The $1/k$-Eulerian Polynomials

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Abstract

We use the theory of lecture hall partitions to define a generalization of the Eulerian polynomials, for each positive integer k. We show that these $1/k$-*Eulerian polynomials* have a simple combinatorial interpretation in terms of a single statistic on *generalized inversion sequences*. The theory provides a geometric realization of the polynomials as the h^*-polynomials of *k-lecture hall polytopes*. Many of the defining relations of the Eulerian polynomials have natural $1/k$-generalizations. In fact, these properties extend to a bivariate generalization obtained by replacing $1/k$ by a continuous variable. The bivariate polynomials have appeared in the work of Carlitz, Dillon, and Roselle on Eulerian numbers of higher order and, more recently, in the theory of rook polynomials.

1 Overview

The Eulerian polynomials, $A_n(x)$, can be defined, for $n \geq 0$, by any of the following relations:

$$A_n(x) = \sum_{\pi \in S_n} x^{\mathrm{des}(\pi)}; \tag{1}$$

$$\sum_{t \geq 0} (t+1)^n x^t = \frac{A_n(x)}{(1-x)^{n+1}}; \tag{2}$$

$$\sum_{n \geq 0} A_n(x) \frac{z^n}{n!} = \frac{(1-x)}{e^{z(x-1)} - x}. \tag{3}$$

In (1), S_n is the set of permutations $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and $\mathrm{des}(\pi)$ is the number of i such that $\pi(i) > \pi(i+1)$. Referring to Foata's survey [9] on the history of the Eulerian polynomials, (2) and (3) are due to Euler [8] and (1) is due to Riordan [16].

In this paper, for positive integers k, we define the $1/k$-Eulerian polynomial combinatorially, as the distribution of a certain statistic "asc" over a set of "k-inversion sequences", $I_{n,k}$, specified in the next subsection. These polynomials arise naturally in the theory of lecture hall partitions, via an associated "k-lecture hall polytope". We will show that the Ehrhart polynomial of the k-lecture hall polytope can be computed explicitly. Consequently, the exponential generating of the $1/k$-Eulerian polynomials can be derived to establish the following relations analogous to (1) - (3):

The $1/k$-Eulerian polynomials, $A_n^{(k)}(x)$, can be defined for $n \geq 0$ by any of the following relations:

$$A_n^{(k)}(x) = \sum_{e \in I_{n,k}} x^{\mathrm{asc}(e)}; \tag{4}$$

$$\sum_{t \geq 0} \binom{t - 1 + \frac{1}{k}}{t} (kt+1)^n x^t = \frac{A_n^{(k)}(x)}{(1-x)^{n+\frac{1}{k}}}; \tag{5}$$

$$\sum_{n \geq 0} A_n^{(k)}(x) \frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)} - x} \right)^{\frac{1}{k}}. \tag{6}$$

Their name is derived from (6) where their exponential generating function is the $1/k$-th power of a k-generalization of (3). Our main contribution is to show that the $1/k$-Eulerian polynomials have a simple combinatorial interpretation in terms of inversion sequences and a geometric realization in terms of lecture hall polytopes.

1.1 Inversion sequences and ascents

In eq. (4), the sum is over the set $I_{n,k}$ of *k-inversion sequences* defined by

$$I_{n,k} = \{e \in \mathbb{Z}^n \mid 0 \leq e_i \leq (i-1)k\}. \tag{7}$$

For $e \in I_{n,k}$, $\mathrm{asc}(e)$ is the number of *ascents* of e, defined as

$$\mathrm{asc}(e) = \#\left\{ i : 1 \leq i \leq n-1 \;\middle|\; \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1} \right\}.$$

Note the somewhat unusual definition of "ascent". See Figure 1 for an example of the computation of $A_3^{(2)}(x)$ from (4) using these definitions.

1.2 The $1/k$-Eulerian numbers

Define the $1/k$-*Eulerian numbers* $a_{n,j}^{(k)}$ by $a_{n,j}^{(k)} = \#\{e \in I_{n,k} \mid \mathrm{asc}(e) = j\}$.

The triangle of 1/2-Eulerian numbers is shown below, for $1 \leq n \leq 7$.

1						
1	2					
1	10	4				
1	36	60	8			
1	116	516	296	16		
1	358	3508	5168	1328	32	
1	1086	21120	64240	42960	5664	64

The third row corresponds to the polynomial in Figure 1. Of course, from the definition, $\sum_{j=0}^{n-1} a_{n,j}^{(k)} = \prod_{i=0}^{n-1}(ik+1)$.

1.3 Lecture hall polytopes

It is a bit surprising that the polynomials $A_n^{(k)}(x)$ defined by (6) have the simple combinatorial interpretation (4). This interpretation has its roots in the theory of *lecture hall partitions* [1, 2]. In Section 2, the equivalence of (4) and (5) is established using recent work of Savage and Schuster relating lecture hall polytopes to statistics on inversion sequences [17]. It follows from Theorem 5 in [17] that $A_n^{(k)}(x)$, defined by (4) has a geometric interpretation as the h^*-vector of the *k-lecture hall polytope*, $P_{n,k}$, defined by

$$P_{n,k} \;=\; \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \frac{\lambda_3}{2k+1} \leq \ldots \leq \frac{\lambda_n}{(n-1)k+1} \leq 1 \right\}. \qquad (8)$$

A key part of the proof of the equivalence of (4) and (5) in Section 2 is a rather involved, explicit computation of the Ehrhart polynomial of $P_{n,k}$. All definitions and background are provided in Section 2.

1.4 Generalizing properties of the Eulerian polynomials

In addition to properties (1) - (3), the Eulerian polynomials satisfy many relations, including: a recursive definition as an n-term sum; a two-term differential recurrence; a differential operator definition; a recurrence for the coefficients; an explicit formula for the coefficients; and a Worpitzky identity (See [9]). All of these properties can be generalized to the $1/k$-Eulerian polynomials.

e	ascents	asc(e)	e	ascents	asc(e)	e	ascents	asc(e)
000	$\{\}$	0	010	$\{1\}$	1	020	$\{1\}$	1
001	$\{2\}$	1	011	$\{1\}$	1	021	$\{1\}$	1
002	$\{2\}$	1	012	$\{1,2\}$	2	022	$\{1\}$	1
003	$\{2\}$	1	013	$\{1,2\}$	2	023	$\{1\}$	1
004	$\{2\}$	1	014	$\{1,2\}$	2	024	$\{1,2\}$	2

Figure 1: Computation of $A_3^{(2)}(x) = 1 + 10x + 4x^2$ using (4).

In order to do so, in Section 3, we view them in a more general setting. In the process, we will uncover a connection between the $1/k$-Eulerian polynomials and previous work.

To this end, for $n \geq 0$, define the bivariate polynomial $F_n(x, y)$ by

$$\sum_{t \geq 0} \binom{t+y-1}{t} (t+y)^n x^t \quad = \quad \frac{F_n(x,y)}{(1-x)^{n+y}}. \tag{9}$$

Then from (5), the relationship between $A_n^{(k)}(x)$ and $F_n(x, y)$ is given by

$$A_n^{(k)}(x) \quad = \quad k^n F_n(x, 1/k). \tag{10}$$

This provides further motivation for the term "$1/k$-Eulerian polynomials". In Section 3, we prove that all of the aforementioned properties of the Eulerian polynomials generalize to $F_n(x, y)$ and thereby to the $1/k$-Eulerian polynomials. Most of these identities appear in some form in earlier work and we make the connections in Section 4. However, it is unexpected that the non-integral case of $y = 1/k$ should be so interesting.

1.5 The combinatorics of $F_n(x, y)$

In concluding this overview, we highlight one particular outcome of Section 3. It turns out that $F_n(x, y)$ has a simple interpretation in terms of the statistics "excedance" and "number of cycles" on permutations. The *excedance* of a permutation $\pi \in S_n$ is defined by

$$\mathrm{exc}(\pi) = \#\{i \mid \pi(i) > i\}.$$

Recall that every $\pi \in S_n$ can be decomposed uniquely as the product of disjoint cycles. The number of such cycles is denoted by $\#\mathrm{cyc}(\pi)$. The last relation we prove in Section 3 is that

$$F_n(x, y) \quad = \quad \sum_{\pi \in S_n} x^{\mathrm{exc}(\pi)} y^{\#\mathrm{cyc}(\pi)}. \tag{11}$$

As discussed in Section 4, this relationship has appeared in various forms elsewhere in the literature. However, the following two consequences of (11) are relevant here. First,

combining (11), (10) and (4), we have

$$\sum_{e \in I_{n,k}} x^{\mathrm{asc}(e)} \quad = \quad \sum_{\pi \in S_n} x^{\mathrm{exc}(\pi)} k^{n-\#\mathrm{cyc}(\pi)}.$$

This gives further evidence that the k-inversion sequences and their associated ascent statistic are encoding something of combinatorial significance.

Secondly, our results provide a *geometric* interpretation of the joint distribution (11) in terms of the k-lecture hall polytope defined by (8), in the special case that y is the reciprocal of an integer. The variable y that tracks the number of cycles in a permutation is related to the angles at which the faces of the polytope meet.

In Section 4 we discuss connections between Section 3 and other work in the literature and suggest some further directions for inquiry.

2 The geometry of the $1/k$-Eulerian polynomials

2.1 Lecture hall polytopes and inversion sequences

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, the $\mathbf{s}$-*lecture hall polytope* $\mathcal{P}_n^{(\mathbf{s})}$ is defined by

$$\mathcal{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$

These polytopes, introduced in [17], were named after the *lecture hall partitions* [1, 2] of Bousquet-Melou and Eriksson.

Let $t\mathcal{P}_n^{(\mathbf{s})} = \{t\lambda \mid \lambda \in \mathcal{P}_n^{(\mathbf{s})}\}$ denote the t-th *dilation* of $\mathcal{P}_n^{(\mathbf{s})}$. Define $i(\mathcal{P}_n^{(\mathbf{s})}, t)$ by

$$i(\mathcal{P}_n^{(\mathbf{s})}, t) \quad = \quad |t\mathcal{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|.$$

Since all vertices of $\mathcal{P}_n^{(\mathbf{s})}$ have integer coordinates, $i(\mathcal{P}_n^{(\mathbf{s})}, t)$ is a rational polynomial in t, known as the *Ehrhart polynomial* of $\mathcal{P}_n^{(\mathbf{s})}$ [6, 7]. There is a relationship between the Ehrhart polynomial of $\mathcal{P}_n^{(\mathbf{s})}$ and the distribution of a certain statistic on $\mathbf{s}$-inversion sequences, as we now describe.

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, define the set $I_n^{(\mathbf{s})}$ of $\mathbf{s}$-*inversion sequences* of length n by

$$I_n^{(\mathbf{s})} \quad = \quad \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \;\mid\; 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

When $\mathbf{s} = (1, 2, \ldots, n)$, $I_n^{(\mathbf{s})}$ is the familiar set of inversion sequences in bijection with S_n.

For $e \in I_n^{(\mathbf{s})}$, an *ascent* of e is a position i such that $1 \leq i < n$ and

$$\frac{e_i}{s_i} \quad < \quad \frac{e_{i+1}}{s_{i+1}}.$$

In addition, if $e_1 > 0$ then 0 is an ascent of e. Let $\mathrm{asc}(e)$ be the number of ascents of e.

For example, if $\mathbf{s} = (5,3,7)$, then $e = (3,2,3)$ is an $\mathbf{s}$-inversion sequence with ascents in positions 0 and 1, but not in position 2. As another example, if $\mathbf{s} = (1,4,7)$, then $e = (0,3,4)$, as an $\mathbf{s}$-inversion sequence, has an ascent only in position 1. In fact, no $(1,4,7)$-inversion sequence will have 0 as as ascent.

The following relationship between the $\mathbf{s}$-inversion sequences and the $\mathbf{s}$-lecture hall polytopes was established in [17].

Theorem 1. [17] *Let* $\mathbf{s}$ *be any sequence of positive integers. For integer* $n \geq 0$,

$$\sum_{t\geq 0} i(\mathcal{P}_n^{(\mathbf{s})}, t)\ x^t \quad = \quad \frac{\sum_{e\in I_n^{(\mathbf{s})}} x^{\mathrm{asc}(e)}}{(1-x)^{n+1}}. \tag{12}$$

Stanley [18] has shown that for any convex lattice polytope, $\mathcal{P}$, of dimension n, there is a polynomial, $h_n^*(x)$, with nonnegative integer coefficients satisfying

$$\sum_{t\geq 0} i(\mathcal{P}, t)x^t \quad = \quad \frac{h_n^*(x)}{(1-x)^{n+1}}.$$

The $\mathbf{s}$-lecture hall polytopes are all convex, in fact they are simplices. So Theorem 1 says, in other words:

> *The* h^**-polynomial of the* $\mathbf{s}$*-lecture hall polytope* $\mathcal{P}_n^{(\mathbf{s})}$ *is the ascent polynomial of the set of* $\mathbf{s}$*-inversion sequences* $I_n^{(\mathbf{s})}$.

In this paper, our focus is on sequences $\mathbf{s}$ of the form

$$\mathbf{s} = (1,\ k+1,\ 2k+1,\ \ldots,\ (n-1)k+1), \tag{13}$$

where k is a positive integer. Recalling $I_{n,k}$ and $P_{n,k}$ from (7) and (8) in Section 1, we have

$$I_{n,k} = I_n^{(\mathbf{s})} \quad \text{and} \quad P_{n,k} = \mathcal{P}_n^{(\mathbf{s})},$$

where $\mathbf{s}$ is defined by (13), giving the following corollary.

Corollary 1. *For integers* $k \geq 1$ *and* $n \geq 0$,

$$\sum_{t\geq 0} i(P_{n,k}, t)x^t \quad = \quad \frac{\sum_{e\in I_{n,k}} x^{\mathrm{asc}(e)}}{(1-x)^{n+1}}.$$

For completeness, we include a proof of Corollary 1 in the Appendix.

2.2 The main result

We will show in Section 2.3 that when the sequence $\mathbf{s}$ has the special form (13), the Ehrhart polynomial of the $\mathbf{s}$-lecture hall polytope has the following closed form.

Theorem 2. *For integers $k \geq 1$ and $n, t \geq 0$,*

$$i(P_{n,k}, t) \;=\; (-1)^t \sum_{p=0}^{t} \binom{\frac{1}{k}-1}{t-p} \binom{-1/k}{p} (kp+1)^n,$$

where

$$i(P_{n,k}, t) \;=\; \#\left\{ \lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \frac{\lambda_3}{2k+1} \leq \ldots \leq \frac{\lambda_n}{(n-1)k+1} \leq t \right\}.$$

Our main result, Theorem 3 below, is a consequence of Theorem 2 and Corollary 1.

Theorem 3. *For integers $k \geq 1$ and $n \geq 0$, let*

$$A_n^{(k)}(x) \;=\; \sum_{e \in I_{n,k}} x^{\mathrm{asc}(e)}.$$

Then

$$\sum_{t \geq 0} \binom{t-1+\frac{1}{k}}{t} (kt+1)^n x^t \;=\; \frac{A_n^{(k)}(x)}{(1-x)^{n+\frac{1}{k}}}.$$

Proof. In Theorem 2, sum over $t \geq 0$ and apply the binomial theorem to get

$$\begin{aligned}
\sum_{t \geq 0} i(P_{n,k}, t) x^t &= \sum_{t \geq 0} (-x)^t \sum_{p=0}^{t} \binom{\frac{1}{k}-1}{t-p} \binom{-1/k}{p} (kp+1)^n \\
&= (1-x)^{1/k-1} \sum_{p \geq 0} \binom{-1/k}{p} (kp+1)^n (-x)^p. \\
&= (1-x)^{1/k-1} \sum_{p \geq 0} \binom{p-1+\frac{1}{k}}{p} (kp+1)^n x^p.
\end{aligned}$$

From Corollary 1,

$$\begin{aligned}
\frac{A_n^{(k)}(x)}{(1-x)^{n+\frac{1}{k}}} &= \frac{\sum_{t \geq 0} i(P_{n,k}, t) x^t}{(1-x)^{1/k-1}} \\
&= \sum_{p \geq 0} \binom{p-1+\frac{1}{k}}{p} (kp+1)^n x^p.
\end{aligned}$$

□

2.3 Proof of Theorem 2: computation of $i(P_{n,k}, t)$

Definition 1. *Let $G_{n,k}^{(j,d,r)}$ be the number of $\lambda \in \mathbb{Z}^n$ satisfying both*

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \ldots \leq \frac{\lambda_n}{(n-1)k+1}$$

and

$$\lambda_n \leq j((n-1)k+1) + d(n-1) + r.$$

Note that

$$i(P_{n,k}, t) \;\; = \;\; G_{n,k}^{(t,0,0)}.$$

For suitable conditions on n, k, j, d, r, we will find and prove a recurrence for $G_{n,k}^{(j,d,r)}$, solve the recurrence, and then set $d = r = 0$ to get $i(P_{n,k}, t)$, thereby proving Theorem 2.

Theorem 4. *For integers $n \geq 0$, $k \geq 1$, $j \geq 0$ and nonnegative integers d, r satisfying $d = r = 0$ if $n = 0$ and otherwise $r \leq n-1$ and $(n-1)d + r < k(n-1) + 1$,*

$$G_{n,k}^{(j,d,r)} = \begin{cases} 1 & \text{if } n = 0 \text{ or } j = d = r = 0, \text{ else} \\ G_{n,k}^{(j-1,k,0)} + G_{n-1,k}^{(j,0,0)} & \text{if } d = r = 0, \text{ else} \\ G_{n,k}^{(j,d-1,n-1)} & \text{if } r = 0, \text{ else} \\ G_{n,k}^{(j,d,r-1)} + G_{n-1,k}^{(j,d,r-1)} & \text{otherwise.} \end{cases}$$

We will need a technical lemma.

Lemma 1. *If $0 < r \leq n-1$ and $0 \leq d < k$ then*

$$\left\lfloor \frac{(n-2)k+1}{(n-1)k+1}((n-1)d + r) \right\rfloor \;\; = \;\; (n-2)d + r - 1.$$

Proof. It suffices to verify that

$$\frac{(n-2)k+1}{(n-1)k+1}((n-1)d+r) - 1 \;\; < \;\; (n-2)d + r - 1 \;\; \leq \;\; \frac{(n-2)k+1}{(n-1)k+1}((n-1)d+r),$$

which is straightforward. □

Proof of Theorem 4. Let $S_{n,k}^{(j,d,r)}$ denote the set counted by $G_{n,k}^{(j,d,r)}$ in Definition 1 and let $\lambda \in S_{n,k}^{(j,d,r)}$. When $n = 0$, or when $j = d = r = 0$, $S_{n,k}^{(j,d,r)}$ contains only the empty sequence. Otherwise, $n > 0$, with $j + d + r > 0$.

If $d = r = 0$, then $j > 0$ and $\lambda_n \le j(k(n-1)+1)$, either $\lambda_n = j(k(n-1)+1)$ or

$$\lambda_n \;\le\; j(k(n-1)+1) - 1 \;=\; (j-1)(k(n-1)+1) + (n-1)k + 0.$$

In the latter case $\lambda \in S_{n,k}^{(j-1,k,0)}$. In the former case, λ satisfies

$$\lambda_{n-1} \;\le\; (k(n-2)+1)\frac{j(k(n-1)+1)}{(k(n-1)+1)} \;=\; j(k(n-2)+1),$$

so $(\lambda_1, \ldots, \lambda_{n-1}) \in S_{n-1,k}^{(j,0,0)}$. Otherwise, if $r = 0$, but $d > 0$, then

$$\lambda_n \;\le\; j(k(n-1)+1) + (n-1)d \;=\; \lambda_n \le j(k(n-1)+1) + (n-1)(d-1) + n - 1.$$

Therefore $\lambda \in S_{n,k}^{(j,d-1,n-1)}$. Otherwise, $r > 0$ and $n > 0$ and either

$$\lambda_n \;=\; j(k(n-1)+1) + (n-1)d + r$$

or

$$\lambda_n \;\le\; j(k(n-1)+1) + (n-1)d + r - 1.$$

Clearly in the latter case $\lambda \in S_{n,k}^{(j,d,r-1)}$. In the former case, $0 < r \le n-1$ and $0 \le d < k$ and λ satisfies

$$\begin{aligned}
\lambda_{n-1} &\le (k(n-2)+1)\frac{(j(k(n-1)+1) + (n-1)d + r)}{(k(n-1)+1)} \\
&\le j(k(n-2)+1) + \left\lfloor \frac{(k(n-2)+1)((n-1)d + r)}{(k(n-1)+1)} \right\rfloor \\
&\le j(k(n-2)+1) + (n-2)d + (r-1),
\end{aligned}$$

where we have used Lemma 1 for the last inequality. Since $n \ge 2$ and $r > 0$ we can conclude that $(\lambda_1, \ldots, \lambda_{n-1}) \in S_{n-1,k}^{(j,d,r-1)}$. □

Now we solve the recurrence.

Theorem 5. *Let integers $k \ge 1$ and $n, j, d, r \ge 0$ satisfy $d = r = 0$ if $n = 0$ and otherwise $r \le n-1$ and $(n-1)d + r < k(n-1)+1$. Then $G_{n,k}^{(j,d,r)}$, defined by Definition 1, has the closed form*

$$G_{n,k}^{(j,d,r)} \;=\; (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k} - 1}{j-p} \binom{\frac{-1}{k}}{p} (kp+1)(kp+d+1)^{n-1-r}(kp+d+2)^r. \quad (14)$$

Proof. Let $f_{n,k}^{(j,d,r)}$ denote the expression on the right-hand side of (14). We prove it satisfies the recurrence of Theorem 4.

When $n = 0$, then $d = r = 0$, and

$$f_{0,k}^{(j,0,0)} \;=\; (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p} \;=\; (-1)^j \binom{-1}{j} \;=\; 1,$$

by the Chu-Vandermonde convolution identity. When $j = d = r = 0$, clearly $f_{n,k}^{(0,0,0)} = 1$. Otherwise, $n > 0$ with $j + d + r > 0$. If $d = r = 0$, then $j > 0$ and

$$\begin{aligned}
&f_{n,k}^{(j-1,k,0)} + f_{n-1,k}^{(j,0,0)} \\
&= (-1)^{j-1} \sum_{p=0}^{j-1} \binom{\frac{1}{k}-1}{j-1-p}\binom{\frac{-1}{k}}{p}(kp+1)(kp+k+1)^{n-1} \\
&\qquad +(-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)^{n-1} \\
&= (-1)^j \sum_{p=1}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p} kp(kp+1)^{n-1} + (-1)^j \binom{\frac{1}{k}-1}{j} \\
&\qquad +(-1)^j \sum_{p=1}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)^{n-1} \\
&= (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)^{n} \;=\; f_{n,k}^{(j,0,0)}.
\end{aligned}$$

Otherwise, if $r = 0$, but $d > 0$, then

$$f_{n,k}^{(j,d-1,n-1)} \;=\; (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)(kp+d+1)^{n-1} \;=\; f_{n,k}^{(j,d,0)}.$$

Otherwise, $r > 0$ and $n > 0$, and

$$\begin{aligned}
&f_{n,k}^{(j,d,r-1)} + f_{n-1,k}^{(j,d,r-1)} \\
&= (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)(kp+d+2)^{(r-1)}((kp+d+1)^{n-r} \\
&\qquad +(-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)(kp+d+2)^{(r-1)}(kp+d+1)^{n-r-1} \\
&= (-1)^j \sum_{p=0}^{j} \binom{\frac{1}{k}-1}{j-p}\binom{\frac{-1}{k}}{p}(kp+1)(kp+d+1)^{n-r-1}(kp+d+2)^{(r)} \;=\; f_{n,k}^{(j,d,r)}.
\end{aligned}$$

□

Setting $d = r = 0$ in Theorem 5 gives Theorem 2. Note that in computing $G_{n,k}^{(j,d,r)}$, we have actually computed the Ehrhart *quasi-polynomial* of the *rational lecture hall polytope* $R_{n,k}$:

$$R_{n,k} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \frac{\lambda_3}{2k+1} \leq \ldots \leq \frac{\lambda_n}{(n-1)k+1} \leq \frac{1}{(n-1)k+1} \right\}.$$

3 A bivariate generalization of the Eulerian polynomials

In Theorem 6 of this section, we extend the properties of the Eulerian polynomials to $F_n(x,y)$, defined by (9), and therefore, in view of (10), to $A_n^{(k)}(x)$. We first introduce a y-generalization of the binomial coefficient that will play a role.

For any real value y, define the *y-binomial coefficient* $\binom{n}{i}_y$ by

$$\binom{n}{i}_y = \binom{n-1}{i}_y + \binom{n-1}{i-1}_y, \tag{15}$$

with initial conditions $\binom{0}{0}_y = 1/y$, $\binom{n}{0}_y = 1$ for $n > 0$, and $\binom{n}{i}_y = 0$ for $i > n$. Then $\binom{n}{i}_1 = \binom{n}{i}$ and it is easy to show that

$$\binom{n}{i}_y = \binom{n-1}{i} + y^{-1}\binom{n-1}{i-1}.$$

In particular, observe that

$$\binom{n}{n}_y = 1/y.$$

When $y = p/r$ for positive integers p and r, the numbers $p\binom{n}{i}_{p/r}$ are referred to as the (p,r) binomial coefficients. We will make use of the following y-generalization of the binomial theorem.

Lemma 2. *For positive integer n,*

$$\sum_{i=0}^{n} y\binom{n}{i}_y w^i = (y+w)(1+w)^{n-1}.$$

(*The sum is equal to* 1 *if* $n = 0$.)

Proof. This is clear for $n = 0$ and $n = 1$. For $n > 1$, assume the lemma is true for integers smaller than n. Then using (15),

$$\begin{aligned}\sum_{i=0}^{n} y\binom{n}{i}_y w^i &= y + \sum_{i=1}^{n-1} y\binom{n-1}{i}_y w^i + \sum_{i=1}^{n-1} y\binom{n-1}{i-1}_y w^i + w^n \\ &= (y+w)(1+w)^{n-2} + w(y+w)(1+w)^{n-2} = (y+w)(1+w)^{n-1}.\end{aligned}$$

□

Theorem 6. *For $n \geq 0$, define $F_n(x,y)$ by*

$$\sum_{t\geq 0} \binom{t+y-1}{t} (t+y)^n x^t \;=\; \frac{F_n(x,y)}{(1-x)^{n+y}}. \tag{16}$$

Then $F_n(x,y)$ satisfies each of the following relations (17) - (25)*:*

Exponential generating function:

$$\sum_{n\geq 0} F_n(x,y)\frac{z^n}{n!} \;=\; \left(\frac{1-x}{e^{z(x-1)}-x}\right)^y. \tag{17}$$

Recursive definition via y-binomial coefficient

$$F_n(x,y) \;=\; \sum_{j=0}^{n-1} y\binom{n}{j}_y F_j(x,y)(x-1)^{n-j-1}. \tag{18}$$

with initial conditions $F_0(x,y) = 1$, $F_1(x,y) = y$.

Two-term differential recurrence:

$$F_{n+1}(x,y) \;=\; x(1-x)\frac{d}{dx}F_n(x,y) + (y+nx)F_n(x,y), \tag{19}$$

with initial conditions $F_0(x,y) = 1$, $F_1(x,y) = y$.

Differential operator definition

Define the operator D_y by $D_y = \frac{x}{y}\frac{d}{dx}$. Then

$$D_y^n\left(\frac{1}{(1-x)^y}\right) \;=\; \frac{(x/y)^n F_n(1/x,y)}{(1-x)^{n+y}}. \tag{20}$$

Recurrence for the coefficient of x^j

Let $F_n(x,y) = \sum_{j=0}^{n-1} f_{n,j}(y)x^j$. Then

$$f_{n+1,j}(y) \;=\; (j+y)f_{n,j}(y) \;+\; (n+1-j)f_{n,j-1}(y). \tag{21}$$

Formula for the coefficient of x^j

$$f_{n,j}(y) \;=\; \sum_{t=0}^{j}(-1)^{j-t}\binom{t-1+y}{t}\binom{n+y}{j-t}(t+y)^n. \tag{22}$$

Generalized Worpitzky identity

$$\binom{t-1+y}{t}(t+y)^n \;=\; \sum_{j} f_{n,j}(y)\binom{t+y+n-j-1}{t-j}. \tag{23}$$

Recurrence for the coefficient of $x^j y^k$

Let $f_{n,j,k}$ be the coefficient of $x^j y^k$ in $F_n(x,y)$. Then

$$f_{n,j,k} \;=\; jf_{n-1,j,k} \;+\; (n-j)f_{n-1,j-1,k} \;+\; f_{n-1,j,k-1}, \tag{24}$$

with boundary conditions $f_{0,0,0}=1$ and $f_{n,j,k}=0$ if $j+k>n$.

Combinatorial characterization:

$$F_n(x,y) \;=\; \sum_{\pi\in S_n} x^{\mathrm{exc}(\pi)}y^{\#\mathrm{cyc}(\pi)}, \tag{25}$$

where S_n is the set of permutations $\pi:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$, $\mathrm{exc}(\pi)=\#\{i \mid \pi(i)>i\}$, and $\#\mathrm{cyc}(\pi)$ is the number of cycles in the disjoint cycle representation of π.

Proof of (17). Using (16),

$$\begin{aligned}
\sum_{n\geq 0}F_n(x,y)\frac{z^n}{n!} &= \sum_{n\geq 0}(1-x)^{n+y}\sum_{t\geq 0}\binom{t-1+y}{t}(t+y)^n x^t\frac{z^n}{n!}\\
&= (1-x)^y\sum_{t\geq 0}\binom{t-1+y}{t}x^t\sum_{n\geq 0}\frac{(z(1-x)(t+y))^n}{n!}\\
&= (1-x)^y\sum_{t\geq 0}\binom{t-1+y}{t}x^t e^{z(1-x)(t+y)}\\
&= (1-x)^y e^{yz(1-x)}\sum_{t\geq 0}\binom{t-1+y}{t}(xe^{z(1-x)})^t\\
&= \frac{((1-x)e^{z(1-x)})^y}{(1-xe^{z(1-x)})^y} \;=\; \left(\frac{1-x}{e^{z(x-1)}-x}\right)^y.
\end{aligned}$$

□

Proof of (18). Let $S_n^{(k)}(x) = F_n(x,y)/(1-x)^{n+y}$. We will show, equivalently, that

$$(1-x)S_n(x,y) \;=\; \sum_{i=0}^{n-1} y\binom{n}{i}_y S_i(x,y)(-1)^{n-1-i}.$$

Using our series expansion (16) of $S_i(x)$ and equating coefficients of x^j, it suffices to show that

$$\begin{aligned}\binom{j-1+y}{j}(j+y)^n - \binom{j-2+y}{j-1}(j-1+y)^n \\ = \sum_{i=0}^{n-1} y\binom{n}{i}_y \binom{j-1+y}{j}(j+y)^i(-1)^{n-1-i},\end{aligned}$$

or, more simply, that

$$\sum_{i=0}^{n-1} y\binom{n}{i}_y (j+y)^i(-1)^{n-1-i} \;=\; (j+y)^n - j(j-1+y)^{n-1}.$$

To show this, apply Lemma 2, setting $w = -(j+y)$:

$$(-1)^{n-1}\sum_{i=0}^{n} y\binom{n}{i}_y (-1)^i(j+y)^i \;=\; (-j)(y+j-1)^{n-1}.$$

The result follows now by subtracting the $i=n$ term from both sides. □

Proof of (19). Apply (16) and rewrite as follows:

$$\begin{aligned}F_{n+1}(x,y) &= (1-x)^{n+1+y}\sum_{t\ge 0}\binom{t-1+y}{t}(t+y)^{n+1}x^t \\ &= (1-x)^{n+1+y}\left[\sum_{t\ge 0}\binom{t-1+y}{t}t(t+y)^n x^t + y\sum_{t\ge 0}\binom{t-1+y}{t}(t+y)^n x^t\right] \\ &= x(1-x)\left[(1-x)^{n+y}\sum_{t\ge 0}\binom{t-1+y}{t}t(t+y)^n x^{t-1}\right] + y(1-x)F_n(x,y).\end{aligned}$$

Now note that

$$\frac{d}{dx}F_n(x,y) \;=\; \left[(1-x)^{n+y}\sum_{t\ge 0}\binom{t-1+y}{t}t(t+y)^n x^{t-1}\right] - \frac{(n+y)F_n(x,y)}{1-x}.$$

Combining the two calculations gives the result. □

Proof of (20). We use induction on n. The identity is clearly true when $n=0$. Assume it is true for some $n>0$. Then

$$D_y^{n+1}\left(\frac{1}{(1-x)^y}\right) \;=\; (x/y)\frac{d}{dx}\left[\frac{(x/y)^n F_n(1/x,y)}{(1-x)^{n+y}}\right].$$

Using (19) with the chain rule,

$$\frac{d}{dx}F_n(1/x,y) \;=\; \frac{1}{(1-x)}\left[F_{n+1}(1/x,y)-(y+n/x)(F_n(1/x,y)\right].$$

Thus

$$\begin{aligned}
\frac{x}{y}\frac{d}{dx}\left[\frac{(x/y)^n F_n(1/x,y)}{(1-x)^{n+y}}\right] &= \left(\frac{(x/y)^{n+1}}{(1-x)^{n+y}}\right)\frac{d}{dx}F_n(1/x,y) \\
&\quad + \frac{x}{y}F_n(1/x,y)\frac{d}{dx}\left[\frac{(x/y)^n}{(1-x)^{n+y}}\right] \\
&= \frac{(x/y)^{n+1}(F_{n+1}(1/x,y)-(y+\frac{n}{x})F_n(1/x,y))}{(1-x)^{n+1+y}} \\
&\quad + \frac{x}{y}F_n(1/x,y)\left[\frac{(x/y)^n(y+\frac{n}{x})}{(1-x)^{n+1+y}}\right] \\
&= \frac{(x/y)^{n+1}F_{n+1}(1/x,y)}{(1-x)^{n+1+y}}.
\end{aligned}$$

□

Proof of (21). By (19),

$$\begin{aligned}
\sum_{j=0}^{n} f_{n+1,j}(y)x^j &= x(1-x)\frac{d}{dx}\sum_{j=0}^{n} f_{n,j}(y)x^j + (y+nx)\sum_{j=0}^{n} f_{n,j}(y)x^j \\
&= x(1-x)\sum_{j=0}^{n} j f_{n,j}(y)x^{j-1} + (y+nx)\sum_{j=0}^{n} f_{n,j}(y)x^j \\
&= \sum_{j=0}^{n}(j f_{n,j}(y) - (j-1)f_{n,j-1}(y) + y f_{n,j}(y) + n f_{n,j-1}(y))x^j \\
&= \sum_{j=0}^{n}((j+y)f_{n,j}(y) + (n+1-j)f_{n,j-1}(y))x^j.
\end{aligned}$$

Now equate coefficients of x^j on both sides. □

Proof of (22). Start with (16), apply the binomial theorem, and extract the coefficient of x^j:

$$\begin{aligned}
\sum_{j=0}^{n-1} f_{n,j}(y)x^j &= \sum_{t\geq 0}\binom{t-1+y}{t}(t+y)^n x^t(1-x)^{n+y} \\
&= \sum_{t\geq 0}\binom{t-1+y}{t}(t+y)^n x^t \sum_{p\geq 0}\binom{n+y}{p}(-x)^p \\
&= \sum_{j\geq 0}\sum_{t=0}^{j}(-1)^{j-t}\binom{t-1+y}{t}\binom{n+y}{j-t}(t+y)^n x^j.
\end{aligned}$$

□

Proof of (23): Start with (16), apply the binomial theorem, and extract the coefficient of x^t:

$$\begin{aligned}\sum_{t\geq 0}\binom{t-1+y}{t}(t+y)^n x^t &= \sum_{i=0}^{n-1} f_{n,i}(y)x^i(1-x)^{-(n+y)} \\ &= \sum_{i=0}^{n-1} f_{n,i}(y)x^i \sum_{p\geq 0}\binom{p+n-1+y}{p}x^p \\ &= \sum_{t\geq 0}\sum_{i\geq 0} f_{n,i}(y)\binom{t+y+n-i-1}{t-i}x^t.\end{aligned}$$

□

Proof of (24): The boundary conditions are clear. For $n > 0$, start with the definition of $f_{n,j,k}$ and apply (21):

$$\begin{aligned}F_n(x,y) = \sum_{j=0}^{n-1}\sum_{k=1}^{n} f_{n,j,k}x^j y^k &= \sum_{j=0}^{n-1} f_{n,j}(y)x^j \\ &= \sum_{j=0}^{n-1}((j+y)f_{n-1,j}(y) + (n-j)f_{n-1,j-1}(y))x^j \\ &= \sum_{j=0}^{n-1}\sum_{k=1}^{n}(j f_{n-1,j,k} + (n-j)f_{n-1,j-1,k} + f_{n-1,j,k-1})y^k x^j.\end{aligned}$$

Equating coefficients of $y^k x^j$ gives the result. □

Proof of (25): This follows from (3) and the Exponential Theorem (Chapter 5, [19]), but we give a combinatorial proof using (24). Let $g_{n,j,k}$ be the number of permutations $\pi \in S_n$ with $\mathrm{exc}(\pi) = j$ and $\#\mathrm{cyc}(\pi) = k$. We show that $g_{n,j,k}$ satisfies the recurrence (24) with the same boundary conditions. Let $\mathrm{Exc}(\pi) = \{i \mid \pi(i) > i\}$.

The boundary conditions are clear. When $n = 1$, $g_{n,j,k} = 0$ unless $j = 0$ and $k = 1$, in which case it is 1. This agrees with $F_1(x, y) = y$. For $n > 1$, a permutation $\mu \in S_n$ with k cycles and j excedances can be constructed in one of three mutually exclusive ways from a permutation in S_{n-1}:

(i) From a $\pi \in S_{n-1}$ with $k-1$ cycles and j excedances, by adding the cycle (n) of length 1. Then $\#\mathrm{cyc}(\mu) = \#\mathrm{cyc}(\pi) + 1$ and $\mathrm{Exc}(\mu) = \mathrm{Exc}(\pi)$.

(ii) From a $\pi \in S_{n-1}$ with k cycles and j excedances, by inserting 'n' between t and $\pi(t)$ on the cycle containing t, where $t \in Exc(\pi)$. Then $\#\mathrm{cyc}(\mu) = \#\mathrm{cyc}(\pi)$ and $\mathrm{Exc}(\mu) = \mathrm{Exc}(\pi)$.

(iii) From a $\pi \in S_{n-1}$ with k cycles and $j-1$ excedances, by inserting 'n' between t and $\pi(t)$ on the cycle containing t, where t is one of the $(n-1)-(j-1)$ elements $t \notin Exc(\pi)$. In this case, $\#\mathrm{cyc}(\mu) = \#\mathrm{cyc}(\pi)$ and $\mathrm{Exc}(\mu) = \mathrm{Exc}(\pi) \cup \{t\}$. □

This concludes the proof of Theorem 6.

4 Connections, observations and further directions

From Theorem 6, we have both (17) and (25), facts that have appeared in various forms in the literature. For *integer* y, $F_n(x,y)$, defined by (17), arises as a special case in the work of Carlitz [4]. For real y, Dillon and Roselle [5] use the exponential in (17) to define a reciprocal of $F_n(x,y)$ and prove several properties, including analogs of (21), (22), and (24). They also provide a combinatorial characterization of their polynomial as the joint distribution, over S_n, of the ascent statistic and the left-to-right-maximum. Since their polynomial is the reciprocal of ours, this implies (25), by the fundamental transformation of Foata and Schützenberger [10].

In [10], Foata and Schützenberger directly compute the exponential generating function of the distribution defined by (25) and obtain a different, but equivalent, form of (17). A refinement appears in the work of Ksavrelof and Zeng [14], where the number of fixed points is also included as a parameter. Variations of (16) and (23) based on weak excedances appear in work in progress of Gessel [11].

The thesis of Butler [3], on rook theory, contains q-analogs of the identities (16), (21), and (22). When $q = 1$, his corresponding polynomial is the joint distribution over S_n of the statistics (descent, left-to-right-min), which have the same joint distribution as (exc, #cyc).

We have not found (18) in the literature in quite this form. Of course the $y = 1$ case is well known. Variations for the cases $y = 2$ and $y = 3$ appear in Sloane (see the 2- and 3-restricted numbers below), which also suggested to us the differential operator definition (20) for the general case.

Several special cases of $F_n(x,y)$ have been studied. Note that from (16) we get a nice proof of the fact (see [14]) that

$$F_n(x,-1) \;=\; -(x-1)^{n-1},$$

since only the $t = 0$ term is nonzero and it is $(-1)^n$. Also, in (19), setting $x = 1$ gives the recurrence $F_{n+1}(1,y) = (y+n)F_n(1,y)$, which has solution

$$F_{n+1}(1,y) \;=\; (y+1)(y+2)\cdots(y+n-1),$$

the generating function for the unsigned Stirling cycle numbers.

The "second order Eulerian numbers" appearing in [13] and as entry A008517 in Sloane's EIS have the same row sums as our 1/2-Eulerian polynomial, $A_n^{(2)}(x)$, but the

entries are not the same. On the other hand, the reciprocal of $A_n^{(2)}(x)$ is given by the rows of the triangle in entry A156919 of Sloane's EIS, so the row polynomial is

$$R_n(x) \;=\; x^n A_n^{(2)}(1/x) \;=\; (2x)^n F_n(1/x, 1/2).$$

That entry is referred to as a "Table of coefficients of polynomials related to the Dirchlet eta function".

The rows of the triangle of "2-restricted Eulerian numbers" in entry A014496 of Sloane's EIS defines a polynomial $P_n(x)$ that is the reciprocal of a multiple of $F_n(x, 2)$, namely

$$P_n(x) \;= x^n F_n(1/x, 2)/2.$$

More generally, the rows of the "r-restricted Eulerian number" triangle satisfy

$$P_n^{(r)}(x) \;= x^n F_n(1/x, r)/r.$$

We close with a few questions. First, is it possible to interpret $F_n(x, y)$ in terms of lecture hall polytopes for other values of y, for example, when y is rational?

Secondly, can we define a meaningful q-analog of the $1/k$-Eulerian polynomial? We could have q track the "amaj" statistic defined in [17]. Another possibility is to adapt the q-analogs of $F_n(x, y)$ that are used in rook theory.

In [15], the work of [17] is being extended to rational lecture hall polytopes. This should allow an interpretation of the h^* polynomial of the rational k-lecture hall polytope whose Ehrhart quasi-polynomial was computed at the end of Section 2. Will this have a continuous analog corresponding to anything that has been studied before?

Finally, is there a more direct connection between lecture hall partitions and rook theory that could provide insight into either area?

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5 Appendix: Proof of Corollary 1 : the h^*-vector of $\mathcal{P}_n^{(k)}$

We are to show that

$$\sum_{t\geq 0} \#\left\{\lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{k+1} \leq \ldots \leq \frac{\lambda_n}{(n-1)k+1} \leq t\right\} x^t = \frac{\sum_{e\in I_{n,k}} x^{\mathrm{asc}(e)}}{(1-x)^{n+1}}. \quad (26)$$

Proof. The proof uses the notion of barred inversion sequences, adapted from the method of barred permutations from [12]. A *barred k-inversion sequence* is a sequence $e \in I_{n,k}$ in which one or more vertical bars are inserted before and/or after elements e_i, with the stipulation that if i is an *ascent* of e, there is at least one bar in position i, the space immediately preceding e_{i+1}. We show that both sides of (26) count the barred k-inversion sequences of length n.

For the right-hand side, consider a barred k-inversion sequence $e \in I_{n,k}$. It must must have at least one bar following each ascent position, but additional bars may be distributed among all of the $n+1$ " spaces " of e. The number of ways to place j identical bars into $n+1$ spaces is the coefficient of x^j in $1/(1-x)^{n+1}$, so, summing over all $e \in I_{n,k}$, the right-hand side of (26) counts the number of barred k-inversion sequences in $I_{n,k}$.

For the left-hand side, for each t, we establish a bijection between the barred k-inversion sequences with t bars and the set $tP_{n,k} \cap \mathbb{Z}^n$ counted by the summand. Let e be a barred k-inversion sequence with t bars. For $1 \le i \le n$, let b_i be the total number of bars preceding e_i in any position. Then $b_1 \le b_2 \le \ldots \le b_n$. Define $\lambda = (\lambda_1, \ldots, \lambda_n)$ by

$$\lambda_i = (k(i-1)+1)b_i - e_i.$$

We show that $\lambda \in (tP_{n,k} \cap \mathbb{Z}^n)$. First note that $\lambda_i \ge 0$, since $e_i < k(i-1)+1$ and if $b_i = 0$, then no position $j < i$ is an ascent, so $e_1 = \ldots = e_i = 0$. Since e is a k-inversion sequence, $\frac{e_j}{k(j-1)+1} < 1$ for all j. Now, if i is an ascent of e, there is at least one bar between e_i and e_{i+1}, so $b_i < b_{i+1}$ and

$$\frac{\lambda_i}{k(i-1)+1} \;=\; b_i - \frac{e_i}{k(i-1)+1} \;\le\; b_i \;\le\; b_{i+1} - 1 \;< b_{i+1} - \frac{e_{i+1}}{ki+1} \;=\; \frac{\lambda_{i+1}}{ki+1}.$$

On the other hand, if i is not an ascent of e, then

$$\frac{e_i}{k(i-1)+1} \;\ge\; \frac{e_{i+1}}{ki+1},$$

so

$$\frac{\lambda_i}{k(i-1)+1} \;=\; b_i - \frac{e_i}{k(i-1)+1} \;\le\; b_{i+1} - \frac{e_{i+1}}{ki+1} \;=\; \frac{\lambda_{i+1}}{ki+1}.$$

To complete the proof that $\lambda \in (tP_{n,k} \cap \mathbb{Z}^n)$ note that since $b_n \le t$ (the total number of bars in e),

$$\frac{\lambda_n}{k(n-1)+1} = b_n - \frac{e_n}{k(n-1)+1} \le t.$$

To prove that this is a bijection, we define the inverse. If $\lambda \in (tP_{n,k} \cap \mathbb{Z}^n)$, let

$$b = \left(\left\lceil \frac{\lambda_1}{1} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{k(n-1)+1} \right\rceil \right) = (b_1, \ldots, b_n).$$

Then $b_n \le t$. Let $e = (e_1, \ldots, e_n)$, where $e_i = s_i b_i - \lambda_i$. Then $e \in I_{n,k}$ and we "bar" it by placing b_1 bars before e_1, $b_i - b_{i-1}$ bars before e_i for $2 \le i \le n$, and $t - b_n$ bars after e_n.

By definition of b and e, $e_1 = 0$, so 0 is not an ascent of e. If, for some i with $1 \leq i < n$, there is no bar following e_i, then $b_i = b_{i+1}$ and therefore, since $\lambda \in (t\mathcal{P}_n^{(k)} \cap \mathbb{Z}^n)$,

$$\frac{e_i}{k(i-1)+1} \;=\; b_i - \frac{\lambda_i}{k(i-1)+1} \;\geq\; b_i - \frac{\lambda_{i+1}}{ki+1} \;=\; b_{i+1} - \frac{\lambda_{i+1}}{ki+1} \;=\; \frac{e_{i+1}}{ki+1}.$$

Thus i is not an ascent of e and the "barring" is valid. □

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