Descents of Permutations in a Ferrers Board

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Abstract

The classical Eulerian polynomials are defined by setting

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{1+\mathrm{des}(\sigma)} = \sum_{k=1}^{n} A_{n,k}t^k$$

where $A_{n,k}$ is the number of permutations of length n with $k-1$ descents. Let $A_n(t,q) = \sum_{\pi\in\mathfrak{S}_n} t^{1+\mathrm{des}(\pi)}q^{\mathrm{inv}(\pi)}$ be the inv q-analogue of the classical Eulerian polynomials whose generating function is well known:

$$\sum_{n\geq 0}\frac{u^n A_n(t,q)}{[n]_q!} = \frac{1}{1-t\sum_{k\geq 1}\frac{(1-t)^k u^k}{[k]_q!}}. \tag{0.1}$$

In this paper we consider permutations restricted in a Ferrers board and study their descent polynomials. For a general Ferrers board F, we derive a formula in the form of permanent for the restricted q-Eulerian polynomial

$$A_F(t,q) := \sum_{\sigma\in F} t^{1+\mathrm{des}(\sigma)}q^{\mathrm{inv}(\sigma)}.$$

If the Ferrers board has the special shape of an $n\times n$ square with a triangular board of size s removed, we prove that $A_F(t,q)$ is a sum of $s+1$ terms, each satisfying an equation that is similar to (0.1). Then we apply our results to permutations with bounded drop (or excedance) size, for which the descent polynomial was first studied by Chung et al. (*European J. Combin.*, 31(7) (2010): 1853-1867). Our method presents an alternative approach.

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1 Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of order n. Given a permutation $\sigma \in \mathfrak{S}_n$, let $\mathrm{Des}(\sigma)$ be the *descent set* of σ, i.e., $\mathrm{Des}(\sigma) = \{i | \sigma_i > \sigma_{i+1}, 1 \le i \le n-1\}$, and let $\mathrm{des}(\sigma) = |Des(\sigma)|$ denote the number of descents of σ. For $D \subseteq \{1, 2, \ldots, n-1\}$, we denote by $\alpha_n(D)$ the number of permutations $\pi \in \mathfrak{S}_n$ whose descent set is *contained* in D, and by $\beta_n(D)$ the number of permutations $\pi \in \mathfrak{S}_n$ whose descent set is *equal* to D. In symbols,

$$\alpha_n(D) := |\{\sigma \in \mathfrak{S}_n : Des(\sigma) \subseteq D\}|, \qquad \beta_n(D) := |\{\sigma \in \mathfrak{S}_n : Des(\sigma) = D\}|.$$

Let $D = \{d_1, d_2, \ldots, d_k\}$ where $1 \le d_1 < \cdots < d_k \le n-1$. For convenience, also let $d_0 = 0$ and $d_{k+1} = n$. Then the following formulas for $\alpha_n(D)$ and $\beta_n(D)$ are well-known (see, for example, [14, p.69]):

$$\alpha_n(D) = \binom{n}{d_1, d_2 - d_1, \ldots, n - d_k} \tag{1.1}$$

$$\beta_n(D) = n! \det\left[\frac{1}{(d_{j+1} - d_i)!}\right] = \det\left[\binom{n - d_i}{d_{j+1} - d_i}\right], \tag{1.2}$$

where $(i, j) \in [0, k] \times [0, k]$ in the matrix of equation (1.2).

A q-analogue of the above formulas is given by considering the permutation statistic $\mathrm{inv}(\sigma)$, where $\mathrm{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j)$. By convention, the symbol $\chi(P)$ is 1 if the statement P is true and 0 if not. See [14, Example 2.2.5]. Explicitly, let

$$\alpha_n(D, q) = \sum_{\pi \in \mathfrak{S}_n : \mathrm{Des}(\pi) \subseteq D} q^{\mathrm{inv}(\pi)}, \qquad \beta_n(D, q) = \sum_{\pi \in \mathfrak{S}_n : \mathrm{Des}(\pi) = D} q^{\mathrm{inv}(\pi)}.$$

Then

$$\alpha_n(D, q) = \begin{bmatrix} n \\ d_1, d_2 - d_1, \ldots, n - d_k \end{bmatrix} = \frac{[n]!}{[d_1]![d_2 - d_1]! \cdots [n - d_k]!} \tag{1.3}$$

$$\beta_n(D, q) = [n]! \det\left[\frac{1}{[d_{j+1} - d_i]!}\right] = \det\left[\begin{bmatrix} n - d_i \\ d_{j+1} - d_i \end{bmatrix}\right], \tag{1.4}$$

where $(i, j) \in [0, k] \times [0, k]$ as before. Here we use the standard notation

$$[n] := (1 - q^n)/(1 - q), \qquad [n]! := [1][2] \cdots [n], \qquad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]_q!}{[k]![n - k]!}$$

for the q-analogue of the integer n, the q-factorial, and the q-binomial coefficient, respectively. Sometimes it is necessary to write the base q explicitly as in $[n]_q, [n]_q!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$, etc., but we omit q in this paper as we do not use the analogues of any other variables.

The classical Eulerian polynomials are defined by setting

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{1 + \mathrm{des}(\sigma)} = \sum_{k=1}^{n} A_{n,k} t^k,$$

where $A_{n,k}$ is called the Eulerian number that denotes the number of permutations of length n with $k-1$ descents. Let $A_0(t)=1$. The polynomials $A_n(t)$ have the generating function (see e.g. Riordan [12])

$$\sum_{n\geq 0} A_n(t)\frac{u^n}{n!} = \frac{1}{1-t\sum_{k\geq 1}\frac{(1-t)^{k-1}u^k}{k!}} = \frac{1-t}{1-te^{u(1-t)}}. \tag{1.5}$$

Let $A_n(t,q)=\sum_{\pi\in\mathfrak{S}_n} t^{1+\mathrm{des}(\pi)}q^{\mathrm{inv}(\pi)}$ be the inv q-analogue of the Eulerian polynomials. Stanley [13] showed that

$$\sum_{n\geq 0}\frac{u^nA_n(t,q)}{[n]!} = \frac{1-t}{1-tE(u(1-t);q)}, \tag{1.6}$$

where

$$E(z;q)=\sum_{n\geq 0}\frac{z^n}{[n]!}.$$

By simple manipulations we can see that an equivalent form of (1.6) is

$$\sum_{n\geq 0}\frac{u^nA_n(t,q)}{[n]!} = \frac{1}{1-t\sum_{k\geq 1}\frac{(1-t)^{k-1}u^k}{[k]!}}. \tag{1.7}$$

Alternative proofs of (1.7) have been given by Gessel [9] and Garsia [8].

In this paper we consider permutations with restricted positions, and extend the above results to descent polynomials of permutations in a Ferrers board. Traditionally a permutation $\sigma\in\mathfrak{S}_n$ is also represented as a 01-filling of an n by n square board: Reading from left to right and bottom to top, we simply put a 1 in the ith row and the jth column whenever $\sigma_i=j$ for $i=1,\ldots,n$. Given integers $0<r_1\leq r_2\leq\cdots\leq r_n$, the Ferrers board of shape $(r_1,\ldots,r_n)$ is defined by

$$F=\{(i,j): 1\leq i\leq n, 1\leq j\leq r_i\}.$$

In the following we identify a permutation σ with its 01-filling representation, and say that σ is in a Ferrers board F if all the cells (i,σ_i) are in F.

In Section 2 we extend the formulas (1.3) and (1.4) to the set of permutations on a fixed Ferrers shape with n rows and n columns, and derive a permanent formula for the restricted q-Eulerian polynomial

$$A_F(t,q):=\sum_{\sigma\in F} t^{1+\mathrm{des}(\sigma)}q^{\mathrm{inv}(\sigma)}.$$

In Section 3 we focus on the Ferrers board that is obtained from the $n\times n$ square by removing a triangular board of size s, and prove that the restricted q-Eularian polynomial is a sum of $s+1$ terms, each determined by an equation that generalizes (1.7).

Finally in Section 4, we apply our results to permutations with bounded drop (or excedance) size, for which the descent polynomial was first studied by Chung, Claesson, Dukes and Graham [4]. Our method presents an alternative approach to the results in [4].

Notation on lattice path

Here we recall some notation and results about the counting of lattice paths with a general right boundary. These results offer the main tool to describe permutations restricted in a Ferrers board. For a reference on lattice path counting, see Mohanty [11].

A lattice path P is a path in the plane with two kinds of steps: a unit north step N or a unit east step E. If x is a positive integer, a lattice path from the origin $(0,0)$ to the point (x,n) can be coded by a length n non-decreasing sequence $(x_1, x_2, \ldots, x_n)$, where $0 \le x_i \le x$ and x_i is the x-coordinate of the ith north step. For example, let $x = 5$ and $n = 3$. Then the path $EENENNEE$ is coded by $(2,3,3)$.

In general, let $\mathbf{s}$ be a non-decreasing sequence with positive integer terms $s_1, s_2, \ldots, s_n$. A lattice path from $(0,0)$ to (x,n) is one with the right boundary $\mathbf{s}$ if $x_i < s_i$ for $1 \le i \le n$. If $x \ge s_n$, then the number of lattice paths from $(0,0)$ to (x,n) with the right boundary $\mathbf{s}$ does not depend on x. Let $Path_n(\mathbf{s})$ be the set of lattice paths from $(0,0)$ to (s_n,n) with the right boundary $\mathbf{s}$, and $LP_n(\mathbf{s})$ be the cardinality of $Path_n(\mathbf{s})$. For a given sequence $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, let

$$LP_n(\mathbf{s};q) = \sum_{P \in Path_n(\mathbf{s})} q^{area(P)},$$

where $area(P) = \sum_{i=1}^{n} x_i$ is the area enclosed by the path P, the y-axis, and the line $y = n$. Hence $LP_n(\mathbf{s}) = LP_n(\mathbf{s};1)$. In this paper we will also allow the entries s_i to satisfy $s_1 \ge s_2 \ge \cdots \ge s_n$, in which case

$$LP_n(\mathbf{s};q) = LP_n((s_n, s_n, \ldots, s_n);q) = \begin{bmatrix} s_n + n - 1 \\ n \end{bmatrix}.$$

In particular $LP_n((n+1, n+1, \ldots, n+1);q) = \begin{bmatrix} 2n \\ n \end{bmatrix}$. It is also easy to see that

$$LP_n((1,2,\ldots,n);q) = C_n(q),$$

where $C_n(q)$ is Carlitz-Riordan's q-Catalan number [2].

2 Descents of permutations in Ferrers boards

Let F be a Ferrers board with n rows and n columns, which is aligned on the top and left. Index the rows from bottom to top, and columns from left to right. Let r_i be the size of row i. Hence $1 \le r_1 \le r_2 \le \cdots \le r_n = n$.

For a set $D = \{d_1, d_2, \ldots, d_k\}$ with $1 \le d_1 < \cdots < d_k \le n-1$, let $\beta_F(D)$ be the number of permutations in F with the descent set D, and $\alpha_F(D)$ be the number of permutations

in F whose descent set is contained in D. The inv q-analogues of $\alpha_F(D)$ and $\beta_F(D)$ are defined by

$$\alpha_F(D,q) = \sum_{\sigma \in F : \mathrm{Des}(\sigma) \subseteq D} q^{\mathrm{inv}(\sigma)}, \qquad \beta_F(D,q) = \sum_{\sigma \in F : \mathrm{Des}(\sigma) = D} q^{\mathrm{inv}(\sigma)}.$$

Clearly $\alpha_F(D,1) = \alpha_F(D)$ and $\beta_F(D,1) = \beta_F(D)$. The Inclusion-Exclusion Principle implies that

$$\alpha_F(D,q) = \sum_{T \subseteq D} \beta_F(T,q), \qquad \beta_F(D,q) = \sum_{T \subseteq D} (-1)^{|D-T|} \alpha_F(T,q).$$

We shall show that $\alpha_F(D,q)$ and $\beta_F(D,q)$ can be expressed in terms of $LP_n(\mathbf{s},q)$, the area enumerator of lattice paths with proper right boundaries and lengths.

Let's first compute $\alpha_F(D,q)$. To get a permutation σ in F satisfying $\mathrm{Des}(\sigma) \subseteq D$, we first choose $x_1 < x_2 < \cdots < x_{d_1}$ such that $1 \le x_i \le r_i$, and put a 1 in the cell (x_i, i) for $1 \le i \le d_1$. Then choose $x_{d_1+1} < x_{d_1+2} < \cdots < x_{d_2}$ such that $1 \le x_i \le r_i$, and put a 1 in the cell (x_i, i) for $d_1 < i \le d_2$, and so on.

We say that the cell (i,j) is a 1-cell if it is filled with a 1. It is clear that an inversion of σ corresponds to a southeast chain of size 2 in the filling, i.e. a pair of 1-cells $\{ (x_{i_1}, i_1), (x_{i_2}, i_2) \}$ such that $i_1 < i_2$ while $x_{i_1} > x_{i_2}$.

For $1 \le i \le d_1$, the 1-cell in the ith row (i.e. $y = i$) has exactly $x_i - i$ many other 1-cells lying above it and to its left. Hence the 1-cell in the ith row contributes $x_i - i$ to the statistic $\mathrm{inv}(\sigma)$, and all the 1-cells in the first d_1 rows contributed

$$(x_1 - 1) + (x_2 - 2) + \cdots + (x_{d_1} - d_1)$$

to the statistic $\mathrm{inv}(\sigma)$.

Note that $0 \le x_1 - 1 \le x_2 - 2 \le \cdots \le x_{d_1} - d_1$, and $x_i - i < r_i - i + 1$. Hence the number of choices for the sequence $(x_1, \ldots, x_{d_1})$ is exactly the number of lattice paths from $(0,0)$ to $(r_{d_1} - d_1 + 1, d_1)$ with the right boundary $(r_1, r_2 - 1, \ldots, r_{d_1} - d_1 + 1)$, and $\sum_{i=1}^{d_1} (x_i - i)$ is the area of the corresponding lattice path. Therefore the first d_1 rows of F contribute a factor of $LP_{d_1}((h_1, \ldots, h_{d_1}); q)$ to $\alpha_F(D,q)$.

Let $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ where $h_i = r_i - i + 1$. Let the i-th block of F consist of rows $d_{i-1} + 1$ to d_i. Applying the above analysis to the i-th block of the Ferrers board F for $i = 2, \ldots, k+1$, we get that

Theorem 2.1

$$\alpha_F(D,q) = \sum_{\sigma \in F : \mathrm{Des}(\sigma) \subseteq D} q^{\mathrm{inv}(\sigma)} = \prod_{i=0}^{k} LP_{d_{i+1} - d_i}((h_{d_i+1}, \ldots, h_{d_{i+1}}); q) \tag{2.1}$$

where we use the convention that $d_0 = 0$ and $d_{k+1} = n$.

Accordingly,

$$\begin{aligned}\beta_F(D,q) &= \sum_{T\subseteq D}(-1)^{|D-T|}\alpha_F(T,q)\\ &= \sum_{1\le i_1<i_2<\cdots<i_j\le k}(-1)^{k-j}f(0,i_1)f(i_1,i_2)\dots f(i_j,k+1)\end{aligned}$$

where

$$f(i,j)=\begin{cases} LP_{d_j-d_i}(h_{d_i+1},\dots,h_{d_j});q) & \text{if } i<j\\ 1 & \text{if } i=j\\ 0 & \text{if } i>j.\end{cases} \tag{2.2}$$

Following the discussion of Stanley [14, p.69], we obtain that

Theorem 2.2 *$\beta_F(D,q)$ is the determinant of a $(k+1)\times(k+1)$ matrix with its (i,j) entry $f(i,j+1)$, $0\le i,j\le k$. That is,*

$$\beta_F(D,q)=\det[f_{i,j+1}]_0^k \tag{2.3}$$

where $f(i,j)$ is given by (2.2).

When the Ferrers board F is an $n\times n$ square, $h_i=n-i+1$, and

$$LP_{j-i}((h_{i+1},\dots,h_j);q)=\begin{bmatrix}n-i\\ j-i\end{bmatrix}.$$

Theorems 2.1 and 2.2 reduce to the classical results

$$\sum_{\substack{\pi\in\mathfrak{S}_n\\ Des(\pi)\subseteq D}} q^{\mathrm{inv}(\pi)}=\begin{bmatrix}n\\ d_1,d_2-d_1,\dots,n-d_k\end{bmatrix}$$

and

$$\sum_{\substack{\pi\in\mathfrak{S}_n\\ Des(\pi)=D}} q^{\mathrm{inv}(\pi)}=\det\left[\begin{bmatrix}n-d_i\\ d_{j+1}-d_i\end{bmatrix}\right]_0^k.$$

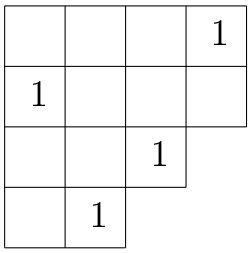
For a general Ferrers board with n rows and n columns, let's check two extreme cases: $D=\{1,2,\dots,n-1\}$ and $D=\emptyset$.

- Case 1. $D=\{1,2,\dots,n-1\}$: Theorem 2.1 yields the identity

$$\sum_{\sigma\in F}q^{\mathrm{inv}(\sigma)}=\alpha_F(=\{1,2,\dots,n-1\},q)=\prod_{i=1}^{n}LP_1((h_i);q)=\prod_{i=1}^{n}[h_i]. \tag{2.4}$$

  Note that permutation fillings of a Ferrers board with n rows and n columns correspond to complete matchings of $\{1,\dots,2n\}$ with fixed sets of left endpoints and right endpoints, and an inversion of the permutation is exactly a nesting of the

matching; See de Mier [6] and Kasraoui [10]. To see this, for a given Ferrers board F with n rows and n columns, one traverses the path from the lower-left corner to the top-right corner, and records the path by its steps $a_1, a_2, \ldots, a_{2n}$ where $a_i = E$ if the ith step is East, and $a_i = N$ if the ith step is North. Let $L = \{i : a_i = E\}$ and $R = \{i : a_i = N\}$. Then 01-fillings of F considered here are in one-to-one correspondence with the matchings of $\{1, \ldots, 2n\}$ for which the set of left endpoints is L and the set of right endpoints is R. For example, in the following Ferrers board F, traversing from the lower-left corner to the top-right corner, we get the sequence $EENENENN$. Thus $L = 1, 2, 4, 6$ and $R = 3, 5, 7, 8$. The filling given in the figure corresponds to the matching $\{(1, 7), (2, 3), (4, 5), (6, 8)\}$.



It follows that equation (2.4) is exactly the generating function of the statistic $ne_2(M)$, which is the number of nestings in a matching M, counted over all the matchings with given sets of left and right endpoints. That is,

$$\sum_M q^{ne_2(M)} = \prod_{i=1}^{n} [h_i],$$

which matches the known results in [7, 10].

- Case 2, $D = \emptyset$: We have

$$\alpha_F(\emptyset, q) = \beta_F(\emptyset, q) = LP_n((h_1, \ldots, h_n), q)$$

Note that $h_n = 1$, hence $LP_n((h_1, \ldots, h_n), q) = 1$ iff $r_i \geq i$ for all i, where the only permutation in the Ferrers board F with no descents is the identity permutation; otherwise $Path_n(h_1, \ldots, h_n) = \emptyset$ and $LP_n((h_1, \ldots, h_n), q) = 0$.

Theorems 2.1 and 2.2 can be used to get a formula for the joint distribution of $\mathrm{des}(\sigma)$ and $\mathrm{inv}(\sigma)$ over permutations in F. Let

$$A_F(t, q) = \sum_{\sigma \in F} t^{1+\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)}. \tag{2.5}$$

Theorem 2.3

$$A_F(t, q) = (1 - t)^n per(M), \tag{2.6}$$

where M is an $n \times n$ matrix whose (i,j)-entry is given by

$$M_{ij} = \begin{cases} \frac{t}{1-t} LP_{j-i+1}((h_i, \ldots, h_j); q) & \text{if } i \leq j \\ 1 & \text{if } i = j+1 \\ 0 & \text{if } i > j+1, \end{cases}$$

and $per(M)$ is the permanent of the matrix M.

Proof. We have

$$\begin{aligned} \sum_{\sigma \in F} t^{1+\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)} &= \sum_{D \subseteq \{1,2,\ldots,n-1\}} t^{1+|D|} \beta_F(D,q) \\ &= \sum_{D \subseteq \{1,2,\ldots,n-1\}} t^{1+|D|} \sum_{T:T \subseteq D} (-1)^{|D-T|} \alpha_F(T,q) \\ &= \sum_{T \subseteq \{1,2,\ldots,n-1\}} \alpha_F(T,q) \sum_{D:T \subseteq D} (-1)^{|D-T|} t^{1+|T|+|D-T|} \\ &= (1-t)^n \sum_{T=\{t_1,\ldots,t_k\}_<} \left(\frac{t}{1-t}\right)^{1+k} \Delta_T(LP_n(\mathbf{h})) \\ &= (1-t)^n per(M), \end{aligned}$$

where M is an $n \times n$ matrix as described in Theorem , and $\Delta_D(LP_n(\mathbf{h}))$ denotes the right-hand side of (2.1). □

Remark 2.1 We remark that descents of permutations in a Ferrers board provide an example of one-dependent determinantal point processes, as studied by Borodin, Diaconis and Fulman [1]. Let U be a finite set. A point process on U is a probability measure P on the $2^{|U|}$ subsets of U. One simple way to specify P is via its correlation functions $\rho(A)$, where for $A \subseteq U$,

$$\rho(A) = P\{S : S \supseteq A\}.$$

A point process is *determinantal* with kernel $K(x,y)$ if

$$\rho(A) = \det[K(x,y)]_{x,y \in A}.$$

It is *one-dependent* if $\rho(X \cup Y) = \rho(X)\rho(Y)$ whenever $\mathrm{dist}(X,Y) \geq 2$.

Borodin et al. showed that many examples from combinatorics, algebra and group theory are determinantal one-dependent point processes, for example, the carries process, the descent set of uniformly random permutations, and the descent set in Mallows model [1]. For these three cases, the point processes are stationary, while the descent set of permutations in a Ferrers board corresponds to a determinantal one-dependent point process that is not stationary. Explicitly, for any set $D = \{d_1, \ldots, d_k\}$ with $1 \leq d_1 < \cdots < d_k \leq n-1$, let $P_F(D) = \beta_F(D)/(\prod_{i=1}^n h_i)$. Using [1, Theorem 7.5] we obtain that P_F is a determinantal, one-dependent process with correlation functions

$$\rho(D) = \alpha_F(D) = \det[K(d_i, d_j)]_{i,j=1}^k$$

and with correlation kernel

$$K(x,y) = \delta_{x,y} + (E^{-1})_{x,y+1},$$

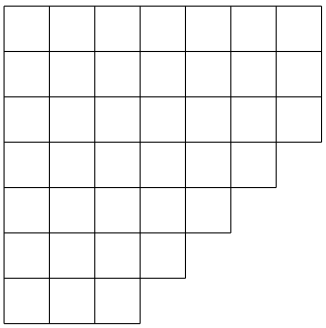
where E is the upper triangular matrix $E = [e(i-1,j)]_{i,j=1}^{n}$ whose entries are given by

$$e(i,j) = \begin{cases} LP_{j-i}(h_{i+1},\ldots,h_j) & \text{if } i<j \\ 1 & \text{if } i=j \\ 0 & \text{if } i>j. \end{cases}$$

3 Permutations in the truncated board $n \times n - \Delta_s$

For a general non-decreasing sequence of positive integers $\mathbf{s}$, $LP_n(\mathbf{s},q)$ can be computed by a determinant formula (see, for example, [11]). But there is no simple closed formula. In the special cases that the Ferrers board F is obtained from truncating the $n \times n$ square board by a triangular board in the corner, we can describe the joint distribution of the statistics $\mathrm{des}(\sigma)$ and $\mathrm{inv}(\sigma)$ by identities of their bi-variate generating functions.

Let Δ_s be the triangular board with row size $(s, s-1, \ldots, 1)$. For $n \geq s$, let $\Lambda_{n,s}$ be the truncated board $n \times n - \Delta_s$ consisting of cells that are lying in $0 \leq x, y \leq n$ and above the line $y = x - (n-s)$. In other words, $\Lambda_{n,s}$ is the Ferrers board whose row lengths are $(n-s, n-s+1, \ldots, n, \ldots, n)$. See the following figure for $\Lambda_{n,s}$ with with $n=7$ and $s=4$.



Now let $D = \{d_1, \ldots, d_k\}$ with $1 \leq d_1 < \cdots < d_k \leq n-1$. We shall compute the joint distribution of $1 + \mathrm{des}$ and inv over all permutations in $\Lambda_{n,s}$ using the formulas obtained in Section 2. Again let $d_0 = 0$ and $d_{k+1} = n$. Let $\delta_i = d_i - d_{i-1}$ for $i = 1, \ldots, k+1$, and assume that j is the particular index to make $d_j \leq s < d_{j+1}$ occur.

First we compute $\alpha_{\Lambda_{n,s}}(D,q)$. Let r_i be the size of row i in $\Lambda_{n,s}$. Then

$$r_i = \begin{cases} n-s-1+i & \text{if } i \leq s \\ n & \text{if } s < i \leq n. \end{cases}$$

Let $h_i = r_i - i + 1$. Then

1. For $0 \leq i < j$,

$$LP_{d_{i+1}-d_i}((h_{d_i+1},\ldots,h_{d_{i+1}}),q) = LP_{\delta_{i+1}}((n-s,\ldots,n-s),q) = \begin{bmatrix} n-s-1+\delta_{i+1} \\ \delta_{i+1} \end{bmatrix}.$$

2. For $i = j$,

$$\begin{aligned} LP_{d_{j+1}-d_j}((h_{d_j+1}, \ldots, h_{d_{i+1}}), q) &= LP_{\delta_{j+1}}(((n-s)^{s-d_j}, n-s-1, \ldots, n-d_{j+1}+1), q) \\ &= \begin{bmatrix} n-d_{j+1}+\delta_{j+1} \\ \delta_{j+1} \end{bmatrix} \\ &= \begin{bmatrix} n-d_j \\ \delta_{j+1} \end{bmatrix}. \end{aligned}$$

3. For $i > j$,

$$\begin{aligned} LP_{d_{i+1}-d_i}((h_{d_i+1}, \ldots, h_{d_{i+1}}), q) &= LP_{\delta_{i+1}}(((n-d_i, n-d_i-1, \ldots, n-d_{i+1}+1), q) \\ &= \begin{bmatrix} n-d_{i+1}+\delta_{i+1} \\ \delta_{i+1} \end{bmatrix} \\ &= \begin{bmatrix} n-d_i \\ \delta_{i+1} \end{bmatrix}. \end{aligned}$$

Summing over all permutations σ with $\mathrm{Des}(\sigma) \subseteq D$ in the Ferrers board $\Lambda_{n,s}$, we obtain

$$\begin{aligned} \alpha_{\Lambda_{n,s}}(D, q) = \sum_{\substack{\sigma \in \Lambda_{n,s} \\ Des(\sigma) \subseteq D}} q^{\mathrm{inv}(\sigma)} &= \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} \cdot \prod_{i=j}^{k} \begin{bmatrix} n-d_i \\ \delta_{i+1} \end{bmatrix} \\ &= \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} \cdot \begin{bmatrix} n-d_j \\ \Delta(D_j) \end{bmatrix}, \end{aligned}$$

where $\Delta(D_j)$ represents the sequence $\delta_{j+1}, \ldots, \delta_{k+1}$.

Hence the Principle of Inclusion-Exclusion leads to

$$\beta_{\Lambda_{n,s}}(I, q) = \sum_{\substack{\sigma \in \Lambda_{n,s} \\ Des(\sigma) = I}} q^{\mathrm{inv}(\sigma)} = \sum_{D \subseteq I} (-1)^{|I|-|D|} \begin{bmatrix} n-d_j \\ \Delta(D_j) \end{bmatrix} \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix}. \tag{3.1}$$

Let $F_{n,s}(q,t)$ be the bi-variate generating function of the statistics inv and des over all permutations in the board $\Lambda_{n,s}$. That is,

$$\begin{aligned} F_{n,s}(q,t) &= \sum_{\sigma \in \Lambda_s} t^{1+des(\sigma)} q^{\mathrm{inv}(\sigma)} \\ &= \sum_{I \subseteq \{1,2,\ldots,n-1\}} t^{1+|I|} \sum_{\substack{\sigma \in \Lambda_{n,s} \\ Des(\sigma) = I}} q^{\mathrm{inv}(\sigma)}. \end{aligned}$$

Let $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a,q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. Our main result here is an analog of the formula (1.7). Explicitly, we show that $F_{n,s}(q,t)$ can be expressed as a linear combination of $s+1$ terms, each of which satisfies a q-identity similar to (1.7).

Theorem 3.1 *For $n \le s$, $F_{n,s}(q,t) = 0$. For $n > s$, we have*

$$F_{n,s}(q,t) = \theta_0 F_{n,s}^{(0)}(q,t) + \theta_1 F_{n,s}^{(1)}(q,t) + \cdots + \theta_s F_{n,s}^{(s)}(q,t), \tag{3.2}$$

where the θ_k's are defined by the formal power series

$$\sum_{k=0}^{\infty} \theta_k z^k = \left(1 - \frac{t}{1-t}\left(\frac{1}{(z;q)_{n-s}} - 1\right)\right)^{-1}, \tag{3.3}$$

and for each $i = 0, 1, \ldots, s$, the term $F_{n,s}^{(i)}(q,t)$ is given by the identity

$$\sum_{n \ge s+1} \frac{z^n}{[n-i]!} \frac{F_{n,s}^{(i)}(q,t)}{(1-t)^n} = \frac{\dfrac{t}{1-t} \displaystyle\sum_{k \ge s+1-i} \frac{z^k}{[k]!}}{1 - \dfrac{t}{1-t} \displaystyle\sum_{k \ge 1} \frac{z^k}{[k]!}}. \tag{3.4}$$

Proof. As before assume $D = \{d_1, d_2, \ldots, d_k\}$ with $d_0 = 0$ and $d_{k+1} = n$. Let j be the index uniquely decided by $d_j \le s < d_{j+1}$. For $n \ge s+1$, by the equation (3.1) we have

$$\begin{aligned}
F_{n,s}(q,t) &= \sum_{I \subseteq \{1,2,\ldots,n-1\}} t^{1+|I|} \sum_{D \subseteq I} (-1)^{|I|-|D|} \begin{bmatrix} n-d_j \\ \Delta(D_j) \end{bmatrix} \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} \\
&= \sum_{D \subseteq \{1,2,\ldots,n-1\}} \begin{bmatrix} n-d_j \\ \Delta(D_j) \end{bmatrix} \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} \sum_{I : D \subseteq I} (-1)^{|I|-|D|} t^{1+|I|} \\
&= \sum_{D \subseteq \{1,2,\ldots,n-1\}} \begin{bmatrix} n-d_j \\ \Delta(D_j) \end{bmatrix} \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} (1-t)^{n-1-|D|} t^{1+|D|} \\
&= \sum_{k=0}^{n-1} t^{1+k} (1-t)^{n-1-k} \sum_{\substack{\delta_1+\delta_2+\ldots+\delta_{k+1}=n \\ \delta_1+\ldots+\delta_j \le s < \delta_1+\ldots+\delta_{j+1}}} \frac{[n-d_j]! \prod_{i=1}^{j} [n-s-1+\delta_i]!}{[\delta_1]! \cdots [\delta_{k+1}]! ([n-s-1]!)^j}.
\end{aligned}$$

Let $d_i = l$ and $\gamma = \frac{t}{1-t}$. Then,

$$\begin{aligned}
&\frac{F_{n,s}(q,t)}{(1-t)^n} \\
&= \sum_{l=0}^{s} \sum_{k \ge 0} \gamma^{1+k} \sum_{\substack{\delta_1+\delta_2+\ldots+\delta_{k+1}=n \\ l=\delta_1+\ldots+\delta_j \le s < \delta_1+\ldots+\delta_{j+1}}} \frac{[n-l]! \prod_{i=1}^{j} [n-s-1+\delta_i]!}{[\delta_1]! \cdots [\delta_{k+1}]! ([n-s-1]!)^j}. \\
&= \sum_{l=0}^{s} \left(\sum_{\substack{j \\ \delta_1+\ldots+\delta_j=l}} \gamma^j \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix} \right) \left(\sum_{\substack{k ; \delta_{j+1} \ge s+1-l \\ \delta_{j+1}+\delta_{j+2}+\ldots+\delta_{k+1}=n-l}} \frac{\gamma^{k+1-j} [n-l]!}{[\delta_{j+1}]! \cdots [\delta_{k+1}]!} \right)
\end{aligned} \tag{3.5}$$

Let $\theta_0 = 1$ and for $l = 1, \ldots, s$,

$$\theta_l := \sum_{\substack{j \\ \delta_1+\ldots+\delta_j=l}} \gamma^j \prod_{i=1}^{j} \begin{bmatrix} n-s-1+\delta_i \\ \delta_i \end{bmatrix},$$

and

$$F_{n,s}^{(l)} := (1-t)^n \sum_{\substack{k;\tau_0 \geq s+1-l \\ \tau_0+\tau_1+\ldots+\tau_k=n-l}} \frac{\gamma^{k+1}[n-l]!}{[\tau_0]!\cdots[\tau_k]!}. \tag{3.6}$$

Then

$$F_{n,s}(q,t) = \theta_0 F_{n,s}^{(0)}(q,t) + \theta_1 F_{n,s}^{(1)}(q,t) + \cdots + \theta_s F_{n,s}^{(s)}(q,t).$$

We show that θ_l and $F_{n,s}^{(l)}(q,t)$ satisfy (3.3) and (3.4).

First, observe that for $l > 0$, θ_l is the coefficient of z^l in the formal power series

$$\sum_{j=0}^{\infty} \left(\gamma \sum_{k=1}^{\infty} \begin{bmatrix} n-s-1+k \\ k \end{bmatrix} z^k \right)^j. \tag{3.7}$$

Using the q-analog of the binomial theorem

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty},$$

we have

$$\begin{aligned}
\sum_{l=0}^{\infty} \theta_l z^l &= \sum_{j=0}^{\infty} \left(\gamma \sum_{k=1}^{\infty} \frac{[n-s+k-1][n-s+k-2]\cdots[n-s]}{[k]!} z^k \right)^j \\
&= \sum_{j=0}^{\infty} \left(\gamma \cdot \left(\frac{(q^{n-s}z;q)_\infty}{(z;q)_\infty} - 1 \right) \right)^j \\
&= \left(1 - \gamma \left(\frac{1}{(z;q)_{n-s}} - 1 \right) \right)^{-1}.
\end{aligned}$$

This proves the formula (3.3).

To get the formula (3.4), observe that (3.6) can be written as

$$\frac{1}{[n-l]!} \frac{F_{n,s}^{(l)}}{(1-t)^n} = [z^{n-l}] \left(\gamma \sum_{\tau_0 \geq s+1-l} \frac{z^{\tau_0}}{[\tau_0]!} \cdot \sum_{k=0}^{\infty} \gamma^k \left(\sum_{\tau=1}^{\infty} \frac{z^\tau}{[\tau]!} \right)^k \right).$$

This leads to the identity

$$\sum_{n\geq s+1} \frac{z^{n-l}}{[n-l]!} \frac{F_{n,s}^{(l)}(q,t)}{(1-t)^n} = \frac{\dfrac{t}{1-t} \displaystyle\sum_{k\geq s+1-l} \frac{z^k}{[k]!}}{1 - \dfrac{t}{1-t} \displaystyle\sum_{k\geq 1} \frac{z^k}{[k]!}}.$$

In the case that $s = 0$, $F_{n,s}(q,t) = F_{n,s}^{(0)}(q,t) = A_n(t,q)$, and equation (3.4) reduces to the well-known identity (1.7) by letting $u = \frac{z}{1-t}$.

□

4 Permutations with bounded drop or excedance size

Permutations with bounded drop size is related to the bubble sort and sequences that can be translated into juggling patterns [5], whose enumeration was first studied by Chung, Claesson, Dukes, and Graham [4]. For a permutation σ, we say that i is a *drop* of σ if $\sigma_i < i$ and the *drop size* is $i - \sigma_i$. Similarly, we say that i is an *excedance* of σ if $\sigma_i > i$, and the *excedance size* is $\sigma_i - i$. It is well-known that the number of excedances is an Eulerian statistic, i.e., has the same distribution as des over the set of permutations.

Following [4], we use $\mathrm{maxdrop}(\sigma)$ to denote the maximum drop of σ,

$$\mathrm{maxdrop}(\sigma) := \max\{i - \sigma_i | 1 \leq i \leq n\},$$

and similarly, $\mathrm{maxexc}(\sigma)$ to denote the maximum excedance size of σ,

$$\mathrm{maxexc}(\sigma) := \max\{\sigma_i - i | 1 \leq i \leq n\}.$$

Let $\mathcal{B}_{n,k} = \{\sigma \in \mathfrak{S}_n | \mathrm{maxdrop}(\sigma) \leq k\}$. It is easy to see that $|\mathcal{B}_{n,k}| = k!(k+1)^{n-k}$: Just note that there are $(k+1)^{n-k}$ ways to determine $\sigma_n, \cdots, \sigma_{k+1}$ in the correct order, one after another, and the remaining is clear (e.g., see [5, Thm.1]). In [4], Chung et al. defined the k-maxdrop descent polynomials

$$B_{n,k}(t) := \sum_{\sigma\in\mathcal{B}_{n,k}} t^{des(\sigma)}$$

and obtained recurrences as well as a formula for the generating function $B_k(t,z) := \sum_{n\geq 0} B_{n,k}(t) z^n$.

In this section, we will use the analysis in the previous section to derive a variant generating function for $B_k(t,z)$. Explicitly, we get an exact formula for

$$E_k(t,z) := \sum_{n\geq k} B_{n,k}(t) z^n = \sum_{n\geq k} \left(\sum_{\sigma\in\mathcal{B}_{n,k}} t^{des(\sigma)} \right) z^n. \tag{4.1}$$

First, let $\mathcal{B}'_{n,k} = \{\sigma \in \mathfrak{S}_n | \text{maxexc}(\sigma) \leq k\}$. It is clear that the map $a_1 a_2 \dots a_n \mapsto (n+1-a_n)(n+1-a_{n-1}) \dots (n+1-a_1)$ is a bijection from $\mathcal{B}_{n,k}$ to $\mathcal{B}'_{n,k}$ that preserves the statistic $\text{des}(\sigma)$ and $\text{inv}(\sigma)$. It follows that

$$B_{n,k}(t) = \sum_{\sigma \in \mathcal{B}'_{n,s}} t^{des(\sigma)} \qquad \text{and hence} \qquad E_k(t,z) = \sum_{n \geq k} \left(\sum_{\sigma \in \mathcal{B}'_{n,k}} t^{des(\sigma)} \right) z^n.$$

Note that $\mathcal{B}'_{n,k}$ is the set of permutations $\sigma \in \mathfrak{S}_n$ satisfying $\sigma_i \leq i+k$. It is easy to check that it is exactly the set of permutations on the truncated board $\Lambda_{n,n-k-1}$. Hence for $n \geq k+1$, we have

$$\sum_{\sigma \in \mathcal{B}_{n,k}} t^{1+des(\sigma)} q^{\text{inv}(\sigma)} = F_{n,n-k-1}(q,t)$$

and Theorem 3.1 with $s = n-k-1$ gives a description of $F_{n,n-k-1}(q,t)$.

To obtain the ordinary generating function for $B_{n,k}(t)$, set $q = 1$. As before, let $\gamma = t/(1-t)$. Then formula (3.5) becomes the following equation for $n \geq k+1$:

$$\frac{tB_{n,k}(t)}{(1-t)^n} = \sum_{l=0}^{n-k} \left(\sum_{\substack{j \\ \delta_1 + \dots + \delta_j = l}} \gamma^j \prod_{i=1}^{j} \binom{k+\delta_i}{\delta_i} \right) \left(\sum_{\substack{p; \delta_{j+1} > n-k-l \\ \delta_{j+1} + \delta_{j+2} + \dots + \delta_{p+1} = n-l}} \frac{\gamma^{p+1-j}(n-l)!}{\delta_{j+1}! \cdots \delta_{k+1}!} \right).$$

(Note that from the analysis, the upper limit of l can include $n-k$.)

Let $\theta_0^{(k)} = 1$ and for $l \geq 1$,

$$\theta_l^{(k)} := \sum_{\substack{j; \delta_1 + \dots + \delta_j = l \\ \delta_i > 0}} \gamma^j \prod_{i=1}^{j} \binom{k+\delta_i}{\delta_i},$$

and

$$c_{n-l}^{(k)} := \sum_{\tau_0 > n-k-l} \gamma \binom{n-l}{\tau_0} \cdot \sum_{\substack{\tau_1 + \dots + \tau_p = n-l-\tau_0 \\ \tau_i \geq 1}} \gamma^p \binom{n-l-\tau_0}{\tau_1, \dots, \tau_p}, \qquad \text{for } k > 0.$$

For $k = 0$, let $c_n^{(0)} = \delta_{n,0}$. Then for any fixed $k \geq 0$ and $n \geq k+1$, we have

$$\frac{tB_{n,k}(t)}{(1-t)^n} = \sum_{l=0}^{n-k} \theta_l^{(k)} c_{n-l}^{(k)}, \tag{4.2}$$

Letting $q = 1$ and $k = n-s-1$ in equation (3.3), we obtain

$$\Theta_k(z) = \sum_{n \geq 0} \theta_n^{(k)} z^n = \left(1 - \frac{t}{1-t} \left(\frac{1}{(1-z)^{k+1}} - 1 \right) \right)^{-1}. \tag{4.3}$$

For the coefficient $c_i^{(k)}$, using formula (3.5) with $s=0$, we get that

$$\sum_{\substack{\tau_1+\cdots+\tau_p=n\\ \tau_i\geq 1}} \gamma^p \binom{n}{\tau_1,\ldots,\tau_p} = \frac{A_n(t)}{(1-t)^n},$$

where $A_n(t)$ is the classical Eulerian polynomial defined by

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{1+des(\sigma)}, \qquad n>0.$$

By convention, we set $A_0(t)=1$. It follows that for $k>0$,

$$c_n^{(k)} = \gamma \sum_{p>n-k} \binom{n}{p} \frac{A_{n-p}(t)}{(1-t)^{n-p}} = \gamma \sum_{p=0}^{k-1} \binom{n}{p} \frac{A_p(t)}{(1-t)^p}. \tag{4.4}$$

Writing as a generating function, we obtain that for $k>0$,

$$C_k(z) = \sum_{n\geq k} c_n^{(k)} z^n = \gamma \sum_{p=0}^{k-1} \frac{A_p(t)}{(1-t)^p} \sum_{n\geq k} \binom{n}{p} z^n$$

which leads to

$$C_k(z) = \frac{t}{1-t} \sum_{p=0}^{k-1} \frac{A_p(t)}{(1-t)^p} \left(\frac{z^p}{(1-z)^{p+1}} - \sum_{n=p}^{k-1} \binom{n}{p} z^n \right). \tag{4.5}$$

For $k=0$, $C_0(z)=1$.

Observe that equation (4.2) is true for $n=k$ as well. In fact it is equivalent to the identity

$$A_k(t) = t \sum_{p=0}^{k-1} \binom{k}{p} (1-t)^{k-p-1} A_p(t),$$

which can be readily checked by using the following expression of the Eulerian polynomial, see, for example [3, Lemma 14.1, p.517],

$$A_n(t) = (1-t)^{n+1} \sum_{j=0}^{\infty} j^n t^j, \qquad n\geq 0.$$

Therefore for all $n\geq k$,

$$B_{n,k}(t) = \frac{(1-t)^n}{t} \sum_{l=0}^{n-k} \theta_l^{(k)} c_{n-l}^{(k)},$$

Multiplying both sides by z^n and summing over $n\geq k$, we have obtained the generating function $E_k(t,z)$.

Theorem 4.1 *Let*

$$E_k(t,z) = \sum_{n\geq k} B_{n,k}(t) z^n = \sum_{n\geq k} \left(\sum_{\sigma \in \mathcal{B}_{n,k}} t^{des(\sigma)} \right) z^n.$$

Then $E_0(t,z) = 1/(1-z)$ *and for* $k \geq 1$,

$$E_k(t,z) = \frac{1}{t} \Theta_k((1-t)z) C_k((1-t)z), \tag{4.6}$$

where $\Theta_k(z)$ *and* $C_k(z)$ *are given in formulas* (4.3) *and* (4.5). *Explicitly,*

$$E_k(t,z) = \frac{\displaystyle\sum_{p=0}^{k-1} A_p(t) \left(\frac{z^p}{(1-(1-t)z)^{p+1}} - \sum_{n=p}^{k-1} \binom{n}{p} (1-t)^{n-p} z^n \right)}{1 - \frac{t}{(1-(1-t)z)^{k+1}}}. \tag{4.7}$$

Example 4.1 *For the case* $k = 1$, *formula* (4.7) *gives*

$$E_1(t,z) = \frac{\frac{1}{1-(1-t)z} - 1}{1 - \frac{t}{(1-(1-t)z)^2}} = \frac{z(1-(1-t)z)}{1 - z(2-(1-t)z)}.$$

Comparing with equation (5) in [4], and noting that the summation of $B_k(z,y)$ *in [4] starts from* $n = 0$, *one checks easily that the two formulas agree with each other.*

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