On the Length of a Partial Independent Transversal in a Matroidal Latin Square

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Abstract

We suggest and explore a matroidal version of the Brualdi - Ryser conjecture about Latin squares. We prove that any $n \times n$ matrix, whose rows and Columns are bases of a matroid, has an independent partial transversal of length $\lceil 2n/3 \rceil$. We show that for any n, there exists such a matrix with a maximal independent partial transversal of length at most $n-1$.

Keywords: Latin square, matroidal Latin square, partial independent transversal

1 Introduction

A Latin Square of order n is an $n \times n$ array L with entries taken from the set $\{1, \ldots, n\}$, where each entry appears exactly once in each row or column of L. A *partial transversal* of size k of a Latin square L is a subset of k different entries of L, where no two of them lie in the same row or column.

The maximal size of a partial transversal in L will be denoted here by $t(L)$ and the minimal size of $t(L)$, over all Latin squares L of order n, will be denoted by $T(n)$.

It has been conjectured by Ryser [10] that $T(n) = n$ for every odd n and by Brualdi [4] (see also [2] p. 255) that $T(n) = n-1$ for every even n. Although these conjectures are still unsettled, a consistent progress has been made towards its resolution: Koksma [8] proved that for $n \geqslant 3$, $T(n) \geqslant \lceil (2n+1)/3 \rceil$. This bound was improved by Drake [5] to $T(n) \geqslant \lceil 3n/4 \rceil$ for $n > 7$, and again by de Veris and Wieringa [3] who obtained a lower bound of $\lceil (4n-3)/5 \rceil$ for $n \geqslant 12$. Woollbright [14] showed that $T(n) \geqslant \lceil n - \sqrt{n} \rceil$. A similar result was obtained independently by Brouwer, de Vries and Wieringa [1]. Recently, Hatami and Shor [6] proved that $T(n) \geqslant n - O(\log^2 n)$. See also a recent comprehensive survey by Wanless [11].

The aim of this note is to suggest and explore a matroidal version of the Brualdi-Ryser conjectures. For basic texts on matroids the reader is referred to Welsh [12], Oxley [9] and White [13].

Definition 1. Let (M, S) be a matroid M on a ground set S. A matroidal Latin square (abbreviated MLS) of degree n over (M, S) is an $n \times n$ matrix A whose entries are elements of S, where each row or column of A is a base of M.

Notice that a matroidal Latin square reduces to a Latin square if M is a partition matroid. We mention that according to a well-known conjecture of Rota [7] every set of n bases of a matroid of rank n can be arranged to form an MLS of degree n so that its rows consist of the original bases.

Definition 2. An independent partial transversal of an MLS A is an independent subset of entries of A where no two of them lie in the same row or column of A.

We propose the following analogue of Brualdi's conjecture:

Conjecture 3. Every MLS of degree n has an independent partial transversal of size $n-1$.

In view of Ryser's conjecture, it is natural to ask whether in Conjecture 3 an independent transversal of size n exists whenever n is odd. Theorem 6 asserts that this is not the case.

2 A lower bound for a maximal independent partial transversal

Let $A = (a_{ij})_{i,j=1}^{n}$ be an MLS of degree n over a matroid M. Let T be an independent partial transversal of size t. Without loss of generality we may assume that the elements of T are the first t elements of the main diagonal of A. That is

$$A = \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right) \tag{1}$$

where B, C, D and E are sub-matrices of A of dimensions $t \times t$, $t \times (n-t)$, $(n-t) \times t$ and $(n-t) \times (n-t)$ respectively, and T constitutes the main diagonal of B. If T is of maximal length, then $t \geqslant \lceil n/2 \rceil$. Otherwise $dim(E) \geqslant n - t > t = dim(T)$ and thus E would contain an element that is not spanned by T and hence can be added to T, contradicting the maximality of T. In order to show that $t \geqslant \lceil 2n/3 \rceil$ we shall need the following lemma:

Lemma 4. *Let X be a finite set and let $s > |X|/2$. Let $X_1, \ldots, X_s$ be a family of subsets of X, each of size at least s. Then there exists some X_i, all of whose elements appear in other subsets in the family.*

Proof. Let Y_1 be the set of elements in X that appear in exactly one of the subsets $X_1, \ldots, X_s$ and let Y_2 be the set of elements in X that appear in at least two of the subsets $X_1, \ldots, X_s$. Let $k_1 = |Y_1|$ and $k_2 = |Y_2|$. Assume, by contradiction, that each X_i contains at least one element of Y_1. Then $k_1 \geqslant s$ and thus

$$k_2 \leqslant |X| - k_1 \leqslant |X| - s < |X|/2 \tag{2}$$

(since $s > |X|/2$). If, for some i, $|X_i \cap Y_1| = 1$ then $|X_i \cap Y_2| \geqslant s - 1$ and thus $k_2 \geqslant s - 1 > |X|/2 - 1$. It follows that $k_2 \geqslant |X|/2$, contradicting (2). It follows that for all i, $|X_i \cap Y_1| \geqslant 2$. Then $k_1 \geqslant 2s$ and thus $k_2 \leqslant |X| - k_1 \leqslant |X| - 2s < |X| - |X| = 0$, which is absurd. This proves the lemma. □

Theorem 5. *Let A be an MLS of degree n over a matroid M. Then A contains an independent partial transversal of size $\lceil 2n/3 \rceil$.*

Proof. We use the notations from the beginning of Section 2. Since T is maximal, all the elements in the sub-matrix E are spanned by T. Let T_E be the minimal subset of T that spans E (this set is unique since T is independent.) Since $dim(E) \geqslant n - t$ then $|T_E| \geqslant n - t$ and thus $|T \setminus T_E| \leqslant t - (n - t) = 2t - n$. Since each row of A is a base and all the elements of E are spanned by T, each row of the sub-matrix D contains a subset of size $n - t$ that complement T to a base. In particular, each row of D contains at least $n - t$ elements that are not spanned by T. Let $X = \{1, \ldots, t\}$ be the set of indices of the columns of D. For each of the $n - t$ rows in D we define a subset $X_i \subseteq X$, $i = t + 1, \ldots, n$, in the following way: $j \in X_i$ if and only if the jth element of the ith row of A is not spanned by T. It follows that $|X_i| \geqslant n - t$ for all $i = t + 1, \ldots, n$. Now assume, by contradiction, that $t < 2n/3$. Then $n - t > n/3 > t/2$. So we have a set X of size t and $n - t$ subsets $X_{t+1}, \ldots, X_n$, each of size at least $n - t$, such that $n - t > t/2$. Let $s = n - t$. By Lemma 4 we conclude that there exists a subset X_i all of whose elements are contained in other subsets in the family $X_{t+1}, \ldots, X_n$. This means that there is a row in D containing at least $n - t$ elements that are not spanned by T and for each such element there exists another element in the same column in D that is not spanned by T. It follows that D contains at least $n - t$ columns, each containing at least two elements that are not spanned by T. Since $t < 2n/3$ we have that $|T \setminus T_E| \leqslant 2t - n < n/3 < n - t$. So there exists $j \leqslant t$ such that (1) $a_{jj} \in T_E$ and (2) the jth column of D contains at least two elements that are not spanned in T. Let $x \in E$ be such that its support (i.e., its minimal spanning set) in T contains a_{jj} and let y and z be two elements in the jth column of D that are not spanned by T. We may assume that x and y are not in the same row (otherwise we take z instead of y). Since $T \cup \{y\}$ is independent, and the support of x in T contains a_{jj}, it follows that $T \setminus \{a_{jj}\} \cup \{y\}$ does not span x and thus $S \setminus \{a_{jj}\} \cup \{x, y\}$ is an independent partial transversal in A of length $t + 1$, contrary to the maximality of T. Thus t must be at least $\lceil 2n/3 \rceil$. □

3 An upper bound of size $n-1$ for an MLS of degree n

It is well known that for any even n there exist Latin squares of order n with no transversal of size n. The following theorem shows that for any n there exists an MLS of degree n with no independent transversal of size n.

Theorem 6. *Let $v_1, v_2, \ldots, v_n$ be a basis of a vectorial matroid of rank n. Then the matrix $A = (a_{ij})_{i,j=1}^{n}$, whose elements are $a_{ii} = v_1$, for $i = 1, \ldots, n$, and $a_{ij} = v_i - v_j$, for $1 \leqslant i \neq j \leqslant n$, is an MLS of order n with no independent transversal of size n.*

Proof. We leave it to the reader to check that the rows and columns of A are independent. Let T be a transversal of size n in A. We show that T is not independent. If T does not contain elements of the main diagonal of A, then, since each row and column is represented exactly once among the elements of T, the sum of the elements of T is 0, and T is not independent. Thus we may assume that T meets the main diagonal exactly once. Let $a_{ii} = v_1 \in T$. If $i = 1$ then the sum of the elements of $T - a_{11}$ is 0. If $i > 1$, then v_i is not spanned by T, so T is not a basis, and thus, is not independent. □

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