathcal{B}_{1,1})$. As $|\mathrm{Stab}_P(0,0)| = p$ and $\mathrm{Stab}_P(0,0) = \mathrm{Stab}_{\mathrm{fix}_P(\mathcal{B}_{1,1})}(0,0)$ (as $P/\mathcal{B}_{1,1}$ has order p and so is regular) also stabilizes only the points $(0,kp)$, $k \in \mathbb{Z}_p$, $\mathrm{fix}_P(\mathcal{B}_{1,1})$ contains p^2 distinct subgroups of order p that are stabilizers of points, and as the identity is contained in all of them, these subgroups contain $p^3-(p^2-1)$ distinct elements. Thus $\mathrm{fix}_P(\mathcal{B})$ only contains p^2-1 nontrivial semiregular elements, and so $\langle \tau_1, \tau_2^p\rangle$ is the only semiregular elementary abelian subgroup of $\mathrm{fix}_P(\mathcal{B}_{1,1})$. Also observe that if $h \in H$, then $h^{-1}\langle \tau_1, \tau_2^p\rangle h \leq L$ is a semiregular elementary abelian subgroup of $\mathrm{fix}_P(\mathcal{B}_{1,1})$, and so is $\langle \tau_1, \tau_2^p\rangle$. Thus $\langle \tau_1, \tau_2^p\rangle \triangleleft H$.

Let $h \in H$. As $H/\mathcal{B}_{1,1} \leq \mathrm{AGL}(1,p)$, there exists $a \in \mathbb{Z}_p^*$ such that $\tau_2^a h^{-1}\tau_2 h/\mathcal{B}_{1,1} = 1$. As $\langle \tau_1, \tau_2^p\rangle \leq h^{-1}\langle \tau_1, \tau_2\rangle h$ we have that τ_2 and $h^{-1}\tau_2 h$ centralize $\langle \tau_1, \tau_2^p\rangle$. Thus $\tau_2^a h^{-1}\tau_2 h$ centralizes $\langle \tau_1, \tau_2^p\rangle$. As a transitive abelian group is self-centralizing [6, Theorem 4.2A (v)], we have that $\tau_2^a h^{-1}\tau_2 h|_B \in \langle \tau_1, \tau_2^p\rangle|_B$ for every $B \in \mathcal{B}_{1,1}$. Thus $\tau_2^a h^{-1}\tau_2 h \in L$ and so $\tau_2^a h^{-1}\tau_2 h \in P$. Finally, observe that the centralizer in $\mathrm{fix}_P(\mathcal{B}_{1,1})$ of $\langle \tau_1, \tau_2^p\rangle$ is $\langle \tau_1, \tau_2^p\rangle$, and so $\tau_2^a h^{-1}\tau_2 h \in \langle \tau_1, \tau_2^p\rangle$. Thus $h^{-1}\langle \tau_1, \tau_2\rangle h = \langle \tau_1, \tau_2\rangle$ and the result follows. $\square$

Lemma 30 *Let $p \geq 5$ be prime, and $\gamma : \mathbb{Z}_p \times \mathbb{Z}_{p^2} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ by $\gamma(i,j) = (i + [aj \pmod p], j + ibp)$, $a, b \in \mathbb{Z}_p^*$, and $P = \langle \tau_1, \tau_2, \gamma\rangle$ be of order p^4. If $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ with Sylow p-subgroup P, then $\langle \tau_1, \tau_2\rangle \triangleleft H$.*

Proof. In view of Lemma 29, we may assume without loss of generality that H admits a complete block system $\mathcal{B}$ formed by the orbits of $\langle \tau_2^p\rangle$. Then $\langle \tau_2^p\rangle$ is a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$ as $\langle \tau_2^p\rangle$ is a Sylow p-subgroup of $\mathrm{fix}_P(\mathcal{B})$. By [11, Lemma 4.2], one of the following is true:

i. $\mathrm{fix}_H(\mathcal{B})$ is cyclic and semiregular of order p,

ii. H is permutation isomorphic to a subgroup of $S_{p^2} \times S_p$. Furthermore, there exists $J \leq S_{p^2}$ and $K \leq S_p$ such that $J \times K \triangleleft H$, or

iii. $\mathrm{fix}_H(\mathcal{B})$ does not act faithfully on $B \in \mathcal{B}$ and a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$ is not semiregular.

Note that (iii) cannot occur as a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$ is $\langle \tau_2^p \rangle$, while H cannot be permutation isomorphic to a subgroup of $S_{p^2} \times S_p$ as P is not. Thus $\mathrm{fix}_H(\mathcal{B})$ is cyclic and semiregular of order p. Note that $H/\mathcal{B}$ is of degree p^2 and has Sylow p-subgroup of order p^3 with a transitive elementary abelian subgroup $\langle \tau_1, \tau_2 \rangle/\mathcal{B}$. By [15, Theorem 4], we have that $\langle \tau_1, \tau_2 \rangle/\mathcal{B} \triangleleft H/\mathcal{B}$ as $p \geq 5$, and so $\mathrm{fix}_H(\mathcal{B}) = \langle \tau_2^p \rangle$. Thus $\langle \tau_1, \tau_2 \rangle \triangleleft H$ as required. $\square$

3 Automorphism groups of Cayley digraphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$

The following result appears in [13, Lemma 28] with the additional hypothesis that G contains a regular cyclic subgroup. This hypothesis was essentially not used in the proof of [13, Lemma 28], and we have the following result.

Lemma 31 *Let $G \leq S_{mk}$ be 2-closed. If G admits a genuine nontrivial complete block system $\mathcal{B}$ consisting of m blocks of size k such that $\mathrm{fix}_G(\mathcal{B})|_B$ is primitive and $\mathrm{fix}_G(\mathcal{B})$ does not act faithfully on $B \in \mathcal{B}$, then $G = G_1 \cap G_2$, where $G_1 = S_r \wr H_1$ and $G_2 = H_2 \wr S_k$, H_1 is a 2-closed group of degree mk/r, H_2 is a 2-closed group of order m, and $r|m$.*

We now prove the main result of this paper, and note that in the statement of this result, α_2 and $\bar{a}$ are as defined in Definition 24.

Theorem 32 *Let $p \geq 5$ be prime, and $H \leq S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$ be a 2-closed group that contains the left regular representation of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Then one of the following is true for some $\alpha_1 \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$, $A \leq \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ of order relatively prime to p, $D \leq \mathbb{Z}_p^*$, and $E \leq \mathbb{Z}_{p^2}^*$ of order relatively prime to p:*

1. *$H = S_{\mathbb{Z}_p \times \mathbb{Z}_{p^2}}$,*
2. *$(\mathbb{Z}_p \times \mathbb{Z}_{p^2})_L \triangleleft H$,*
3. *$\alpha_1^{-1} H \alpha_1 = S_p \times B$, where $B \leq N(p^2)$ is 2-closed of order dividing $(p-1)p^2$ and so has a cyclic Sylow p-subgroup,*
4. *H is permutation isomorphic to $X \times S_{p^2}$ or $C((X \wr Y) \times Z)$, where $X, Y, Z \leq S_p$ are 2-closed and $C \leq \mathrm{Aut}(\mathbb{Z}_p^3)$.*
5. *$\alpha_1^{-1} H \alpha_1 = H_1 \wr H_2$ or $H_2 \wr H_1$ where H_1 is a 2-closed group of degree p and H_2 is a 2-closed group of degree p^2,*
6. *$\alpha_1^{-1} H \alpha_1 = \{(i,j) \mapsto (\omega(i), \alpha j + a + pb_i) : \omega \in S_p, \alpha \in E, a \in \mathbb{Z}_{p^2}, b_i \in \mathbb{Z}_p\}$,*
7. *$\alpha_1^{-1} H \alpha_1 = A \cdot P$, where $P = \langle \tau_1, \tau_2, \tau_2^p|_B : B \in \mathcal{B}_2 \rangle$,*

8. $\alpha_2^{-c}\alpha_1 H\alpha_1^{-1}\alpha_2^c = \{(i,j)\mapsto(\omega_{j \pmod p}(i),\alpha j+b) : \omega_{j \pmod p}\in S_p, \alpha\in E, b\in\mathbb{Z}_{p^2}\}$, $c=0,1$, *and* $\alpha_1=\bar{a}$ *for some* $a\in\mathbb{Z}_p^*$, *or*

9. $\alpha_2^{-c}\alpha_1^{-1}H\alpha_1\alpha_2^c = A\cdot\{(i,j)\mapsto(d_{j \pmod p}i+c_{j \pmod p}, j+b) : d_{j \pmod p}\in D$, $c_{j \pmod p}\in\mathbb{Z}_p$, and $b\in\mathbb{Z}_{p^2}\}$, *for* $c=0,1$, *and* $\alpha_1=\bar{a}$ *for some* $a\in\mathbb{Z}_p^*$.

Proof. Let P be a Sylow p-subgroup of H that contains $\langle\tau_1,\tau_2\rangle$. By Theorem 12, there exists $\alpha_1\in\mathrm{Aut}(\mathbb{Z}_p\times\mathbb{Z}_{p^2})$ such that one of the following is true:

(i) $H=S_{\mathbb{Z}_p\times\mathbb{Z}_{p^2}}$,

(ii) $P=(\mathbb{Z}_p\times\mathbb{Z}_{p^2})_L$,

(iii) $P=\alpha_1^{-1}\langle\tau_1,\tau_2,\tau_2^p|_B : B\in\mathcal{B}_{1,1}\rangle\alpha_1\cong\mathbb{Z}_p\times(\mathbb{Z}_p\wr\mathbb{Z}_p)$,

(iv) $P=\alpha_1^{-1}\langle\tau_1,\tau_2,\tau_2^p|_B : B\in\mathcal{B}_2\rangle\alpha_1$,

(v) $P=\alpha_1^{-1}\langle\tau_1,\tau_2,\tau_1|_B : B\in\mathcal{B}_{1,1}\rangle\alpha_1$,

(vi) if $\gamma:\mathbb{Z}_p\times\mathbb{Z}_{p^2}\mapsto\mathbb{Z}_p\times\mathbb{Z}_{p^2}$ by $\gamma(i,j)=(i+[aj \pmod p)], j+ibp)$, $a,b\in\mathbb{Z}_p^*$, then $P=\alpha_1^{-1}\langle\tau_1,\tau_2,\gamma\rangle\alpha_1$, and $|P|=p^4$,

(vii) $\alpha_1^{-1}P\alpha_1=P_1\wr P_2$, where P_1 is 2-closed p-group of degree p^2 and contains a regular subgroup isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$, and $P_2\le S_p$ is cyclic of order p,

(viii) $\alpha_1^{-1}P\alpha_1=P_1\wr P_2$, where $P_2\le S_p$ is cyclic of order p, and $P_1\le S_{p^2}$ is 2-closed p-subgroup of degree p^2 and contains a regular subgroup isomorphic to $\mathbb{Z}_{p^2}$.

If (i) occurs, then (1) occurs. If (ii) occurs, then by Theorem 13, either (2) occurs or H is permutation isomorphic to $S_p\times B$, where $B\le N(p^2)$ has order dividing $(p-1)p^2$, and so has a cyclic Sylow p-subgroup. Applying Lemma 11, we see that H is also 2-closed. Note that H admits orthogonal complete block systems $\mathcal{B}$ and $\mathcal{C}$ consisting of p blocks of size p^2 and p blocks of size p formed by the orbits of the subgroups of H permutation isomorphic to $1_{S_p}\times B$ and $S_p\times 1_{S_{p^2}}$, respectively. As $\mathcal{B}$ and $\mathcal{C}$ are genuine, $\mathcal{B}$ is formed by the orbits of $\langle\tau_2\tau_1^a\rangle$ while $\mathcal{C}$ is formed by the orbits of $\langle\tau_1\tau_2^{bp}\rangle$, $a,b\in\mathbb{Z}_p$. Let $\alpha_1\in\mathrm{Aut}(\mathbb{Z}_p\times\mathbb{Z}_{p^2})$ such that $\alpha_1^{-1}\langle\tau_2\tau_1^a\rangle\alpha_1=\langle\tau_2\rangle$ and $\alpha_1^{-1}\langle\tau_2\tau_2^{bp}\rangle\alpha_1=\langle\tau_1\rangle$. Such an α_1 exists by [5, pg. 5]. Then $\alpha_1^{-1}H\alpha_1=S_p\times B$ and (3) occurs. If (iii) occurs, then (4) follows by Lemma 14.

If (iv) occurs, then by Lemma 20, we have that $\alpha_1^{-1}H\alpha_1=K\le\{(i,j)\mapsto(\omega(i),\alpha j+a+pb_i) : \omega\in S_p, \alpha\in\mathbb{Z}_{p^2}^*, a,b_i\in\mathbb{Z}_{p^2}\}$. Then K admits $\mathcal{B}_2$ as a complete block system. If $K/\mathcal{B}_2\le\mathrm{AGL}(1,p)$, then (7) occurs by Lemma 21. If $K/\mathcal{B}_2\not\le\mathrm{AGL}(1,p)$, then by Theorem 6, we have that $K/\mathcal{B}_2$ is doubly-transitive with nonabelian simple socle. Let L be the normal closure of $\langle\tau_1,\tau_2\rangle$ in K. As $K/\mathcal{B}_2$ is a doubly-transitive group with nonabelian simple socle, we have that $L/\mathcal{B}_2$ is a nonabelian simple group, and so by Theorem 6, $L/\mathcal{B}$ is doubly-transitive. By the definition of K, we have that $\mathrm{fix}_L(\mathcal{B}_2)|_{B_2}\cong\mathbb{Z}_{p^2}$ for every $B_2\in\mathcal{B}_2$. Then $\mathrm{fix}_L(\mathcal{B}_2)\le\{(i,j)\mapsto(i,j+a+b_ip) : a,b_i\in\mathbb{Z}_{p^2}\}$. As $\langle\tau_2^p|_{B_2} : B_2\in\mathcal{B}_2\rangle=\{(i,j)\mapsto(i,j+b_ip) : b\in\mathbb{Z}_{p^2}\}$, we conclude that K contains a subgroup M such that $M/\mathcal{B}_2 = L/\mathcal{B}_2$ and $\mathrm{fix}_M(\mathcal{B}_2) = \mathbb{Z}_{p^2}$. By [11, Corollary 6.5], we have that $M = T \times \mathbb{Z}_{p^2}$, there $T \le S_p$ is a doubly-transitive nonabelian group. Then $M^{(2)} = S_p \times \mathbb{Z}_{p^2}$ by Lemma 11 and the map $(i,j) \mapsto (\omega(i), j)$ is in K for every $\omega \in S_p$. Thus if $(i,j) \mapsto (\omega(i), \alpha j + a + pb_i) \in K$, then the map $(i,j) \mapsto (i, \alpha j) \in K$. Letting $E \le \mathbb{Z}^*_{p^2}$ such that $\mathrm{fix}_K(\mathcal{B}_2)|_{B_2} = E \cdot (\mathbb{Z}_{p^2})_L$ for some $B_2 \in \mathcal{B}_2$, (6) holds.

If (v) occurs, then by Lemma 25 for $c = 0$ or 1, and $a \in \mathbb{Z}^*_p$, $K = \alpha_2^{-c}\bar{a}^{-1}H\bar{a}\alpha_2^c \le \{(i,j) \mapsto (\beta_{j \pmod p}(i), \alpha j + b) : \beta_{j \pmod p} \in S_p, \alpha \in \mathbb{Z}^*_{p^2}, b \in \mathbb{Z}_{p^2}\}$. Then K admits a complete block system $\mathcal{B}$ formed by the orbits of $\langle \tau_1 \rangle$. If $\mathrm{fix}_K(\mathcal{B})|_B \le \mathrm{AGL}(1,p)$ for every $B \in \mathcal{B}$, then $K \le \{(i,j) \mapsto (\beta_{j \pmod p}(i), \alpha j + b) : \beta_{j \pmod p} \in \mathrm{AGL}(1,p), \alpha \in \mathbb{Z}^*_{p^2}, b \in \mathbb{Z}_{p^2}\}$. Then (9) follows from Lemma 26. Otherwise, $\mathrm{fix}_K(\mathcal{B})|_B$ is a doubly-transitive group with nonabelian simple socle T by Theorem 6. Define an equivalence relation $\equiv$ on the blocks of $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $k \in \mathrm{fix}_K(\mathcal{B})$ then $k|_B$ is a p-cycle if and only if $k|_{B'}$ is also a p-cycle. Clearly the equivalence classes of $\equiv$ are $\mathcal{B}_{1,1}$ as $\langle \tau_1|_{B_{1,1}} : B_{1,1} \in \mathcal{B}_{1,1} \rangle$ is a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$. By Lemma 10 we have that if $k \in \mathrm{fix}_K(\mathcal{B})$, then $k|_{B_{1,1}} \in \mathrm{fix}_K(\mathcal{B})$ for every $B_{1,1} \in \mathcal{B}_{1,1}$. We conclude that $\{(i,j) \mapsto (t_{j \pmod p}(i), j) : t_{j \pmod p} \in T\} \le K$. Then $L = \{(i,j) \mapsto (t(i), j + b) : t \in T, b \in \mathbb{Z}_{p^2}\} \le K$, and $L = T \times \mathbb{Z}_{p^2}$. By Lemma 11, $L^{(2)} = T^{(2)} \times (\mathbb{Z}_{p^2})^{(2)} = S_p \times \mathbb{Z}_{p^2} \le K$ and so if $(i,j) \mapsto (\beta_{j \pmod p}(i), \alpha j + b) \in K$, then the map $(i,j) \mapsto (i, \alpha j) \in K$. Letting $E \le \mathbb{Z}^*_{p^2}$ such that $K/\mathcal{B} = E \cdot (\mathbb{Z}_{p^2})_L$, (8) occurs.

If (vi) occurs, then (2) occurs by Lemma 30. If (vii) or (viii) occur, then let D be a color digraph such that $\mathrm{Aut}(D) = H$. Then D can be written as a nontrivial wreath product as P is a nontrivial wreath product. We conclude that H is a nontrivial wreath product by [14, Theorem 5.7], and so (5) occurs. $\square$

Acknowledgement: The author is indebted to the anonymous referee for comments and suggestions which made the paper both shorter and easier to read.

References

[1] B. Alspach, Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree, *J. Combinatorial Theory Ser. B* **15** (1973), 12–17.

[2] W. Burnside, On some properties of groups of odd order, *J. London Math. Soc.* **33** (1901), 162–185.

[3] P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), no. 1, 1–22.

[4] P. J. Cameron, M. Giudici, G. A. Jones, W. M. Kantor, M. H. Klin, D. Marušič, and L. A. Nowitz, Transitive permutation groups without semiregular subgroups, *J. London Math. Soc.* (2) **66** (2002), no. 2, 325–333.

[5] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 14, Springer-Verlag, Berlin, 1965.

[6] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996.

[7] E. Dobson, Isomorphism problem for Cayley graphs of $\mathbb{Z}_p^3$, *Discrete Math.* **147** (1995), no. 1-3, 87–94.

[8] E. Dobson, Classification of vertex-transitive graphs of order a prime cubed. I, *Discrete Math.* **224** (2000), no. 1-3, 99–106.

[9] E. Dobson, On groups of odd prime-power degree that contain a full cycle, *Discrete Math.* **299** (2005), no. 1-3, 65–78.

[10] E. Dobson, Automorphism groups of metacirculant graphs of order a product of two distinct primes, *Combin. Probab. Comput.* **15** (2006), no. 1-2, 105–130.

[11] E. Dobson, On overgroups of regular abelian p-groups, *Ars Math. Contemp.* **2** (2009), no. 1, 59–76.

[12] E. Dobson and I. Kovács, Automorphism groups of Cayley digraphs of $\mathbb{Z}_p^3$, *Electron. J. Combin.* **16** (2009), no. 1, Research Paper 149, 20.

[13] E. Dobson and J. Morris, On automorphism groups of circulant digraphs of square-free order, *Discrete Math.* **299** (2005), no. 1-3, 79–98.

[14] E. Dobson and J. Morris, Automorphism groups of wreath product digraphs, *Electron. J. Combin.* **16** (2009), no. 1, Research Paper 17, 30.

[15] E. Dobson and D. Witte, Transitive permutation groups of prime-squared degree, *J. Algebraic Combin.* **16** (2002), no. 1, 43–69.

[16] G. A. Jones, Cyclic regular subgroups of primitive permutation groups, *J. Group Theory* **5** (2002), no. 4, 403–407.

[17] L. A. Kalužnin and M. H. Klin, Some numerical invariants of permutation groups, *Latviĭsk. Mat. Ežegodnik* (1976), no. Vyp. 18, 81–99, 222.

[18] M. H. Klin and R. Pöschel, The König problem, the isomorphism problem for cyclic graphs and the method of Schur rings, Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), *Colloq. Math. Soc. János Bolyai*, vol. 25, North-Holland, Amsterdam, 1981, pp. 405–434.

[19] C. H. Li, The finite primitive permutation groups containing an abelian regular subgroup, *Proc. London Math. Soc.* (3) **87** (2003), no. 3, 725–747.

[20] C. H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.* **21** (2005), no. 2, 131–136.

[21] J. D. P. Meldrum, *Wreath products of groups and semigroups*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 74, Longman, Harlow, 1995.

[22] I. N. Ponomarenko, Determination of the automorphism group of a circulant association scheme in polynomial time, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **321** (2005), Vopr. Teor. Predst. Algebr. i Grupp. 12, 251–267, 301.

[23] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438.

[24] H. Wielandt, *Finite permutation groups*, Translated from the German by R. Bercov, Academic Press, New York, 1964.