On the Unitary Cayley Graph of a Ring

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Abstract

Let R be a ring with identity. The *unitary Cayley graph* of a ring R, denoted by G_R, is the graph, whose vertex set is R, and in which $\{x, y\}$ is an edge if and only if $x - y$ is a unit of R. In this paper we find chromatic, clique and independence number of G_R, where R is a finite ring. Also, we prove that if $G_R \simeq G_S$, then $G_{R/J_R} \simeq G_{S/J_S}$, where $\mathrm{J_R}$ and $\mathrm{J_S}$ are Jacobson radicals of R and S, respectively. Moreover, we prove if $G_R \simeq G_{M_n(F)}$ then $R \simeq M_n(F)$, where R is a ring and F is a finite field. Finally, let R and S be finite commutative rings, we show that if $G_R \simeq G_S$, then $\mathrm{R/J_R} \simeq \mathrm{S/J_S}$.

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1 Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see [1], [2] and [15] .

Throughout this paper, R is a finite ring with identity. We denote the set of unit elements by $R^\times$. The unitary Cayley graph of a ring R, denoted by G_R, is the graph whose vertex set is R, and in which $\{x, y\}$ is an edge if and only if x and y are distinct elements of R such that $x-y \in R^\times$. Let A_G be the adjacency matrix of a simple graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix A_G. The *energy* of G is defined as the sum of absolute values of its eigenvalues, $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This concept was introduced first by Gutman in [6] and afterwards has been studied intensively in the literature [7],

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[8], [10] and [11]. If the distinct eigenvalues of A_G are $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix},$$

that is, the multiset of eigenvalues of the adjacency matrix of G.
The motivation of this paper is the study of interplay between graph theoretic properties of G_R and the ring properties of R. For some other recent papers on unitary Cayley graphs, see [9], [13], [15] and [17]. In this paper we find chromatic, clique and independence number of G_R, where R is a finite ring. Also, we prove that if $G_R \simeq G_S$, then $G_{R/J_R} \simeq G_{S/J_S}$, where $\mathrm{J_R}$ and $\mathrm{J_S}$ are Jacobson radicals of R and S, respectively. Moreover, we prove if $G_R \simeq G_{M_n(F)}$ then $R \simeq M_n(F)$, where R is a ring and F is a finite field. Finally, let R and S be finite commutative rings, we show that if $G_R \simeq G_S$, then $R/\mathrm{J_R} \simeq \mathrm{S/J_S}$.

2 Basic Notations and Properties

Throughout this paper, we use $N(v)$ for the neighborhood of a vertex (that is, the set of vertices adjacent to v). For a graph G, let $V(G)$ denote the set of vertices. The *category product* of G_1 and G_2, $G_1 \otimes G_2$, is the graph with vertex set $V(G_1 \otimes G_2) := V(G_1) \times V(G_2)$, specified by putting (u, v) adjacent to (u', v') if and only if u is adjacent to u' in G_1 and v is adjacent to v' in G_2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G_1, and $\mu_1, \mu_2, \ldots, \mu_m$ be the eigenvalues of G_2. Then the eigenvalues of $G_1 \otimes G_2$ are $\lambda_i \mu_j$, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, by Theorem 2.3.4 of [4]. For a graph G, we denote by $\overline{G}$ its complement, $\omega(G)$ its clique number, $\chi(G)$ its chromatic number and $\alpha(G)$ its independence number. The *Jacobson radical* of a ring R, denoted by J_R, is defined to be the intersection of all the maximal left ideals of R. Let R be commutative ring. We say that R is *local* if R has exactly one maximal ideal. If R is a finite commutative ring, then $R \simeq R_1 \times \cdots \times R_t$ where each R_i is a finite commutative local ring with maximal ideal M_i, by Theorem 8.7 of [3]. It is obvious that $\mathrm{R/J_R} \simeq \mathrm{R_1/M_1} \times \cdots \times \mathrm{R_t/M_t}$, and this decomposition is unique up to permutation of factors, where $\mathrm{J_R}$ is the Jacobson radical of R. We know that $(u_1, \ldots, u_t)$ is a unit of R if and only if each u_i is a unit element in R_i for $i = 1, \ldots, t$. So, we immediately see that G_R is the category product of the graphs $G_{R_1}, \ldots, G_{R_t}$.

Proposition 2.1. [2] *Let R be a finite commutative ring.*
(i) G_R is a regular graph of degree $|R^\times|$.
(ii) If $R \cong R_1 \times \cdots \times R_s$ is a product of local rings, then $G_R = \bigotimes_{i=1}^s G_{R_i}$.
(iii) If R is a commutative local ring with maximal ideal M, then G_R is a complete multipartite graph whose partite sets are the cosets of M.

The ring R is said to be *DU-ring* (determined by unitary Cayley graph) if S is a ring and $G_R \simeq G_S$. Then we have $R \simeq S$.
The commutative ring R is said to be *CDU-ring* (determined by unitary Cayley graph on commutative ring) if S is a commutative ring and $G_R \simeq G_S$. Then we have $R \simeq S$.

A ring R is said to be *reduced* if R has no nonzero nilpotent element. So, a finite commutative reduced ring R is a finite product of finite fields. We show that a finite commutative reduced ring is CDU-ring.

Lemma 2.2. [14] *Let D be a finite division ring. Then D is a finite field.*

Lemma 2.3. [14] **Wedderburn-Artin Theorem**
Let R be a semisimple ring. Then $R \simeq M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k)$ where n_i are integers and D_i are division ring, for $i = 1, \ldots, k$.

Lemma 2.4. [14] *Let R be a finite ring. Then $\mathrm{R/J_R}$ is semisimple ring and $\mathrm{R/J_R} \simeq \mathrm{M_{n_1}(F_1)} \times \ldots \times \mathrm{M_{n_k}(F_k)}$ where n_i are integers and F_i are finite fields, for $i = 1, \ldots, k$.*

3 Chromatic and Clique number

Lemma 3.1. *Let F be a finite field such that $|F| = q$. If N_n is the number of monic irreducible polynomial over F of degree n, then*

$$N_n = \frac{1}{n} \sum_{d|n} \mu(\frac{n}{d}) q^d. \tag{1}$$

Corollary 3.2. *Let F be a finite field. Then there exists at least a monic irreducible polynomial over F of degree n.*

Lemma 3.3. [16, 14.2.] *Let F be a finite field. If $P(x)$ is a monic irreducible polynomial over F of degree n, then there exists a matrix A in $M_n(F)$ such that characteristic and minimal polynomial of A is $P(x)$.*

Theorem 3.4. *Let F be a finite field and $R = M_n(F)$, where n is a positive integer. Then $\omega(G_R) = \chi(G_R) = |F|^n$.*

Proof. By Corollary 3.2, for every $n \in \mathbb{N}$ there is an irreducible $P(x) \in F[x]$ such that $degP(x) = n$. By using Lemma 3.3, there is a matrix $A \in M_n(F)$ such that the minimal polynomial of A is $P(x)$. So $[F[A] : F] = diam_F^{F[A]} = n$ and hence $|F[A]| = |F|^n$. We can see that $F[A] \simeq \frac{F[x]}{(p(x))}$. Thus $F[A]$ is a field. So

$$\chi(G_R) \geqslant \omega(G_R) \geqslant |F|^n. \tag{2}$$

Let $X = \{A_1, A_2, \ldots, A_{|F|^n}\}$ be the set of all vectors in F^n. Let S_i be the set of matrices in $M_n(F)$ such that the first row is A_i for $i = 1, 2, \ldots, |F|^n$. It is obvious that if $A, B \in S_i$, then $det(A - B) = 0$, so A is not adjacent to B. Therefore $\omega(G_R) \leqslant \chi(G_R) \leqslant |F|^n$, thus by formula (2), $\chi(G_R) = \omega(G_R) = |F|^n$. ☐

Remark 1. Let G be a graph. Given an n-coloring c of the graph G, it is straightforward to verify that the mapping $c'((g, h)) = c(g)$ is an n-coloring of the product $G \otimes H$. Therefore, $\chi(G \otimes H) \leqslant \chi(G)$. Similarity, we have $\chi(G \otimes H) \leqslant \chi(H)$, and hence $\chi(G \otimes H) \leqslant min\{\chi(G), \chi(H)\}$.

Theorem 3.5. *Let $R \simeq M_{n_1}(F_1) \times \ldots \times M_{n_k}(F_k)$ be a finite semisimple ring, where n_i are integers and F_i are finite fields, for $i = 1, \ldots, k$. Then $\omega(G_R) = \chi(G_R) = min\{|F_i|^{n_i}\}$.*

Proof. By Theorem 3.4, we see that $\omega(G_{M_{n_i}(F_i)}) = \chi(G_{M_{n_i}(F_i)}) = |F_i|^{n_i}$, for $i = 1, \ldots, k$. Let $\omega_i = \{A_{i1}, A_{i2}, \ldots, A_{i|F_i|^{n_i}}\}$ be a maximal clique set of $G_{M_{n_i}(F_i)}$. It is clear that $\{(A_{11}, A_{21}, \ldots, A_{k1}), (A_{12}, A_{22}, \ldots, A_{k2}), \ldots, (A_{1|F_1|^{n_1}}, A_{2|F_1|^{n_1}}, \ldots, A_{k|F_1|^{n_1}})\}$ is clique set of G_R. Thus $\chi(G_R) \geqslant \omega(G_R) \geqslant min\{|F_i|^{n_i}\}$. This shall complete the proof. □

Theorem 3.6. *Let F be a finite field and $R = M_n(F)$, where n is a positive integer. Thus $\alpha(G_R) = |F|^{n^2-n}$.*

Proof. In the proof of Theorem 3.4 we obtain that there exists a finite field K such that $K \subset M_n(F)$ and $|K| = |F|^n$. It is clear that K is a subgroup of $M_n(F)$, so $M_n(F) = \bigcup_{i=1}^{|F|^{n^2-n}} (K + A_i)$, where $K + A_i$ are distinct cosets of K, for all $i = 1, 2, \ldots, |F|^{n^2-n}$. Since $K + A_i$ are clique in G_R, then

$$\alpha(G_R) \leqslant |F|^{n^2-n}. \tag{3}$$

If S is the set of matrices in $M_n(F)$ such that the first row is zero vector, then S is an independent set of G_R. Therefore $\alpha(G_R) \geqslant |F|^{n^2-n}$. Thus by formula (3),

$$\alpha(G_R) = |F|^{n^2-n}.$$

□

Theorem 3.7. *Let $R \simeq M_{n_1}(F_1) \times \ldots \times M_{n_k}(F_k)$ be a finite semisimple ring, where n_i are integers and F_i are finite fields, for $i = 1, \ldots, k$. Then $\alpha(G_R) = \frac{|R|}{min\{|F_i|^{n_i}\}}$.*

Proof. Without loss of generality, we can assume that $min\{|F_i|^{n_i}\} = |F_1|^{n_1}$. Let $S = \{A_1, A_2, \ldots, A_{|F_1|^{n_1^2-n_1}}\}$ be the set of matrices in $M_{n_1}(F_1)$ such that the first row is zero vector. Then $I = S \times M_{n_2}(F_2) \times \ldots \times M_{n_k}(F_k)$ is an independent set of G_R. Thus $\alpha(G_R) \geqslant \frac{|R|}{min\{|F_i|^{n_i}\}}$. It is clear that I is a right ideal of R. We now construct a coloring of $\overline{G_R}$ by elements of I as follows: given $b = (b_1, \ldots, b_k) \in R$, fix an arbitrary clique C in G_R such that $|C| = |F_1|^{n_1}$, for example a clique which is constructed in Theorem 3.5.
We show that there is a unique element of C such as c_b in such a way that $b - c_b \in I$.
Existence: If c, c' are distinct elements of C, then $c - c'$ is unit element of R, so $c - c' \notin I$, thus $I + c \neq I + c'$, so $I + c$ and $I + c'$ are distinct cosets of I in R. Since $|C| = \frac{|R|}{|I|}$, it follows that $R = \bigcup_{c \in C}(I + c)$. Then there is a unique element of C such as c_b, in such a way that $b \in I + c_b$, so $b - c_b \in I$.
Uniqueness: Let c_b and c'_b be elements of C such that $b - c_b, b - c'_b \in I$. Since I is a right ideal, it follows that $b - c_b - (b - c'_b) \in I$. Then $c'_b - c_b \in I$. If $c'_b \neq c_b$, then $c'_b - c_b \in I$ is a unit element of R. So $I = R$, which is a contradiction.
Define a vertex coloring $f : R \longrightarrow I$ by $f(b) = b - c_b$. Then $f(b) = f(d)$ implies that $b - d = c_d - c_b$. If $c_d = c_b$, then $b = d$; so assume $c_d \neq c_b$. Then by construction, $c_d - c_b \in R^\times$, so $b - d \in R^\times$, and hence b is not adjacent to d in $\overline{G_R}$. Thus f is a proper coloring for $\overline{G_R}$, showing that $\alpha(G_R) \leqslant \frac{|R|}{min\{|F_i|^{n_i}\}}$, as desired. □

Theorem 3.8. *Let R be a ring. If J_R is Jacobson radical of R, then $\omega(G_R) = \chi(G_R) = \chi(G_{R/J_R}) = \omega(G_{R/J_R})$.*

Proof. By [14, Proposition 4.8], $u + J_R$ is unit in R/J_R if and only if u is unit in R and $(R/J_R)^\times = R^\times + J_R$. So, $u_1 + J_R$ is adjacent to $u_2 + J_R$ in G_{R/J_R} if and only if u_1 is adjacent to u_2 in G_R. Therefore, if $j_1, j_2 \in J_R$, then $u_1 + j_1$ is adjacent to $u_2 + j_2$, where $u_1 - u_2 \in R^\times$. Hence the induced graph of G_R on vertices $(a_1 + J_R) \cup (a_2 + J_R)$ is a complete bipartite graph, where $a_1 + J_R$ and $a_2 + J_R$ are distinct. Therefore, $\omega(G_R) = \chi(G_R) = \chi(G_{R/J_R}) = \omega(G_{R/J_R})$. □

Remark 2. Let R be a finite ring. By Lemma 2.4, $\mathrm{R/J_R} \simeq \mathrm{M_{n_1}(F_1)} \times \ldots \times \mathrm{M_{n_k}(F_k)}$. Therefore by Theorems 3.4, 3.5 and 3.8 we see that $\omega(G_R) = \chi(G_R) = min\{|F_i^{n_i}|\}$.

4 The unitary Cayley graphs of semisimple rings

In what follows, we study the interplay between G_R and the structure of R, when R is a finite ring.

Lemma 4.1. *Let R be a finite ring. For $j \in R$, the following statements are equivalent:*
(i) $j \in J_R$;
(ii) $j + u \in R^\times$ for any $u \in R^\times$.

Proof. (i) $\longrightarrow$ (ii) is trivial.

(ii) $\longrightarrow$ (i) Assume R is a finite semisimple ring, then $R \cong M_{n_1}(F_1) \times M_{n_2}(F_2) \times \ldots \times M_{n_t}(F_t)$, where each F_i is a field. Let $j = (A_1, A_2, \ldots, A_t)$, where $A_i \in M_{n_i}(F_i)$ and $j + R^\times = R^\times$. Thus $A_i + (M_{n_i}(F_i))^\times = (M_{n_i}(F_i))^\times$, for all $i = 1, 2, \ldots, t$. Assume to the contrary that $A_i \notin J(M_{n_i}(F_i)) = \{0\}$. Let $B_1, B_2, \ldots, B_{n_i}$ be rows of A_i. Without loss of generality, we can assume that $B_1 \neq 0$. Thus, $-B_1$ can be extended to a basis $\{-B_1, B'_2, \ldots, B'_{n_i}\}$ for $F_i^{n_i}$. Let B be a matrix such that the first row is $-B_1$ and the j-th row is B'_j. Hence, $B \in GL_{n_i}(F_i)$ and $det(A_i + B) = 0$, contradicting our assumption that $A_i \neq 0$.

Now assume that R is not semisimple, $(R/J_R)^\times = R^\times + J_R$ by [14, Proposition 4.8] if $j + R^\times = R^\times$, then $\overline{j} + (R/J_R)^\times = (R/J_R)^\times$. Thus $\overline{j} = 0$, and so $j \in J_R$, since $\overline{R} = R/J_R$ is a semisimple ring. □

Lemma 4.2. *Let R be a finite ring and $x, y \in G_R$. Then $N(x) = N(y)$ if and only if $x - y \in J_R$.*

Proof. It is clear that $N(x) = x + R^\times$ and $N(y) = y + R^\times$. Then $N(x) = N(y)$ if and only if $x + R^\times = y + R^\times$, hence it is equivalent to $x - y + R^\times = R^\times$. Therefore by Lemma 4.1, $x - y \in J_R$. □

Remark 3. Consider two vertices x, y of graph G to be equivalent when $N(x) = N(y)$. Then, following [5], we define the *reduction of G to be the graph G_{red} whose vertex set is the set of equivalence classes of vertices, and whose edges consist of pairs $\{A, B\}$ of equivalence classes with the property that $A \cup B$ induces a complete bipartite subgraph of G.*

Theorem 4.3. *Let R and S be finite rings such that $G_R \cong G_S$. Then $G_{R/J_R} \cong G_{S/J_S}$.*

Proof. It is clear that $(G_R)_{red} \cong G_{R/J_R}$. Since $G_R \cong G_S$, then $(G_R)_{red} \cong (G_S)_{red}$. Thus $G_{R/J_R} \cong G_{S/J_S}$. □

Corollary 4.4. *Let R and S be finite rings such that $G_R \cong G_S$. Then $|J_R| = |J_S|$.*

Corollary 4.5. *Let R and S be finite rings such that $G_R \cong G_S$. If R is semisimple, then S is semisimple.*

Proof. It is clear that $|R| = |S|$. By Theorem 4.3, we see that $|R| = \frac{|S|}{|J_S|}$. Thus $|J_S| = 1$. This shall complete the proof. □

Theorem 4.6. *Let F and E be two finite fields and m, n be two natural numbers. If $G_{M_n(F)} \simeq G_{M_m(E)}$, then $m = n$ and $F \simeq E$.*

Proof. We know that $|F|$ and $|E|$ are prime power numbers, say $|F| = p^r$ and $|E| = p_1^{r_1}$. Since $|F|^{n^2} = |E|^{m^2}$, then $p = p_1$ and $p^{rn^2} = p^{r_1 m^2}$, so

$$rn^2 = r_1 m^2. \tag{4}$$

By Theorem 3.4, $|F|^n = |E|^m$, so $p^{rn} = p^{r_1 m}$ and hence

$$rn = r_1 m. \tag{5}$$

By using (4) and (5), $n = m$ and $r = r_1$. Therefore, $F \simeq E$ and hence the proof is complete. □

Theorem 4.7. *Let $R = M_n(F)$, where F is a finite field and S be a semisimple ring. If $G_R \simeq G_S$, then $S \simeq M_n(F)$.*

Proof. Let $S \simeq M_{n_1}(E_1) \times M_{n_2}(E_2) \times \ldots \times M_{n_k}(E_k)$. We know that $|F|$ is prime power, $|F| = p^r$. It is obvious that

$$|F|^{n^2} = \prod_{i=1}^{i=k} |E_i|^{n_i^2}. \tag{6}$$

Thus $|E_i| = p^{r_i}$, so by above formula,

$$p^{rn^2} = p^{\sum_{i=1}^{i=k} r_i n_i^2}.$$

Therefore,

$$rn^2 = \sum_{i=1}^{i=k} r_i n_i^2. \tag{7}$$

By Theorem 3.5, $\chi(G_S) = min\{|E_i|^{n_i}\}$ and $\chi(G_R) = |F|^n$, so we have

$$|F|^n = min\{|E_i|^{n_i}\}. \tag{8}$$

Without loss of generality, we can assume that $min\{|E_i|^{n_i}\} = |E_1|^{n_1} = p^{r_1 n_1}$. Thus, $p^{rn} = p^{r_1 n_1}$. Therefore,

$$rn = r_1 n_1. \tag{9}$$

Degrees of all vertices of G_R and G_S are $|GL_n(F)|$ and $|\prod_{i=1}^{i=k} GL_{n_i}(E_i)|$, respectively. So,

$$|GL_n(F)| = |\prod_{i=1}^{i=k} GL_{n_i}(E_i)|.$$

Thus,

$$\prod_{i=0}^{n-1}(|F|^n - |F|^i) = \prod_{j=1}^{j=k}\prod_{i=0}^{n_j-1}(|E_j|^{n_j} - |E_j|^i). \tag{10}$$

Hence,

$$|F|^{\frac{n(n-1)}{2}}\prod_{i=1}^{n}(|F|^i - 1) = \prod_{j=1}^{j=k}|E_j|^{\frac{n_j(n_j-1)}{2}}\prod_{j=1}^{j=k}\prod_{i=1}^{n_j}(|E_j|^i - 1). \tag{11}$$

It is clear that if $i > 0$, then $gcd(|F|^i - 1, p) = gcd(|E_j|^i - 1, p) = 1$. Thus,

$$|F|^{\frac{n(n-1)}{2}} = \prod_{j=1}^{j=k}|E_j|^{\frac{n_j(n_j-1)}{2}}.$$

Therefore,

$$p^{\frac{rn(n-1)}{2}} = p^{\sum_{j=1}^{j=k}\frac{r_j n_j(n_j-1)}{2}}.$$

Thus,

$$rn(n-1) = \sum_{j=1}^{j=k} r_j n_j(n_j - 1). \tag{12}$$

By formula (7) and (12), we have that

$$rn = \sum_{j=1}^{j=k} r_j n_j. \tag{13}$$

Thus by formula (9) and (13), we have that

$$\sum_{j=2}^{j=k} r_j n_j = 0. \tag{14}$$

Therefore, $S = M_{n_1}(E_1)$. Thus by Theorem 4.6, the proof is complete. □

Theorem 4.8. *Let $R = M_n(F)$, where F is a finite field and S be a ring. If $G_R \simeq G_S$, then $S \simeq M_n(F)$.*

Proof. It is clear that S is finite and $J_R = \{0\}$, so by corollary 4.4, $J_S = \{0\}$. Thus S is a semisimple ring. Thus by Theorem 4.7, the proof is complete. □

Corollary 4.9. *Let F be a finite field and n be a natural number. Then $M_n(F)$ is a DU-ring.*

5 The unitary Cayley graphs of commutative rings

We recall the results obtained in [12] regarding the spectrum and the energy of unitary Cayley graphs of commutative rings.

Lemma 5.1. [12, Lemma 2.3] *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then the eigenvalues of G_R are*

(i) $(-1)^{|C|}\dfrac{|R^\times|}{\prod_{j\in C}|R_j^\times|/m_j}$ with multiplicity $\displaystyle\prod_{j\in C}\frac{|R_j^\times|}{m_j}$ for all subsets C of $\{1, 2, \ldots, s\}$,

and

(ii) 0 with multiplicity $|R| - \displaystyle\prod_{i=1}^{s}\left(1 + \frac{|R_i^\times|}{m_i}\right)$.

Lemma 5.2. [12, Theorem 2.4] *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then*

$$E(G_R) = 2^s|R^\times|.$$

Theorem 5.3. *Let R and R' be two finite commutative rings. If $G_R \simeq G_{R'}$, then $\mathrm{R}/\mathrm{J}_\mathrm{R} \simeq \mathrm{R}'/\mathrm{J}_{\mathrm{R}'}$.*

Proof. By our assumption, there exist local rings R_i and R_j' with maximal ideals M_i and M_j', where $R_i/M_i = F_i$ and $R_j'/M_j' = F_j'$ are finite fields for $i = 1, \ldots, r$ and $j = 1, \ldots, r'$ such that $R \simeq R_1 \times \cdots \times R_r$ and $R' \simeq R_1' \times \cdots \times R_{r'}'$. Let $|F_i| = q_i$ and $|F_j'| = q_j'$. We know that $|R^\times| = |R'^\times|$. Since G_R is isomorphic to $G_{R'}$, we have $E(G_R) = E(G_{R'})$ and so by Lemma 5.2, $2^r|R^\times| = 2^{r'}|R'^\times|$, hence $2^r = 2^{r'}$ and so $r = r'$. By Theorem 4.3, $G_{R/J_R} \cong G_{R'/J_{R'}}$. So, $G_{F_1\times F_2\times\ldots\times F_r} \cong G_{F_1'\times F_2'\times\ldots\times F_r'}$.

It is clear that $|(F_1 \times F_2 \times \ldots \times F_r)^\times| = |(F_1' \times F_2' \times \ldots \times F_r')^\times|$. So, $\prod_{i=1}^r(q_i - 1) = \prod_{i=1}^r(q_i' - 1)$. By Lemma 5.1, we conclude that the eigenvalues of $G_{F_1\times F_2\times\ldots\times F_r}$ and $G_{F_1'\times F_2'\times\ldots\times F_r'}$ are, respectively, $(-1)^{r-|C|}\prod_{j\in C}(q_j - 1)$ with multiplicity $\dfrac{\prod_{i=1}^r(q_i-1)}{\prod_{j\in C}(q_j-1)}$ and $(-1)^{r-|C|}\prod_{j\in C}(q_j' - 1)$ with multiplicity $\dfrac{\prod_{i=1}^r(q_i'-1)}{\prod_{j\in C}(q_j'-1)}$, for all subsets C of $\{1, 2, \ldots, r\}$.

Without loss of generality, we can assume that $q_1 \leqslant q_2 \leqslant \ldots \leqslant q_r$ and $q_1' \leqslant q_2' \leqslant \ldots \leqslant q_r'$. We want to prove that $q_i = q_i'$. We know that G_{R/J_R} has an eigenvalue $(-1)^{(r-1)}(q_1 - 1)$. From $Spec(G_{R/J_R}) = Spec(G_{R'/J_{R'}})$, we deduce that $q_1 - 1 = q_1' - 1$. Assume the contrary and let i be the smallest number such that $q_i \neq q_i'$. Without loss of generality, we can assume that $q_i < q_i'$. So,

$$q_i < q_j'. \tag{15}$$

for all $j \in \{i, i + 1, i + 2, \ldots, r\}$. It is clear that if C is a subset of $\{1, 2, \ldots, i - 1\}$, then

$$\frac{\prod_{i=1}^r(q_i-1)}{\prod_{j\in C}(q_j-1)} = \frac{\prod_{i=1}^r(q_i'-1)}{\prod_{j\in C}(q_j'-1)} \quad , \quad (-1)^{r-|C|}\prod_{j\in C}(q_j - 1) = (-1)^{r-|C|}\prod_{j\in C}(q_j' - 1). \tag{16}$$

Let, A and B be multisets of $(-1)^{r-|C|}\prod_{j\in C}(q_j-1)$ with multiplicities of $\dfrac{\prod_{i=1}^{r}(q_i-1)}{\prod_{j\in C}(q_j-1)}$ and $(-1)^{r-|C|}\prod_{j\in C}(q'_j-1)$ with multiplicities of $\dfrac{\prod_{i=1}^{r}(q'_i-1)}{\prod_{j\in C}(q'_j-1)}$, for all $C\subseteq\{1,2,\ldots,i-1\}$, respectively.

By formula (16), $A=B$. So, $Spec(G_{R/J_R})\backslash A = Spec(G_{R'/J_{R'}})\backslash B$ (as multiset). We can see that $(-1)^{r-1}(q_i-1)\in Spec(G_{R/J_R})-A$. Then we deduce that there is a subset C of $\{1,2,\ldots,r\}$, such that $C\nsubseteq\{1,2,\ldots,i-1\}$ and $((-1)^{r-1})(q_i-1)=(-1)^{r-|C|}\prod_{j\in C}(q'_j-1)$. So, Thus $q_i-1=\prod_{j\in C}(q'_j-1)$, which contradicts our assumption (15). □

The following corollary follows directly from Theorem 5.3.

Corollary 5.4. *Let R be a commutative reduced ring. Then R is a CDU-ring.*

Proof. Let S be a commutative ring. If $G_R\simeq G_S$, then $|S|=|R|$. Since R is reduced, we have $\mathrm{J_R}=0$. So by Theorem 5.3, $\mathrm{R}\simeq \mathrm{S/J_S}$ and hence $|\mathrm{J_S}|=1$. Therefore $R\simeq S$. □

Remark 4. The ring of polynomials with coefficients in $\mathbb{Z}_2$ be denoted by $\mathbb{Z}_2[x]$. Let $R=\mathbb{Z}_4$ and S be the quotient ring $\mathbb{Z}_2[x]/(x^2)$. Obviously, $G_R\simeq G_S$, but R is not isomorphic to S, which means that $\mathbb{Z}_4$ is not a CDU-ring. Therefore Corollary 5.4 does not hold for an arbitrary commutative ring.

Conjecture 1. Let R and S be finite rings such that $G_R\simeq G_S$. Then $R/J_R\simeq S/J_S$.

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