Avoiding (m, m, m)-arrays of order $n = 2^k$

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Abstract

An (m, m, m)-array of order n is an $n \times n$ array such that each cell is assigned a set of at most m symbols from $\{1, \ldots, n\}$ such that no symbol occurs more than m times in any row or column. An (m, m, m)-array is called *avoidable* if there exists a Latin square such that no cell in the Latin square contains a symbol that also belongs to the set assigned to the corresponding cell in the array. We show that there is a constant γ such that if $m \leq \gamma 2^k$ and $k \geq 14$, then any (m, m, m)-array of order $n = 2^k$ is avoidable. Such a constant γ has been conjectured to exist for all n by Häggkvist.

1 Introduction

Consider an $n \times n$ array A such that each cell contains a subset of the symbols in the set $\{1, \ldots, n\}$. We call such an array an *(m, m, m)-array of order n* if the following conditions are fulfilled:

(a) No cell contains a set with more than m symbols.

(b) Each symbol occurs at most m times in each column.

(c) Each symbol occurs at most m times in each row.

Note that this definition means that if $\ell < m$, then any (ℓ, ℓ, ℓ)-array is an (m, m, m)-array.

Let $A(i, j)$ denote the set of symbols in the cell (i, j) of A. To simplify the notation, if the set of symbols in cell (i, j) of A is $\{k\}$, we will write $A(i, j) = k$. A *Latin square* L is a $(1, 1, 1)$-array with no empty cell. The question addressed in this paper is whether an (m, m, m)-array is avoidable by a Latin square or not. To make this more precise, let A be an (m, m, m)-array of order n, and consider an $n \times n$ Latin square L with cells (i, j), $1 \leq i, j \leq n$. We say that L *avoids* A if there is no cell (i, j) of L such that $L(i, j) \in A(i, j)$. We call A *avoidable* if there exists a Latin square L that avoids A.

The question of avoidability for arrays was first raised by Häggkvist in [4]. That article contains the first positive result on the subject, namely that any $2^k \times 2^k$ array with empty last column such that each symbol occurs at most once in every column, and such that every cell contains at most one symbol, is avoidable. In addition, the following conjecture was stated:

Conjecture 1. (Häggkvist) Let A be an $n \times n$-array of m-sets from a set of symbols $\{1, 2, \ldots, n\}$ where every symbol is used at most $m \leq \gamma n$ times in each row and column.

(a) There is a constant γ such that for any such array A there exists an $n \times n$ Latin square L on $1, 2, \ldots, n$ that avoids A.

(b) The largest value of γ so that (a) holds is $\frac{1}{3}$.

If the conjecture (b) is true it would be sharp, since the largest examples of unavoidable arrays use m very close to $n/3$. In [1], Bryant *et al.* (see also [5]) studied whether, for a given set $\mathcal{L}_n$ of $n \times n$ Latin squares such that for any cell, no two of them have the same symbol in that cell, there exists another Latin square L_n^1 such that $\mathcal{L}_n \cup \{L_n^1\}$ has the same property. For all $n \equiv 3 \mod 4$ with $n > 3$, they give examples of sets $\mathcal{L}_n$ of size $(n-1)/2$ which do not have this property. Those examples can be seen as (m, m, m)-arrays that cannot be avoided, with $m = (n-1)/2$. A general example of an unavoidable $(\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 1)$-array attributed to Pebody can be found in [3]: Let $k = \lfloor \frac{n}{3} \rfloor + 1$ and consider the array A:

$$\left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right)$$

where the subarrays are constructed according to the following:

- A_{11} is a $k \times k$ array with each cell containing the symbols $1, \ldots, k$ in all cells.
- A_{12} and A_{21} are empty $k \times k$ arrays.
- A_{13} and A_{23} are empty $k \times (n-2k)$ arrays.
- A_{22} is a $k \times k$ array with each cell containing the symbols $k+1, \ldots, 2k$.
- A_{31} and A_{32} are empty $(n-2k) \times k$ arrays.
- A_{33} is a $(n-2k) \times (n-2k)$ array with each cell containing symbols $2k+1, \ldots, n$.

In addition to the original article by Häggkvist, the subject was studied further by Chetwynd and Rhodes who in [2] proved that any $(2, 2, 2)$-array of order $4k$ is avoidable if k is a natural number such that $k > 3240$. This result was generalized by Cutler and Öhman [3] who proved the following theorem:

Theorem 2. *(Cutler and Öhman) Let $m \in \mathbb{N}$. There exists a number $c = c(m)$ such that if $k > c$ and if F is a $2mk \times 2mk$ array on the symbols $[2mk]$ in which every cell contains at most m symbols and every symbol appears at most m times in every row and column, then F is avoidable.*

The proof of Theorem 2 gives a value for the constant c, namely

$$c = (2m+1)\left(m^2 + 4m^2\binom{m}{2} + 2m^5 + \binom{2m^3}{2} + 4m^6 + 4m^7\right).$$

Hence, the conclusion is that an (m, m, m)-array of order n is avoidable if m is $o(n^{\frac{1}{9}})$.

In this paper we prove that Conjecture 1 (a) holds in the special case when the order of the array is a power of 2 and $n \geq 2^{14}$.

2 Definitions and examples

In the next section we will work with a specific Latin square which has some useful properties, namely the *Boolean Latin square*. We start by providing a definition and a few useful properties of the Boolean Latin square.

Definition 3. The Boolean Latin square of order 2^k is the Latin square with entries as in the addition table of $\mathbb{Z}_2^k$ with the elements of $\mathbb{Z}_2^k$ mapped to the integers $1, \ldots, 2^k$. We will denote the Boolean Latin square of order n by B_n, where n will be omitted whenever this can be done without confusion.

Definition 4. A 4-cycle in a Latin square L is a set c of four cells $\{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$ such that $L(i_1, j_1) = L(i_2, j_2)$ and $L(i_1, j_2) = L(i_2, j_1)$. Other names for a 4-cycle used in the literature are *subsquare of order* 2 and *intercalate*.

Example 5. The Boolean Latin square of order 8.

1	2	3	**4**	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	**1**	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

A 4-cycle would, for example, be the set $\{(1, 1), (1, 4), (4, 1), (4, 4)\}$.

Property 1. *Each cell in B_n belongs to $n - 1$ 4-cycles.*

Property 2. *Permuting the rows or the columns of B_n does not affect the number of 4-cycles to which a cell belongs.*

Definition 6. Given a 4-cycle $\{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$ in a Latin square L, a *swap on it* (or simply a *swap*) denotes the transformation $L \mapsto L'$ which retains all other parts of the Latin square, but if $L(i_1, j_1) = L(i_2, j_2) = k_1$ and $L(i_1, j_2) = L(i_2, j_1) = k_2$ then $L'(i_1, j_1) = L'(i_2, j_2) = k_2$ and $L'(i_1, j_2) = L'(i_2, j_1) = k_1$.

Example 7. To perform a swap on the 4-cycle in Example 5 would result in the following Latin square B':

4	2	3	**1**	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
1	3	2	**4**	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

Note that performing a swap does not interfere with the property of being a Latin square, but the number of 4-cycles to which a given cell belongs will change for some cells.

Property 3. *An $n \times n$ array which is a Latin square will still be a Latin square if we permute the rows, the columns, or the symbols.*

Definition 8. Given an (m, m, m)-array A and a Latin square L that does not avoid A, we say that those cells (i, j) of L where $L(i, j) \in A(i, j)$ are *conflict cells* or simply *conflicts*.

Definition 9. Given a Latin square L and an (m, m, m)-array A an *allowed* 4*-cycle* is a 4-cycle $c = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$ in L such that swapping on c gives a Latin square L' where none of $(i_1, j_1), (i_1, j_2), (i_2, j_1)$ or (i_2, j_2) is a conflict.

Property 4. *The intersection of two* 4*-cycles is either empty, or it contains* 1 *or* 4 *cells.*

Proof. A cycle will be uniquely determined if two cells are known, since if we assume that two cells (a, b) and (a, c) of a Latin square L are in the same row and both belong to a cycle, then the other two cells are (d, b) and (d, c) such that d is the row where $L(a, b)$ is in column c and $L(a, c)$ is in column b. Because we have assumed that (a, b) and (a, c) belong to a cycle, there is such a row d. For two cells in the same column, or two cells containing the same symbol, the argument is similar. □

Finally, in all that follows, we will denote the basis of the natural logarithm by e.

3 Main theorem

The main object of this paper is to prove Conjecture 1 (a) for $n = 2^k$ if $k \geq 14$, that is, to prove that there is a constant γ such that for $n = 2^k$, if $m \leq \gamma n$ then any (m, m, m)-array of order n can be avoided. The idea used for the proof is to start with the Boolean Latin square, and to permute its rows and columns so that the resulting Latin square does not have too many conflicts with a given (m, m, m)-array A. We can then find a set of allowed 4-cycles such that each conflict belongs to one of them, with no two of the 4-cycles intersecting, and swap on those. We want to prove:

Theorem 10. *There is a positive constant γ such that if $t \geq 14$ and $m \leq \gamma 2^t$, then any (m, m, m)-array A of order 2^t is avoidable.*

From now on, let $n = 2^t$, for some $t \geq 14$ and assume that γ is such that $\gamma 2^t \in \mathbb{N}$. Let A be an $(\gamma n, \gamma n, \gamma n)$-array of order n, and let L be a Latin square that can be obtained from the Boolean Latin square of order n by permuting both the rows and the columns.

The value for γ used in the proof is $1/512$, but better approximations at some stages of the proof are likely to show that a larger value could be used. Furthermore, a different proof could still prove that Conjecture 1 (b) holds.

Lemma 11. *Let α and κ be constants, $0 < \kappa < 1$, and $6\kappa + 6\sqrt{2\kappa} < \alpha < 1$ such that αn and κn are integers, and*

$$n \geq \max\left\{\frac{1}{2\sqrt{\left(\frac{\alpha}{12}-\frac{\kappa}{2}\right)^2-\frac{\kappa}{2}}}, \frac{1}{\frac{\alpha}{12}-\frac{\kappa}{2}-\sqrt{\left(\frac{\alpha}{12}-\frac{\kappa}{2}\right)^2-\frac{\kappa}{2}}}\right\}.$$

If L has the following properties:

(a) no row in L contains more than κn conflicts;

(b) no column in L contains more than κn conflicts;

(c) no symbol in L appears in more than κn conflicts;

(d) each conflict belongs to at least αn allowed cycles;

there is a set of disjoint allowed cycles such that each conflict belongs to one of them. Thus, by performing a number of swaps on L, we can obtain a Latin square L' that avoids A.

Proof. To find L', start with L and iterate through the conflicts creating a set S of disjoint allowed cycles. For each conflict, which does not already belong to a cycle in S, choose an allowed cycle to which it belongs and add to S. Each chosen cycle will intersect with other cycles. Those cycles can no longer be added to S. This means that choosing cycles randomly could lead to a conflict with no allowed cycle which was disjoint from the previously chosen cycles. In order to avoid this, choices will be made according to the following Choosing Rule:

The Choosing Rule. Each choice of a 4-cycle $\{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$ for S chosen to resolve a conflict (i_1, j_1) must be done so that

- the number of cycles in S that contain a cell in row i_2, but do not contain a conflict in row i_2, is at most νn;
- the number of cycles in S that contain a cell in column j_2, but does not contain a conflict in column j_2, is at most νn;

- the number of cycles in S containing the symbol $L(i_1, j_2) = L(i_2, j_1)$, but does not contain a conflict in this symbol is at most νn;

for some number $\nu,\ 0 < \nu < 1$, such that νn is a positive integer.

If there is a value for ν such that using this value in the Choosing Rule would guarantee that we can construct a set S of 4-cycles such that every conflict belongs to a cycle in S, the Lemma is true. Those acceptable values for ν are determined by the values used for κ and α: Assume that choices of cycles are made consecutively, one conflict at a time. Suppose that there is a conflict in cell (i, j) of L. We need to find an allowed cycle that contains cell (i, j), but which is disjoint from all cycles already in S. When we started to add cycles to S there were αn allowed cycles for each conflict. Some of these may intersect with cycles already in S: There are four cells where a cycle in S could intersect one of the allowed cycles containing (i, j). If, by a choice made earlier for some other conflict, there is a cycle in S containing (i, j) no more choice needs to be done for (i, j). Assume that this is not the case. If S contains cycles that intersect any of the allowed cycles containing (i, j) in one of the other three cells this must be taken into account. We count the number of cycles not useful because they either intersect with a cycle already in S or because they break the Choosing Rule:

- *Cycles in S that intersect allowed cycles in $(i, j+k)$ for some k.* Those of the cycles already in S that contain a conflict in row i will remove at most 2 allowed cycles each, by Property 4. There are at most $\kappa n - 1$ other conflicts in this row, and by the Choosing Rule, this row can contain 2 cells from each of νn cycles in S resolving conflicts on other rows. This amounts to a total of $2\kappa n - 2 + 2\nu n$ forbidden cycles.

- *Cycles in S that intersect allowed cycles in $(i+k, j)$ for some k.* By similar reasoning, at most $2\kappa n - 2 + 2\nu n$ cycles may intersect with the cycles chosen for conflicts in the same column, and are we are thus forbidden to use these.

- *Cycles in S that intersect allowed cycles in $(i + k, j + \ell)$ for some k, ℓ.* We have that $L(i + k, j + \ell) = L(i, j)$. The number of other conflicts involving the symbol $L(i, j)$ can be at most $\kappa n - 1$, and each of these cycles can remove up to 2 of the allowed cycles for (i, j). Furthermore, the symbol $L(i, j)$ might have been used in νn cycles that were added to S to resolve conflicts in other symbols. Thus at most $2\kappa n - 2 + 2\nu n$ of the cycles allowed for (i, j) may intersect with the cycles that were put in S to resolve conflicts for symbol $L(i, j)$, and thus are not useful now.

Hence, at most $6\kappa n - 6 + 6\nu n$ of the possible cycles could be disallowed because of earlier choices for cycles in S. Consider how many possible cycles could be forbidden by the Choosing Rule. The maximum number of allowed cycles that are forbidden by the Choosing Rule can be found by calculating the number of rows, columns and symbols which could have been used more than νn times for conflicts in another row, column or symbol. Since there were at most $\kappa n \cdot n$ conflicts originally, we know that we have made at most $\kappa n^2 - 1$ choices already. Hence, the number of rows that are forbidden by the Choosing Rule is at most

$$\frac{\kappa n^2 - 1}{\nu n} = \frac{\kappa n}{\nu} - \frac{1}{\nu n}.$$

The same calculation works for columns and symbols, so up to $\frac{3\kappa n}{\nu} - \frac{3}{\nu n}$ allowed cycles may be unavailable because they contain a cell from a row, from a column, or containing a particular symbol that the Choosing Rule forbids us to use. This gives that at least

$$\alpha n - (6\kappa n - 6 + 6\nu n) - \left(\frac{3\kappa n}{\nu} - \frac{3}{\nu n}\right) = n\left(\alpha - 6\kappa - 6\nu - \frac{3\kappa}{\nu}\right) + 6 + \frac{3}{\nu n}$$

cycles that contain cell (i, j) are still allowed.

If we choose ν so that

$$\left(\alpha - 6\kappa - 6\nu - \frac{3\kappa}{\nu}\right) \geq 0, \tag{1}$$

there are at least 6 allowed cycles from which to choose. In order to be able to choose ν so that (1) holds we need $\frac{\alpha}{12} - \frac{\kappa}{2} - \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}} \leq \nu \leq \frac{\alpha}{12} - \frac{\kappa}{2} + \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}}$. Furthermore, we need ν so that $\nu n \geq 1$ and $0 < \nu < 1$. Hence, if we require that the lower limit $\frac{\alpha}{12} - \frac{\kappa}{2} - \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}} < 1$, and that the upper limit $\frac{\alpha}{12} - \frac{\kappa}{2} + \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}} > 0$ there will be a a value for ν between these bounds and between 0 and 1. Those conditions will both hold since we have assumed $\alpha > 6\kappa + 6\sqrt{2\kappa}$. Now, let ν be as large as possible with $\nu \leq \frac{\alpha}{12} - \frac{\kappa}{2} + \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}}$ and $\nu 2^t$ an integer. If $2\sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}} > \frac{1}{2^t}$, there will be an acceptable value for ν within the interval required and if $\frac{\alpha}{12} - \frac{\kappa}{2} - \sqrt{\left(\frac{\alpha}{12} - \frac{\kappa}{2}\right)^2 - \frac{\kappa}{2}} > \frac{1}{2^t}$ we will certainly have $\nu 2^t \geq 1$.

Because of the calculations above, we know that there is still a cycle that can be added to S (actually, at least 6 of them), and we can proceed adding cycles to S untill all conflicts are covered. Thus, there is a set S of disjoint allowed cycles such that each conflict cell belongs to one of them. Starting with L and swapping on all the cycles in S will yield a Latin square L' as in the statement of the lemma. □

In order to prove Theorem 10 we need to show that there is a γ such that if $m \leq \gamma n$ we can find a pair of permutations, one for permuting the rows of B and one for permuting the columns of B, which satisfies the conditions of Lemma 11. Given a pair of permutations (σ, τ) on $n = 2^t$ symbols each, let L be the Latin square obtained from B by permuting the rows according to σ and the columns according to τ.

Lemma 12. *Let $\alpha, \gamma, \kappa, t, m$ be constants such that $n = 2^t, t, m, \alpha n, \gamma n, \kappa n \in \mathbb{N}$ and*

$$\left(\frac{3n\,(\gamma n)^{\kappa n}}{(\kappa n)!} + \frac{n^2\,(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)\,n)!}\right) < 1 \tag{2}$$

Then for any $(\gamma n, \gamma n, \gamma n)$-array A of order n there is a pair of permutations (σ, τ) such that:

(a) No row in L contains more than κn conflicts with A.

(b) No symbol in L appears in more than κn conflicts with A.

(c) No column in L contains more than κn conflicts with A.

(d) Each cell of L belongs to at least αn allowed cycles.

Proof. To bound the number of pairs of permutations that do not fulfill these conditions, let C be an upper bound for the number of permutations τ of the columns breaking the conditions (a) or (b), given a fixed permutation of the rows. Let R be an upper bound for the number of permutations σ of the rows breaking conditions (c) or (d), given a fixed permutation of the columns. There are $n!$ ways to permute the rows, and similarly $n!$ ways to permute the columns. Let X_C be the event that either (a) or (b) do not hold, and X_R be the event that either (c) or (d) do not hold. We prove that the probability of a randomly chosen pair of permutations (σ, τ) to belong to neither X_C nor X_R is strictly greater than 0. But for $\mathbb{P}[\overline{X_C \vee X_R}] = \mathbb{P}[\overline{X_C} \wedge \overline{X_R}]$ we have

$$\mathbb{P}[\overline{X_C} \wedge \overline{X_R}] \geq 1 - \mathbb{P}[X_C] - \mathbb{P}[X_R].$$

To estimate $\mathbb{P}[X_C]$, we have assumed that for any fixed permutation σ of the rows at most C choices of a permutation τ of the columns give a pair (σ, τ) of permutations of the rows and columns that break at least one of the conditions (a) and (b). Hence, at most $n!C$ pairs (σ, τ) of permutations break at least one of these conditions. Similarly, for any fixed permutation τ of the columns, at most R choices of a permutation σ of the rows give a pair (σ, τ) of permutations that break at least one of the conditions (c) and (d), yielding at most $n!R$ pairs (σ, τ) for which at least one of these condition does not hold. Thus

$$\mathbb{P}[\overline{X_C} \wedge \overline{X_R}] \geq 1 - \mathbb{P}[X_C] - \mathbb{P}[X_R] \geq 1 - \frac{n!C}{n!n!} - \frac{n!R}{n!n!}$$

so if

$$1 - \frac{C}{n!} - \frac{R}{n!} > 0 \tag{3}$$

there is a pair of permutations (σ, τ) such that conditions (a)-(d) all hold.

To estimate C note that $C \leq C_a + C_b$, where C_a is the number of permutations of the columns that break condition (a), and C_b is the number of permutations of the columns that break condition (b). First, we estimate C_a by counting the maximum number of permutations τ such that condition (a) does not hold. Let r be a fixed row, chosen arbitrarily. For (a) not to hold on row r means that there are more than κn conflicts between L and A on that row. We count the number of ways a permutation τ of the columns can be constructed so that (a) does not hold on row r:

In order to have more than κn conflicts, choose a set S of size κn of columns of A where we want L to have conflicts with A on row r. There are $\binom{n}{\kappa n}$ ways to choose S. The columns of B should now be permuted so that in L, the columns in S all contain conflicts on row r. Each cell (r, s) of A belonging to a column $s \in S$ contains at most γn

symbols, hence there are at most γn columns b of B that would cause a conflict in cell (r, s) if we chose τ so that $\tau(s) = b$. Thus, there is at the very most $(\gamma n)^{\kappa n}$ ways to chose which columns of B are mapped by τ to columns in S so that all the cells on row r in S are conflicts. The rest of the columns can be arranged in any of the $(n - \kappa n)!$ possible ways. In total this gives at most

$$\binom{n}{\kappa n} (\gamma n)^{\kappa n} (n - \kappa n)! = \frac{n! \, (\gamma n)^{\kappa n}}{(\kappa n)!}$$

permutations τ that do not fulfill condition (a) on row r. The number of permutations τ that do not fulfill (a) in at least one row would thus be bounded by n times this number. An analogous argument gives the same bound for C_b, so in total, we have that

$$C \leq 2n \frac{n! \, (\gamma n)^{\kappa n}}{(\kappa n)!}. \tag{4}$$

Furthermore, if we let the number of permutations of the rows that, for a fixed permutation of the columns, do not fulfill condition (c) be denoted by R_c, and the number of permutations of the rows that do not fulfill condition (d) be denoted by R_d, we have that $R \leq R_c + R_d$. An upper bound on R can then be obtained by bounding R_c and R_d separately. But R_c can be bounded similarly to how we bound C_a; there are at most

$$n \frac{n! \, (\gamma n)^{\kappa n}}{(\kappa n)!}$$

permutations σ of the rows such that condition (c) does not hold.

Finally, we need a bound on R_d. Fix an arbitrary cell (r, c) of A. There are n ways to choose a row u in B_n so that $r = \sigma(u)$. Each cycle $\mathcal{C}$ containing (r, c) is uniquely defined by the value of s where $(r, s) \in \mathcal{C}$ is the other cell in $\mathcal{C}$ on row r. A permutation σ puts the pair (σ, τ) in R_d if and only if there are more than $(1 - \alpha) \, n$ choices for s so that the swap along $\mathcal{C}$ is not allowed. We count the number of ways σ could be constructed for this to happen. First, note that for each choice of what row of B_n σ maps to row r of L, there are up to $2\gamma n$ cycles that are not allowed because of this choice. The reason for this is that since cell (r, c) of A contains γn symbols, all cycles where cell (r, s) of L contains one of those are disallowed. Furthermore, there are γn values for s such that cell (r, s) of A contains $L \, (r, c)$. Thus there are up to $2\gamma n$ cycles that are disallowed because of what row of B_n σ chooses as row r of L, so for a pair (σ, τ) to belong to R_d, σ must be such that at least $(1 - \alpha - 2\gamma) \, n$ cycles containing cell (r, c) are forbidden because of the restrictions on other rows of A. To bound the number of ways σ could be chosen so that this happens, we start by choosing a set S of columns, $|S| = (1 - \alpha - 2\gamma) \, n$, such that for all columns $s \in S$ the cycle that contains (r, c) and (r, s) is not allowed because of what happens in other rows than row r. There are $\binom{n-1}{(1-\alpha-2\gamma)n}$ ways to choose S. Fix a column $s \in S$. There is a unique row t in B_n such that $\mathcal{C} = \{(u, c), (u, s), (t, c), (t, s)\}$ is a 4-cycle, and there are up to $2\gamma n$ choices for $\sigma(t)$ in L that would make $\mathcal{C}$ disallowed because of row $\sigma(t)$ in A. Note that S defines a set R of rows, with $|S| = |R|$ by

$R = \{t \in [n] \mid \{(r,c),(r,s),(t,c),(t,s)\}$ is a 4-cycle, $s \in S\}$. For all rows, this gives at most $(2\gamma n)^{(1-\alpha-2\gamma)n}$ options for choosing the values of σ on R. The rows that do not belong to R can be permuted in $(n-1-(1-\alpha-2\gamma)\,n)! = ((\alpha+2\gamma)\,n-1)!$ ways. Hence, for a fixed permutation of the columns, the number of permutations σ of the rows with not enough allowed cycles for a given cell is bounded from above by

$$n\binom{n-1}{(1-\alpha-2\gamma)\,n}(2\gamma n)^{(1-\alpha-2\gamma)n}((\alpha+2\gamma)\,n-1)! = \frac{n!\,(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)\,n)!}$$

and the total number of permutations σ that have too few allowed cycles for at least one cell is bounded from above by

$$n^2\frac{n!\,(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)\,n)!}.$$

The upper bound R of the number of permutations of the rows that, for a fixed permutation of the columns, break either (c) or (d) is thus

$$R \leq R_c + R_d \leq n\frac{n!\,(\gamma n)^{\kappa n}}{(\kappa n)!} + n^2\frac{n!\,(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)\,n)!}. \tag{5}$$

We can now proceed to see that inequality (3) does indeed hold by considering the expression

$$\begin{aligned}1-\frac{R}{n!}-\frac{C}{n!} &\geq 1-\frac{\left(n\frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}+n^2\frac{n!(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)n)!}\right)}{n!}-\frac{\left(2n\frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}\right)}{n!}\\ &= 1-\left(\frac{n\,(\gamma n)^{\kappa n}}{(\kappa n)!}+\frac{n^2\,(2\gamma n)^{(1-\alpha-2\gamma)n}}{((1-\alpha-2\gamma)\,n)!}+\frac{2n\,(\gamma n)^{\kappa n}}{(\kappa n)!}\right)\end{aligned}$$

which is, by assumption, strictly greater than 0. □

We now conclude with a proof of the original theorem:

Proof. (Proof of Theorem 10) Let B be the Boolean Latin square of appropriate order. Let L be the Latin square obtained from B by permuting the rows according to some permutation σ, and the columns according to some permutation τ.

Let $\gamma = \frac{1}{512}, \kappa = \frac{3}{512}, \alpha = \frac{502}{512}$ and $t \geq 14$. Note that $\gamma n \in \mathbb{N}, \kappa n \in \mathbb{N}$ and $\alpha n \in \mathbb{N}$ if $n = 2^t$. We need to prove that for these values of γ, κ and α and any $t \geq 14$, inequality (2) holds. But in this case, the left hand side of inequality (2) simplifies to

$$\frac{3n\left(\frac{n}{512}\right)^{\frac{3n}{512}}}{\frac{3n}{512}!}+\frac{n^2\left(\frac{2n}{512}\right)^{\frac{8n}{512}}}{\left(\frac{8n}{512}\right)!} \tag{6}$$

By using the version of Stirling's formula found in Robbins [6], we obtain that the asymptotic behaviour of these terms are determined by $\left(\frac{e}{3}\right)^{\frac{3n}{512}}$ and $\left(\frac{e}{4}\right)^{\frac{8n}{512}}$ respectively. Thus, the sum is decreasing for large values of n. For $n = 2^{14}$, the sum (6) evaluates to 0.1547, and 2^{14} is already large enough that the behaviour of $\left(\frac{e}{3}\right)^{\frac{3n}{512}}$ determines the behaviour of the whole sum. Hence, we can check the conditions for Lemma 11: By Lemma 12 there is a Latin square L satisfying the conditions (a)-(d). Furthermore, we have that $0 < \kappa < 1$, that $6\kappa + 6\sqrt{2\kappa} < \alpha < 1$ and that

$$\max\left\{\frac{1}{2\sqrt{\left(\frac{\alpha}{12}-\frac{\kappa}{2}\right)^2-\frac{\kappa}{2}}}, \frac{1}{\frac{\alpha}{12}-\frac{\kappa}{2}-\sqrt{\left(\frac{\alpha}{12}-\frac{\kappa}{2}\right)^2-\frac{\kappa}{2}}}\right\} = \frac{768}{\sqrt{7729}} < 9 < 2^{14},$$

meaning that Lemma 11 yields a Latin square L' that avoids A. $\square$

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