Arithmetic Properties of Overpartition Pairs into Odd Parts

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Abstract

In this work, we investigate various arithmetic properties of the function $\overline{pp}_o(n)$, the number of overpartition pairs of n into odd parts. We obtain a number of Ramanujan type congruences modulo small powers of 2 for $\overline{pp}_o(n)$. For a fixed positive integer k, we further show that $\overline{pp}_o(n)$ is divisible by 2^k for almost all n. We also find several infinite families of congruences for $\overline{pp}_o(n)$ modulo 3 and two formulae for $\overline{pp}_o(6n+3)$ and $\overline{pp}_o(12n)$ modulo 3.

Keywords: congruence, modular forms

1 Introduction and statement of results

An overpartition of the positive integer n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. For convenience, we assume that there is only one overpartition of zero denoted by $\emptyset$. Properties of $\overline{p}(n)$ have been the subject of many recent studies [5, 6, 8, 9, 11, 13, 14].

Recently, Hirschhorn and Sellers [10] studied the arithmetic properties of overpartitions using only odd parts. More recently, arithmetic properties of overpartition pairs have been considered by Bringmann and Lovejoy [3], Chen and the author [4], and Kim [12]. In this paper, we are concerned with the arithmetic properties of the number of overpartition pairs of n into odd parts. An overpartition pair into odd parts is a pair of overpartitions (λ, μ) such that the parts of both overpartitions λ and μ are restricted to be odd integers. For example, there are 8 overpartition pairs of 2 into odd parts:

$$(1+1, \emptyset),\ (\overline{1}+1, \emptyset),\ (\overline{1}, 1),\ (\overline{1}, \overline{1}),\ (1, 1),\ (1, \overline{1}),\ (\emptyset, 1+1),\ (\emptyset, \overline{1}+1).$$

Let $\overline{pp}_o(n)$ denote the number of overpartition pairs of n into odd parts. Then the generating function for $\overline{pp}_o(n)$ is

$$\overline{PP}_o(q) = \sum_{n=0}^{\infty} \overline{pp}_o(n) q^n = \frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}. \tag{1.1}$$

Throughout this paper, we assume that q is a complex number with $|q| < 1$ and we adopt the following customary q-series notation:

$$(a;q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

In Section 2, we investigate arithmetic behavior of $\overline{pp}_o(n)$ modulo powers of 2. In particular, we show the following two results.

Theorem 1.1. *For any $n \geqslant 1$,*

$$\overline{pp}_o(n) \equiv \begin{cases} 4 \pmod 8, & \text{if } n \text{ is an odd square number;} \\ 0 \pmod 8, & \text{otherwise.} \end{cases}$$

Theorem 1.2. *Assume the prime factorization of n is given by*

$$n = 2^\alpha \prod p_i^{u_i} \prod q_j^{v_j}$$

where $p_i \equiv 1 \pmod 4$ and $q_j \equiv 3 \pmod 4$. Then $\overline{pp}_o(n) \equiv 0 \pmod{16}$ if and only if one of the following holds:

- $\alpha \geqslant 2$,
- $\alpha = 1$ *and at least one number among u_i's and v_j's is odd,*
- $\alpha = 0$ *and at least one v_j is odd,*
- $\alpha = 0$ *and at least one u_i is congruent to 3 modulo 4,*
- $\alpha = 0$ *and at least two u_i are congruent to 1 modulo 4;*

At the end of Section 2, we prove the following theorem.

Theorem 1.3. *Let k be a positive integer. Then $\overline{pp}_o(n)$ is almost always divisible by 2^k, namely,*

$$\lim_{X \to \infty} \frac{\sharp\{n \leqslant X : \overline{pp}_o(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

In Section 3, we aim to show divisibilities satisfied by $\overline{pp}_o(n)$ with modulus 3.

Theorem 1.4. *For $\alpha \geqslant 0$ and all $n \geqslant 0$,*

$$\overline{pp}_o(9^\alpha(9n+6)) \equiv 0 \pmod 3, \tag{1.2}$$

$$\overline{pp}_o(9^\alpha(27n+18)) \equiv 0 \pmod 3. \tag{1.3}$$

Theorem 1.5. *For all $n \geqslant 0$,*

$$\overline{pp}_o(6n+3) \equiv (-1)^n \sigma(2n+1) \pmod 3, \tag{1.4}$$

$$\overline{pp}_o(12n) \equiv (-1)^{n+1}(\sigma(n) - \sigma(n/4)) \pmod 3. \tag{1.5}$$

Here $\sigma(n)$ denotes the sum of positive divisors of n and $\sigma(x) = 0$ if $x \notin \mathbb{N}$.

2 Congruences for $\overline{pp}_o(n)$ modulo powers of 2

In this section, we want to establish congruences for $\overline{pp}_o(n)$ modulo small powers of 2. We will require a number of properties of Ramanujan's functions $\varphi(q)$ and $\psi(q)$, namely,

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2-n}.$$

The necessary properties of $\varphi(q)$ and $\psi(q)$ are given in the following lemmas.

Lemma 2.1.

$$\varphi(q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{2.1}$$

$$\psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}. \tag{2.2}$$

Proof. These two identities follow from Jacobi's triple product identity [1, p.35]. ∎

Lemma 2.2.

$$\varphi(-q^2)^2 = \varphi(q)\varphi(-q), \tag{2.3}$$

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2. \tag{2.4}$$

$$\psi(q)^2 = \varphi(q)\psi(q^2). \tag{2.5}$$

Proof. The first identity follows from (2.1). The last two identities can be proved by using series manipulations, see [1, pp. 40-41] for a proof. ∎

To prove the congruences in this paper, we will frequently use the following congruence relations without explicitly mentioning it.

Lemma 2.3. *For positive prime p, we have*

$$(q; q)_\infty^p \equiv (q^p; q^p)_\infty \pmod p,$$

$$\varphi(-q)^p \equiv \varphi(-q^p) \pmod p.$$

Proof. The first congruence identity follows from the following fact

$$(1-q)^p \equiv 1 - q^p \pmod{p}.$$

The second congruence identity follows from the first congruence identity and the product representation for $\varphi(-q)$. ∎

By the product formula (2.1) for $\varphi(q)$, we have

$$\overline{PP}_o(q) = \sum_{n=0}^{\infty} \overline{pp}_o(n) q^n = \frac{\varphi(q)}{\varphi(-q)}. \tag{2.6}$$

We shall begin by proving the following Ramanujan type identities, which are essential to congruences modulo small powers of two in this section.

Theorem 2.1.

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n) q^n = \frac{\varphi(q)^2}{\varphi(-q)^2}, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n+1) q^n = 4\frac{\psi(q^2)^2}{\varphi(-q)^2}. \tag{2.8}$$

Proof. By (2.3) and (2.6), we have

$$\overline{PP}_o(q) = \frac{\varphi(q)^2}{\varphi(q)\varphi(-q)} = \frac{\varphi(q)^2}{\varphi(-q^2)^2}.$$

Applying (2.4), we find that

$$\overline{PP}_o(q) = \frac{\varphi(q^2)^2}{\varphi(-q^2)^2} + 4q\frac{\psi(q^4)^2}{\varphi(-q^2)^2},$$

which is equivalent to identities (2.7) and (2.8). ∎

Next, we wish to derive the generating function for $\overline{pp}_o(4n+2)$ from (2.7).

Theorem 2.2.

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2) q^n = 8\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}}. \tag{2.9}$$

Proof. Applying (2.3) in (2.7), we obtain that

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n) q^n = \frac{\varphi(q)^4}{\varphi(-q^2)^4}.$$

Choosing the terms for which the power of q is odd, we see that

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2) q^{2n+1} = \frac{\varphi(q)^4 - \varphi(-q)^4}{2\varphi(-q^2)^4}.$$

By (2.4), we have

$$\varphi(q)^4 - \varphi(-q)^4 = (\varphi(q)^2 + \varphi(-q)^2)(\varphi(q)^2 - \varphi(-q)^2) = 16q\varphi(q^2)^2\psi(q^4)^2.$$

Combining these two identities together, we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^{2n+1} = 8q\frac{\varphi(q^2)^2\psi(q^4)^2}{\varphi(-q^2)^4}.$$

Dividing both sides by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^{n} = 8\frac{\varphi(q)^2\psi(q^2)^2}{\varphi(-q)^4},$$

which implies the desired result. ∎

As an immediate consequence of the above theorem, we obtain the following congruences.

Corollary 2.1. *For all $n \geqslant 0$,*

$$\overline{pp}_o(12n+6) \equiv 0 \pmod{24}, \tag{2.10}$$

$$\overline{pp}_o(12n+10) \equiv 0 \pmod{24}. \tag{2.11}$$

We now want to prove the following theorem with the aid of Theorem 2.1.

Theorem 2.3. *Let $d(n)$ denote the number of positive divisors of n. Then for all $n \geqslant 1$,*

$$\overline{pp}_o(2n) \equiv 8(d(n) - d(n/4)) \pmod{16}, \tag{2.12}$$

$$\overline{pp}_o(2n-1) \equiv 4(-1)^{n-1}\sigma(2n-1) \pmod{32}. \tag{2.13}$$

Here $d(x) = 0$ if $x \notin \mathbb{N}$.

Proof. Using (2.5) in (2.8), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n+1)q^{n} = 4\frac{\psi(-q)^4}{\varphi(-q)^4}.$$

Now it is known that (see, e.g., Berndt [2, Chapter 3])

$$\varphi(q)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n}, \tag{2.14}$$

$$\psi(q)^4 = \sum_{n=0}^{\infty} \sigma(2n+1)q^n. \tag{2.15}$$

This gives

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n+1)q^n \equiv 4\sum_{n=0}^{\infty} \sigma(2n+1)(-q)^n \pmod{32}.$$

Equating the coefficients q^n, we obtain (2.13). By (2.3), we see that

$$\sum_{n=0}^{\infty} \overline{pp}(2n)q^n = \frac{\varphi(-q)^4\varphi(-q^2)^4}{\varphi(-q)^8}.$$

Applying (2.14) and the following congruence relation

$$\varphi(-q)^8 \equiv 1 \pmod{16},$$

we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pp}(2n)q^n &\equiv 1 + 8\sum_{n=1}^{\infty} \frac{n(-q)^n}{1-(-q)^n} + 8\sum_{n=1}^{\infty} \frac{n(-q^2)^n}{1-(-q^2)^n} \pmod{16} \\
&\equiv 1 + 8\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} + 8\sum_{n=0}^{\infty} \frac{q^{4n+2}}{1+q^{4n+2}} \pmod{16} \\
&\equiv 1 + 8\sum_{4\nmid n} \frac{q^n}{1-q^n} \pmod{16} \\
&\equiv 1 + 8\sum_{n=1}^{\infty}(d(n) - d(n/4))q^n \pmod{16}.
\end{aligned}$$

This implies (2.12) and thus the proof is complete. ∎

From Theorem 2.3, we have the following Ramanujan type congruences.

Corollary 2.2. *For all $n \geqslant 0$,*

$$\overline{pp}_o(4n+3) \equiv 0 \pmod{16}, \tag{2.16}$$

$$\overline{pp}_o(8n+7) \equiv 0 \pmod{32}. \tag{2.17}$$

Proof. Using the elementary fact that $\sigma(4n+3)$ is divisible by 4 and $\sigma(8n+7)$ is divisible by 8, we immediately have the desired congruences. ∎

With Theorem 2.3 in hand, we are now in a position to give a proof of Theorem 1.1. It is worth mentioning that Kim's [12] combinatorial argument for the characterization of the number of overpartition pairs of n modulo 8 can also work for Theorem 1.1.

Proof of Theorem 1.1. From (2.12) and (2.13), it follows that for any $n \geqslant 1$,

$$\overline{pp}(2n) \equiv 0 \pmod{8}$$

and

$$\overline{pp}(2n-1) \equiv 4\sigma(2n-1) \pmod{8}.$$

Since $\sigma(2n-1)$ is odd if and only if $2n-1$ is a perfect square, we conclude that $\overline{pp}(2n-1)$ is divisible by 8 if $2n-1$ is not an odd perfect square and $\overline{pp}(2n-1)$ is congruent to 4 modulo 8 if $2n-1$ is an odd perfect square. This completes the proof. ∎

By Theorem 2.3 and elementary properties of functions $d(n)$ and $\sigma(n)$, it is not hard to get Theorem 1.2, and we omit the details here. Theorem 1.2 produces infinite many congruences modulo 16. We record two corollaries as follows.

Corollary 2.3. *Let p be an odd prime and let r be an integer with $1 \leqslant r < p$. Then, for all $n \geqslant 0$,*

$$\overline{pp}_o(2p(pn+r)) \equiv 0 \pmod{16}.$$

Corollary 2.4. *Let p be a prime such that $p \equiv 1 \pmod 4$ and let r be an integer with $1 \leqslant r < p$. Then, for all $n \geqslant 0$,*

$$\overline{pp}_o(p^3(pn+r)) \equiv 0 \pmod{16}.$$

To conclude this section, we shall show Theorem 1.3 by modular forms. It is worth mentioning that for any fixed positive integer k, Gordon and Ono [7] have proven that the number of partitions of n into distinct parts is divisible by 2^k for almost all n, and Bringmann and Lovejoy [3] showed that the number of overpatition pairs of n is also divisible by 2^k for almost all n.

Proof of Theorem 1.3. Recall that the Dedekind eta function $\eta(z)$ is defind by

$$\eta(z) = q^{1/14} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i z}$ and z is in the upper plane of complex plane. Now we can rewrite $\overline{PP}_o(q^{24})$ as the following eta quotient

$$F(z) = \frac{\eta(48z)^6}{\eta(24z)^4 \eta(96z)^2}.$$

Let

$$f_k(z) = \frac{\eta(24z)^{2^k}}{\eta(48z)^{2^{k-1}}}.$$

Observing that $f_1(z) \equiv 1 \pmod 2$, it is not hard to establish the following fact by induction

$$f_k(z) \equiv 1 \pmod{2^k}.$$

Define $G_k(z)$ by

$$G_k(z) = F(z) f_k(z) = \frac{\eta(24z)^{2^k-4}}{\eta(48z)^{2^{k-1}-6} \eta(96z)^2}.$$

Thus,

$$G_k(z) \equiv F(z) \pmod{2^k}.$$

Without loss of generality, we may assume that $k \geqslant 3$. By [15, Thm 1.64 and Thm 1.65], it is not hard to check that $G_k(z)$ is a holomorphic modular form of weight 2^{k-2} on the congruence subgroup $\Gamma_0(1152)$. For the background on modular forms, see Ono [15]. From the deep theorem of Serre [15, p. 43], it follows that the Fourier coefficients of $G_k(z)$ is almost always divisible by 2^k and so are the Fourier coefficients of $F(z)$. Now

$$F(z) = \sum_{n=0}^{\infty} \overline{pp}_o(n) q^{24n},$$

we see that almost all n have the property that $\overline{pp}_o(n)$ is a multiple of 2^k. ∎

3 Congruences for $\overline{pp}_o(n)$ modulo 3

In this section, we aim to show Theorems 1.4 and 1.5. Before proving Theorem 1.4, we first give the following theorem.

Theorem 3.1. *For all* $n \geqslant 0$,

$$\overline{pp}_o(3n) \equiv \overline{pp}_o(27n) \pmod 3, \tag{3.1}$$

$$\overline{pp}_o(9n+6) \equiv 0 \pmod 3, \tag{3.2}$$

$$\overline{pp}_o(27n+18) \equiv 0 \pmod 3. \tag{3.3}$$

The following 3-dissections of $\varphi(-q)$ and $\psi(q)$ are useful for the proof of Theorem 3.1.

Lemma 3.1.

$$\psi(q) = A(q^3) + q\psi(q^9), \tag{3.4}$$

$$\varphi(-q) = \varphi(-q^9) - 2qB(q^3), \tag{3.5}$$

where

$$A(q) = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty}, \quad B(q) = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}.$$

Proof. Proofs of these two identities can be found in [1, p. 49]. ∎

Proof of Theorem 3.1. In this proof, all congruences hold to the modulus 3. By Lemma 2.3, we see that

$$\overline{PP}_o(q) \equiv \frac{\varphi(-q)\varphi(-q^2)^2}{\varphi(-q^3)}.$$

Applying 3-dissection (3.5) of $\varphi(-q)$, we have

$$\sum_{n=0}^{\infty} \overline{pp}_o(n) q^n = \frac{1}{\varphi(-q^3)} (\varphi(-q^9) - 2qB(q^3))(\varphi(-q^{18}) - 2q^2 B(q^6))^2.$$

Choosing the terms for which the power of q is a multiple of 3, replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n \equiv \frac{1}{\varphi(-q)}(\varphi(-q^3)\varphi(-q^6)^2 - 4qB(q)B(q^2)\varphi(-q^6)).$$

By the the following identity due to Hirschhorn and Sellers [10, Lemma 3.4]

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} - 4q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty} = \frac{(q;q)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty} \equiv (q^2;q^2)_\infty(q^6;q^6)_\infty,$$

we see that

$$\begin{aligned}
\varphi(-q^3)\varphi(-q^6) - 4qB(q)B(q^2) &= \frac{(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty} - 4q\frac{(q;q)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^3;q^3)_\infty(q^4;q^4)_\infty}\\
&= \frac{(q;q)_\infty(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}\left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} - 4q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}\right)\\
&\equiv \frac{(q;q)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n &\equiv \frac{\varphi(-q^6)}{\varphi(-q)} \cdot \frac{(q;q)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}\\
&\equiv \frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2}\psi(q) \qquad (3.6)\\
&= \frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2}(A(q^3) + q\psi(q^9)), \qquad (3.7)
\end{aligned}$$

where the last equality is obtained by the 3-dissection (3.4) of $\psi(q)$. From (3.7), we immediately deduce that for $n \geqslant 0$,

$$\overline{pp}_o(9n+6) \equiv 0$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{pp}_o(9n)q^n &\equiv \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty(q^4;q^4)_\infty^2}A(q)\\
&\equiv \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty}(q;q^2)_\infty(q^4;q^4)_\infty \qquad (3.8)\\
&= \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}. \qquad (3.9)
\end{aligned}$$

Here the last equality is obtained by Jacobi's triple product identity. Because there are no integers n such that $2n^2+n$ congruent to 2 modulo 3, we conclude that

$$\overline{pp}_o(27n+18) \equiv 0.$$

Since $2n^2+n$ is divisible by 3 unless n is congruent to 2 modulo 3, we have

$$\begin{aligned}\sum_{\substack{n=-\infty\\ 3|2n^2+n}}^{\infty} q^{2n^2+n} &= \sum_{n=-\infty}^{\infty} q^{2(3n)^2+3n} + \sum_{n=-\infty}^{\infty} q^{2(3n+1)^2+3n+1}\\ &= \sum_{n=-\infty}^{\infty} (q^3)^{n(3n+1)/2}.\end{aligned}$$

From (3.9), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(27n)q^{3n} &\equiv \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty} \sum_{\substack{n=-\infty\\ 3|2n^2+n}}^{\infty} (-1)^n q^{2n^2+n}\\ &= \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^{12};q^{12})_\infty} \sum_{n=-\infty}^{\infty} (-q^3)^{n(3n+1)/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(27n)q^{n} &\equiv \frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} (-q)^{n(3n+1)/2}\\ &= \frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^4;q^4)_\infty} \cdot (q;-q^3)_\infty(-q^2;-q^3)_\infty(-q^3;-q^3)_\infty\\ &= \frac{(q^6;q^6)_\infty^5}{(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^2} \cdot \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.\end{aligned}$$

It is easy to check that

$$\frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2} \cdot \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \equiv \frac{(q^6;q^6)_\infty^5}{(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^2} \cdot \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} \pmod 3,$$

from which we have

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n \equiv \sum_{n=0}^{\infty} \overline{pp}_o(27n)q^n \pmod 3.$$

The desired result now follows by equating coefficients of q^n, $n \geqslant 0$, on both sides above. ∎

With Theorem 3.1 in hand, Theorem 1.4 follows immediately by induction on α. Finally, we turn to show Theorem 1.5.

Proof of Theorem 1.5. From (3.6), we see that

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n \equiv \varphi(q)^2\varphi(-q^2)^2 \pmod 3. \tag{3.10}$$

By the 2-dissection (2.3) of $\varphi(q)^2$, it is not hard to get that

$$\varphi(q)^2\varphi(-q^2)^2 = \varphi(-q^4)^4 + 4q\psi(-q^2)^4.$$

It immediately follows that

$$\sum_{n=0}^{\infty} \overline{pp}_0(6n+3)q^n \equiv \psi(-q)^4 \pmod 3, \tag{3.11}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(6n)q^n \equiv \varphi(-q^2)^4 \pmod 3. \tag{3.12}$$

By (2.15) and the following identity (see, e.g., Berndt [2, p. 59])

$$\varphi(q)^4 = 1 + 8\sum_{n=1}^{\infty}(\sigma(n) - 4\sigma(n/4))q^n,$$

we obtain the desired results. ∎

As a consequence of Theorem 1.5, we have the following corollaries.

Corollary 3.1. *Let p be prime with $p \equiv 2 \pmod 3$ and let r be an integer with $1 \leqslant r < p$. Then, for all $s \geqslant 0, n \geqslant 0$,*

$$\overline{pp}_o(3p^{2s+1}(pn+r)) \equiv 0 \pmod 3.$$

Proof. Note that $\sigma(p^{2s+1}) = \frac{p^{2s+2}-1}{p-1}$ is divisible by 3 and $\sigma(p^{2s+1}(pn+r)) = \sigma(p^{2s+1})\sigma(pn+r)$, so the result follows. ∎

Corollary 3.2. *Let p be prime with $p \equiv 1 \pmod 3$ and let r be an integer with $1 \leqslant r < p$. Then, for all $s \geqslant 0, n \geqslant 0$,*

$$\overline{pp}_o(3p^{3s+2}(pn+r)) \equiv 0 \pmod 3.$$

Proof. Since $\sigma(p^{3s+2})$ is divisible by 3, it is not hard to obtain the result by using the fact that $\sigma(p^{3s+2}(pn+r)) = \sigma(p^{3s+2})\sigma(pn+r)$. ∎

Corollary 3.3. *For all $n \geqslant 0$,*

$$(-1)^n\overline{pp}_o(24n+12) \equiv \overline{pp}_o(6n+3) \pmod 3.$$

Proof. From (1.5), we see that

$$\overline{pp}_o(24n+12) \equiv \sigma(2n+1) - \sigma\left(\frac{2n+1}{4}\right) \equiv \sigma(2n+1) \pmod 3,$$

and so

$$(-1)^n \overline{pp}_o(24n+12) \equiv (-1)^n \sigma(2n+1) \equiv \overline{pp}_o(6n+3) \pmod 3.$$

This completes the proof. ∎

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