Strings of length 3 in Grand-Dyck paths and the Chung-Feller property

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Abstract

This paper deals with the enumeration of Grand-Dyck paths according to the statistic "number of occurrences of τ" for every string τ of length 3, taking into account the number of flaws of the path. Consequently, some new refinements of the Chung-Feller theorem are obtained.

1 Introduction

Throughout this paper, a path is considered to be a lattice path on the integer plane, consisting of steps $u = (1, 1)$ (called *rises*) and $d = (1, -1)$ (called *falls*). Since the sequence of steps of a path is encoded by a word in $\{u, d\}^*$, we will make no distinction between these two notions. The *length* of a path is the number of its steps.

A *Grand-Dyck path* is a path that starts and ends at the same height. It is convenient to consider that the starting point of a Grand-Dyck path is the origin of a pair of axes. Obviously, every Grand-Dyck path ends at a point $(0, 2n)$ where n is referred to as the *semilength* of the path.

The set of Grand-Dyck paths of semilength n is denoted by $\mathcal{G}_n$, and we set $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$, where $\mathcal{G}_0 = \{\varepsilon\}$ and ε is the empty path.

Every rise of a path α which lies below (resp. above) the x-axis is called a *flaw* (resp. a *non-flaw*) of α. The number of flaws of α is denoted by $p(\alpha)$.

The set of Grand-Dyck paths of semilength n having m flaws is denoted $\mathcal{G}_{n,m}$. In particular, we set $\mathcal{D}_n = \mathcal{G}_{n,0}$ (resp. $\bar{\mathcal{D}}_n = \mathcal{G}_{n,n}$) the set of *Dyck paths* (resp. *inverted Dyck paths*). It is well known that the set $\mathcal{D}_n$ is counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see A000108 in [22]). Furthermore, the classical Chung-Feller theorem [5, 12] states that $|\mathcal{G}_{n,m}| = C_n$, for all $m \in [0, n]$.

Every $\alpha \in \mathcal{G} \setminus \bar{\mathcal{D}}$ can be uniquely decomposed in the form $a = \beta u \gamma d \delta$ where $\beta \in \bar{\mathcal{D}}$, $\gamma \in \mathcal{D}$ and $\delta \in \mathcal{G}$ (see Figure 1 i); this decomposition will be referred to as the *first non-flaw decomposition* and will be used in the sequel extensively. Similarly, the decomposition $\alpha = \delta d \beta u \gamma$ of $\mathcal{G} \setminus \mathcal{D}$ will be referred to as the *last flaw decomposition* (see Figure 1 ii).

A path $\tau \in \{u, d\}^*$, called in this context *string*, *occurs* in a path α if $\alpha = \beta\tau\gamma$, for some $\beta, \gamma \in \{u, d\}^*$. The number of occurrences of the string τ in α, is denoted by $|\alpha|_\tau$. Given a string τ, the symmetric string of τ with respect to a horizontal (resp. vertical) axis is denoted by $\bar{\tau}$ (resp. τ').

A wide range of articles dealing with the occurrence of strings in Dyck paths appear frequently in the literature [1–4, 6–10, 13–21, 23]. For the occurrences of strings in Grand-Dyck paths it is interesting to take also into account the number of flaws of the paths. In this direction Ma and Yeh [11] have studied the statistic "number of occurrences of τ" in Grand-Dyck paths for all strings τ of length 2. In this work, the same subject is studied for all strings τ of length 3 obtaining some new Chung-Feller type results.

For this we use the generating function

$$F_\tau(x, y, z) = \sum_{\alpha \in \mathcal{G}} x^{|\alpha|_\tau} y^{p(\alpha)} z^{|\alpha|_u}.$$

Clearly, using the two classical bijections of $\mathcal{G}$ according to which every $a \in \mathcal{G}$ is mapped to $\bar{\alpha}$ and α', we obtain that

$$[x^k y^m z^n] F_\tau = [x^k y^{n-m} z^n] F_{\bar{\tau}} = [x^k y^m z^n] F_{\tau'}, \text{ where } m \leqslant n.$$

Thus, among the eight strings of length 3 it is enough to restrict ourselves to the following three: uuu, udu, duu.

In [11], it has been proved, both analytically and bijectively, that the number of Grand-Dyck paths with semilength n, having m flaws and k occurrences of the string $\tau = u^2$, is independent of m, thus satisfying the Chung-Feller property. In fact, the following generalization holds.

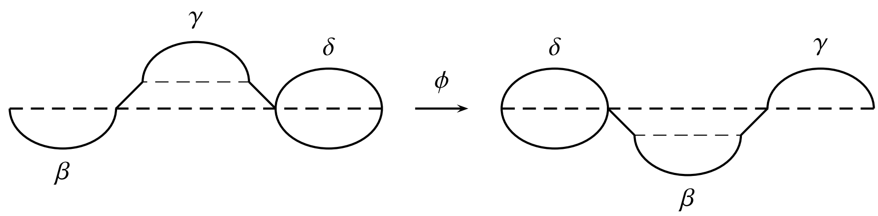
Proposition 1.1. *The number of Grand-Dyck paths with semilength n, having m flaws and k_i occurrences of the string u^i, for every $i \geqslant 2$, is independent of m.*

Although the previous proposition can be proved analytically, it is easier to be justified by using the simple mapping $\phi : \mathcal{G} \setminus \bar{\mathcal{D}} \to \mathcal{G} \setminus \mathcal{D}$ with $\phi(\beta u \gamma d \delta) = \delta d \beta u \gamma$, where $\beta \in \bar{D}$, $\gamma \in \mathcal{D}$, and $\delta \in \mathcal{G}$, which is a length-preserving bijection that increases the number of flaws by one and preserves the number of occurrences of the strings u^i; (see Figure 1).

In particular, the number of Grand-Dyck paths with semilength n, having m flaws and k occurrences of the string u^3 is counted by the sequence A092107 in [22] and it is given by the complex formula [19]

$$\frac{1}{n+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{n+j}{n} \binom{n+1}{k-j} \sum_{i=j}^{[(n+j)/2]} \binom{n+j+1-k}{i+1} \binom{n-i}{i-j}.$$

In the following two sections we will study the strings udu and duu.



i The first non-flaw decomposition of $\mathcal{G} \setminus \bar{\mathcal{D}}$

ii The last flaw decomposition of $\mathcal{G} \setminus \mathcal{D}$

Figure 1: The bijection ϕ

2 The string $\tau = udu$

Let $F(x,y,z) = \sum_{a \in \mathcal{G}} x^{|a|_{udu}} y^{p(a)} z^{|a|_u} = \sum_{m \geqslant 0} f_m(x,z) y^m$, where $f_m(x,z)$ is the generating function of the set of Grand-Dyck paths with m flaws, with respect to the number of occurrences of $\tau = udu$ and to the semilength.

The Dyck path statistic "number of occurrences of udu" has been studied independently by Sun [23] and Merlini, Sprungoli and Verri [17] where it is stated that the corresponding generating function f_0 satisfies the equation

$$z f_0^2(x,z) = (1-(x-1)z)(f_0(x,z)-1) \tag{2.1}$$

and has coefficients

$$[x^k z^n] f_0 = \begin{cases} 1, & k = n = 0 \\ \binom{n-1}{k} M_{n-k-1}, & k \in [0, n-1], \end{cases} \tag{2.2}$$

where M_n is the n-th Motzkin number (see A001006 in [22]).

Theorem 2.1.

$$f_m(x,z) = (1-(x-1)z) \sum_{n \geqslant m} \sum_{k=0}^{n-1} \binom{n-1}{k} M_{n-k-1} x^k z^n, \qquad m \geqslant 1.$$

Proof. We will first show that

$$F(x,y,z) = \frac{(1-(x-1)z) f_0(x,yz)}{1-(x-1)z - z f_0(x,yz) f_0(x,z)}. \tag{2.3}$$

For every $\alpha \in \mathcal{G} \setminus \bar{D}$ with $a = \beta u \gamma d \delta$, $\beta \in \bar{\mathcal{D}}$, $\gamma \in \mathcal{D}$, $\delta \in \mathcal{G}$ we have that

$$|\alpha|_\tau = |\beta|_\tau + |\gamma|_\tau + |\delta|_\tau + [\gamma = \varepsilon][\delta \in \mathcal{A}]$$

where $\mathcal{A} = \{a \in \mathcal{G} : a \text{ starts with } u\}$, and $[P]$ is the Iverson notation: $[P] = 1$ if P is true and $[P] = 0$ if P is false.

From the previous equality, using the first non-flaw decomposition of $\mathcal{G} \setminus \bar{D}$, it follows that

$$\begin{aligned}
&F(x,y,z)\\
&= \sum_{\beta\in\bar{\mathcal{D}}} x^{|\beta|_\tau} y^{p(\beta)} z^{|\beta|_u} + z \sum_{\beta\in\bar{\mathcal{D}}} x^{|\beta|_\tau} y^{p(\beta)} z^{|\beta|_u} \sum_{\substack{\gamma\in\mathcal{D}\\ \delta\in\mathcal{G}}} x^{|\gamma|_\tau+|\delta|_\tau+[\gamma=\varepsilon][\delta\in\mathcal{A}]} y^{p(\delta)} z^{|\gamma|_u+|\delta|_u}\\
&= \sum_{\beta\in\mathcal{D}} x^{|\beta|_{\bar{\tau}}} (yz)^{|\beta|_u} + z \sum_{\beta\in\mathcal{D}} x^{|\beta|_{\bar{\tau}}} (yz)^{|\beta|_u} \left(\sum_{\substack{\gamma\in\mathcal{D}\setminus\{\varepsilon\}\\ \delta\in\mathcal{G}}} x^{|\gamma|_\tau+|\delta|_\tau} y^{p(\delta)} z^{|\gamma|_u+|\delta|_u} + \sum_{\delta\in\mathcal{G}} x^{|\delta|_\tau+[\delta\in A]} y^{p(\delta)} z^{|\delta|_u} \right)\\
&= f_0(x,yz)\Big(1 + z\,(f_0(x,z)-1)\,F(x,y,z) + z\Big(F(x,y,z) + (x-1)A(x,y,z)\Big)\Big),
\end{aligned}$$

where $A(x,y,z)$ is the generating function of the set $\mathcal{A}$.

It follows that

$$F(x,z,y) = f_0(x,yz)\big(1 + zf_0(x,z)F(x,y,z) + z(x-1)A(x,y,z)\big). \tag{2.4}$$

Every $a \in \mathcal{A}$ can be written uniquely in the form $a = u\gamma d\delta$, where $\gamma \in \mathcal{D}$, $\delta \in \mathcal{G}$, so that $|a|_\tau = |\gamma|_\tau + |\delta|_\tau + [\gamma = \varepsilon][\delta \in \mathcal{A}]$, giving

$$A(x,y,z) = z((f_0(x,z)-1)F(x,y,z) + F(x,y,z) + (x-1)A(x,y,z)).$$

Hence,

$$A(x,y,z) = \frac{zf_0(x,z)F(x,y,z)}{1-(x-1)z}.$$

By substituting the previous expression of $A(x,y,z)$ in relation (2.4), we obtain the required relation (2.3).

From relation (2.3), using relation (2.1), we have that

$$\begin{aligned}
&F(x,y,z) - f_0(x,z)\\
&= \frac{(1-(x-1)z)(f_0(x,yz) - f_0(x,z) + f_0(x,yz)(f_0(x,z)-1))}{1-(x-1)z - zf_0(x,yz)f_0(x,z)}\\
&= \frac{y(1-(x-1)z)f_0(x,z)(f_0(x,yz)-1)(f_0(x,z)-f_0(x,yz))}{y(1-(x-1)z)(f_0(x,z)-f_0(x,yz)) - y(1-(x-1)z)(f_0(x,z)-1)f_0(x,yz) + (1-(x-1)yz)(f_0(x,yz)-1)f_0(x,z)}\\
&= \frac{y(1-(x-1)z)f_0(x,z)(f_0(x,yz)-1)(f_0(x,z)-f_0(x,yz))}{(1-y)f_0(x,z)(f_0(x,yz)-1)}.
\end{aligned}$$

Thus, we have that

$$F(x,y,z) = f_0(x,z) + (1-(x-1)z)(f_0(x,z) - f_0(x,yz))\frac{y}{1-y}. \tag{2.5}$$

From relation (2.2), we have that

$$[y^m](f_0(x,z) - f_0(x,yz)) = \begin{cases} f_0(x,z) - 1, & m = 0\\ -\sum\limits_{k=0}^{m-1} \binom{m-1}{k} M_{m-k-1} x^k z^m, & m \geqslant 1. \end{cases}$$

Hence, from relation (2.5), we obtain that, for $m \geqslant 1$,

$$\begin{aligned}
f_m(x,z) &= (1-(x-1)z)[y^{m-1}]\frac{f_0(x,z)-f_0(x,yz)}{1-y} \\
&= (1-(x-1)z)(f_0(x,z)-1-\sum_{n=1}^{m-1}\sum_{k=0}^{n-1}\binom{n-1}{k}M_{n-k-1}x^k z^n) \\
&= (1-(x-1)z)\sum_{n\geqslant m}\sum_{k=0}^{n-1}\binom{n-1}{k}M_{n-k-1}x^k z^n.
\end{aligned}$$

□

Corollary 2.2. *The number of Grand-Dyck paths with semilength n, having m flaws and k occurrences of the string udu is equal to*

$$[x^k z^n]f_m = \begin{cases} \binom{n-1}{k}M_{n-k-1}, & m = 0, n \\ \binom{n-2}{k}(M_{n-k-1}+M_{n-k-2}), & m \in [1, n-1]. \end{cases}$$

Proof. By Theorem 2.1, for $m \in [1, n-1]$, we have that

$$\begin{aligned}
[x^k z^n]f_m &= \binom{n-1}{k}M_{n-k-1} - \binom{n-2}{k-1}M_{n-k-1} + \binom{n-2}{k}M_{n-k-2} \\
&= \binom{n-2}{k}M_{n-k-1} + \binom{n-2}{k}M_{n-k-2} \\
&= \binom{n-2}{k}\left(M_{n-k-1}+M_{n-k-2}\right).
\end{aligned}$$

The cases where $m = 0, n$ are obvious and are omitted. □

Corollary 2.3. *The number of Grand-Dyck paths with semilength n, having k occurrences of the string udu is equal to*

$$\binom{n-1}{k}\left((n-k+1)M_{n-k-1}+(n-k-1)M_{n-k-2}\right).$$

For further information concerning the double sequence described in the previous corollary, see A097692 in [22].

Remark. Corollary 2.2 is a Chung-Feller type theorem, since it shows that the number of Grand-Dyck paths with semilength n having m flaws and k occurrences of the string udu is independent of m, for $m \in [1, n-1]$.

We end this section by giving a combinatorial proof of this result. For this purpose it is enough to construct a bijection from $\mathcal{G}_{n,m}$ to $\mathcal{G}_{n,m+1}$ which preserves the number of occurrences of the string udu, for every $m \in [1, n-2]$.

We observe that every $a \in \mathcal{G} \setminus \bar{\mathcal{D}}$ (resp. $a \in \mathcal{G} \setminus \mathcal{D}$) can be written uniquely in the form $\alpha = \beta_1 u \gamma_1 d \beta_2 \delta \gamma_2$ (resp. $\alpha = \beta_1 \delta \gamma_1 d \beta_2 u \gamma_2$) where $\beta_1, \beta_2 \in \bar{\mathcal{D}}$, $\gamma_1, \gamma_2 \in \mathcal{D}$, $\delta \in \mathcal{G}$ such that $\delta = \varepsilon$ or δ starts and ends with u.

The mapping $\psi : \mathcal{G} \setminus \bar{\mathcal{D}} \to \mathcal{G} \setminus \mathcal{D}$ with

$$\psi(\beta_1 u \gamma_1 d \beta_2 \delta \gamma_2) = \beta_1 \delta \gamma_1 d \beta_2 u \gamma_2$$

(see Figure 2) is a length-preserving bijection that increases the number of flaws by one.

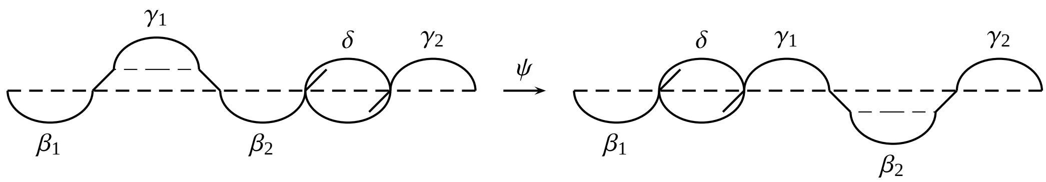


Figure 2: The bijection ψ

Furthermore, $|\psi(\alpha)|_{udu} = |\alpha|_{udu}$ for every $\alpha \in \mathcal{G} \setminus \bar{\mathcal{D}}$ with $1 \leqslant p(\alpha) \leqslant |a|_u - 2$.

Indeed, every occurrence of τ in α (resp. in $\psi(\alpha)$) that does not lie entirely in one of $\beta_1, \beta_2, \gamma_1, \gamma_2$ exists if and only if $\gamma_1 = \beta_2 = \varepsilon$ and $\delta\gamma_2 \neq \varepsilon$ (resp. $\gamma_1 = \beta_2 = \varepsilon$ and $\beta_1 \delta \neq \varepsilon$).

Then, if $\delta \neq \varepsilon$, the required equality is obviously true. On the other hand, if $\delta = \varepsilon$, since $|\alpha|_u - p(\alpha) \geqslant 2$ (resp. $p(\alpha) \geqslant 1$), it follows that, if $\gamma_1 = \varepsilon$ (resp. $\beta_2 = \varepsilon$), then $\gamma_2 \neq \varepsilon$ (resp. $\beta_1 \neq \varepsilon$). Hence, the required equality holds in this case, too.

Thus, the restriction of ψ on the set $\mathcal{G}_{n,m}$ gives the required bijection.

3 The string $\tau = duu$

Let $F(x, y, z) = \sum_{a \in \mathcal{G}} x^{|a|_{duu}} y^{p(a)} z^{|a|_u} = \sum_{m \geqslant 0} f_m(x, z) y^m$, where $f_m(x, z)$ is the generating function of the set of Grand-Dyck paths with m flaws, with respect to the number of occurrences of $\tau = duu$ and to the semilength.

Let $g_0(x, z)$ be the generating function of $\mathcal{D}$ with respect to the number of occurrences of $\bar{\tau} = udd$ and to the semilength. The Dyck path statistics "number of occurrences of duu" and "number of occurences of udd" have been studied by Deutsch [9] and Sapounakis, Tasoulas, Tsikouras [19] respectively, where it is stated that the corresponding generating functions f_0, g_0 satisfy the equations

$$xzf_0^2(x, z) - (1 + 2(x - 1)z)f_0(x, z) + 1 + (x - 1)z = 0 \tag{3.1}$$

and

$$z(1 + (x - 1)z)g_0^2(x, z) - g_0(x, z) + 1 = 0 \tag{3.2}$$

with coefficients

$$a_{n,k} = [x^k z^n] f_0 = \begin{cases} 1, & k = n = 0 \\ 2^{n-2k-1} C_k \binom{n-1}{2k}, & n \geqslant 1 \end{cases} \tag{3.3}$$

and

$$b_{n,k} = [x^k z^n]g_0 = \begin{cases} 1, & k = 0 \\ \frac{1}{n+1}\binom{n+1}{k}\sum\limits_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n+1-k}{n-j}, & k \geqslant 1. \end{cases} \tag{3.4}$$

From relations (3.1), (3.2) it follows that

$$f_0(x,z) = \frac{1 + 2(x-1)z - \sqrt{\Delta}}{2xz} \text{ and } g_0(x,z) = \frac{1 - \sqrt{\Delta}}{2z(1 + (x-1)z)},$$

where $\Delta = 1 - 4z - 4z^2(x-1)$.

From the previous two equalities it follows that

$$f_0(x,z) = (1 - (x-1)z(f_0(x,z) - 1))g_0(x,z) = \frac{1}{1 - zg_0(x,z)}. \tag{3.5}$$

Theorem 3.1.

$$f_m(x,z) = (1 - (x-1)z(f_0(x,z) - 1)) \sum_{n \geqslant m} \sum_{k \geqslant 0} b_{n,k} x^k z^n, \qquad m \geqslant 0.$$

Proof. We will first show that

$$F(x,y,z) = \frac{g_0(x,yz)}{1 - z(1 + (x-1)yz)g_0(x,z)g_0(x,yz)}. \tag{3.6}$$

For every $\alpha \in \mathcal{G} \setminus \bar{D}$ with $a = \beta u \gamma d \delta$, $\beta \in \bar{\mathcal{D}}$, $\gamma \in \mathcal{D}$, $\delta \in \mathcal{G}$ we have that

$$|\alpha|_\tau = |\beta|_\tau + |\gamma|_\tau + |\delta|_\tau + [\beta \text{ ends with } du] + [\delta \in \mathcal{A}]$$

where $\mathcal{A} = \{a \in \mathcal{G} : a \text{ starts with } u^2\}$.

From the previous equality, using the first non-flaw decomposition of $\mathcal{G} \setminus \bar{D}$, it follows that

$$\begin{aligned} F(x,y,z) = &\sum_{\beta \in \bar{\mathcal{D}}} x^{|\beta|_\tau} y^{p(\beta)} z^{|\beta|_u} \\ &+ z \sum_{\beta \in \bar{\mathcal{D}}} x^{|\beta|_\tau + [\beta \text{ ends with } du]} y^{p(\beta)} z^{|\beta|_u} \sum_{\gamma \in \mathcal{D}} x^{|\gamma|_\tau} z^{|\gamma|_u} \sum_{\delta \in \mathcal{G}} x^{|\delta|_\tau + [\delta \in \mathcal{A}]} y^{p(\delta)} z^{|\delta|_u} \\ = &\, g_0(x,yz) + z\Bigg(g_0(x,yz) + (x-1) \sum_{\substack{\beta \in \bar{\mathcal{D}} \\ \beta \text{ ends with } du}} x^{|\beta|_\tau} (yz)^{|\beta|_u}\Bigg) f_0(x,z)\big(F(x,y,z) \\ &+ (x-1)A(x,y,z)\big) \end{aligned}$$

where $A(x,y,z)$ is the generating function of the set $\mathcal{A}$.

It follows that

$$F(x,y,z) = g_0(x,yz)\left(1 + z(1 + (x-1)yz)f_0(x,z)(F(x,y,z) + (x-1)A(x,y,z))\right). \tag{3.7}$$

Every $a \in \mathcal{A}$ can be written uniquely in the form $a = u\gamma d\delta$, where $\gamma \in \mathcal{D} \setminus \{\varepsilon\}$, $\delta \in \mathcal{G}$, so that $|a|_\tau = |\gamma|_\tau + |\delta|_\tau + [\delta \in \mathcal{A}]$, giving

$$A(x, y, z) = z(f_0(x, z) - 1)(F(x, y, z) + (x - 1)A(x, y, z)).$$

Hence,

$$A(x, y, z) = \frac{z(f_0(x, z) - 1)F(x, y, z)}{1 - (x - 1)z(f_0(x, z) - 1)}.$$

By substituting the previous expression of $A(x, y, z)$ in relation (3.7), and using relation (3.5), we obtain the required relation (3.6).

From relation (3.6), and using relations (3.2) and (3.5), we have that

$$\begin{aligned} F(x, y, z) &= \frac{g_0(x, yz)\,(g_0(x, z) - yg_0(x, yz))}{g_0(x, z) - yg_0(x, yz) - z(1 + (x-1)yz)g_0^2(x, z)g_0(x, yz) + g_0(x, z)(g_0(x, yz) - 1)} \\ &= \frac{g_0(x, z) - yg_0(x, yz)}{g_0(x, z) - zg_0^2(x, z) - y(1 + (x - 1)z^2 g_0^2(x, z))} \\ &= (1 - (x - 1)z(f_0(x, z) - 1))\,\frac{g_0(x, z) - yg_0(x, yz)}{1 - y}. \end{aligned}$$

Since

$$[y^m](g_0(x, z) - yg_0(x, yz)) = \begin{cases} g_0(x, z), & m = 0 \\ -\sum\limits_{k \geqslant 0} b_{m-1,k}x^k z^{m-1}, & m \geqslant 1 \end{cases}$$

it follows that

$$\begin{aligned} f_m(x, z) &= (1 - (x - 1)z(f_0(x, z) - 1))\,[y^m]\left(\frac{g_0(x, z) - yg_0(x, yz)}{1 - y}\right) \\ &= (1 - (x - 1)z(f_0(x, z) - 1))\left(g_0(x, z) - \sum_{n=1}^{m}\sum_{k \geqslant 0} b_{n-1,k}x^k z^{n-1}\right) \\ &= (1 - (x - 1)z(f_0(x, z) - 1))\sum_{n \geqslant m}\sum_{k \geqslant 0} b_{n,k}x^k z^n. \end{aligned}$$

□

Corollary 3.2. *The number of Grand-Dyck paths with semilength n, having m flaws and k occurrences of the string duu is equal to*

$$[x^k z^n]f_m = b_{n,k} + \sum_{i=1}^{n-m-1}\sum_{j=0}^{k} a_{i,j}\left(b_{n-i-1,k-j} - b_{n-i-1,k-j-1}\right).$$

Proof. We first observe that

$$(f_0(x, z) - 1)\sum_{n \geqslant m}\sum_{k \geqslant 0} b_{n,k}x^k z^n = \sum_{n \geqslant m+1}\sum_{k \geqslant 0}\sum_{j=1}^{n-m}\sum_{j=0}^{k} a_{i,j}b_{n-i,k-j}x^k z^n$$

and hence, by Theorem 3.1 it follows that

$$[x^k z^n]f_m = b_{n,k} + \sum_{i=1}^{n-m-1} \sum_{j=0}^{k} a_{i,j} \left(b_{n-i-1,k-j} - b_{n-i-1,k-j-1}\right).$$

$\square$

Remark From the previous corollary, it follows that the number of Grand-Dyck paths with semilength n having m flaws and k occurrences of the string $\tau = udd$ is equal to

$$b_{n,k} + \sum_{i=1}^{m-1} \sum_{j=0}^{k} a_{i,j} \left(b_{n-i-1,k-j} - b_{n-i-1,k-j-1}\right).$$

Corollary 3.3. *The number of Grand-Dyck paths with semilength n, having k occurrences of the string duu is equal to*

$$(n+1)b_{n,k} + \sum_{i=1}^{n-1}(n-i) \sum_{j=0}^{k} a_{i,j} \left(b_{n-i-1,k-j} - b_{n-i-1,k-j-1}\right).$$

For further information concerning the double sequence described in the previous corollary, see A051288 in [22].

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