The Coin Exchange Problem and the Structure of Cube Tilings

Andrzej P. Kisielewicz and Krzysztof Przesławski

Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski
ul. Z. Szafrana 4a, 65-516 Zielona Góra, Poland

A.Kisielewicz@wmie.uz.zgora.pl, K.Przeslawski@wmie.uz.zgora.pl

Submitted: Jul 25, 2011; Accepted: Apr 13, 2012; Published: Jun 6, 2012

Abstract

It is shown that if $[0,1)^d+t$, $t\in T$, is a unit cube tiling of $\mathbb{R}^d$, then for every $x\in T$, $y\in\mathbb{R}^d$, and every positive integer m the number $|T\cap(x+\mathbb{Z}^d)\cap([0,m)^d+y)|$ is divisible by m. Furthermore, by a result of Coppersmith and Steinberger on cyclotomic arrays it is proven that for every finite discrete box $D=D_1\times\dots\times D_d\subseteq x+\mathbb{Z}^d$ of size $m_1\times\dots\times m_d$ the number $|D\cap T|$ is a linear combination of $m_1,\dots,m_d$ with non-negative integer coefficients. Several consequences are collected. A generalization is presented.

An interest in cube tilings of $\mathbb{R}^d$ originated from the following question raised by Hermann Minkowski [23]: Characterize lattices $\Lambda\subset\mathbb{R}^d$ such that $[0,1)^d+\lambda$, $\lambda\in\Lambda$, is a cube tiling. Minkowski conjectured that such a lattice Λ is of the form $A\mathbb{Z}^d$, where A is a lower triangular matrix with ones on the main diagonal. By a simple inductive argument, it is equivalent to showing that Λ contains an element of the standard basis. Geometrically, it means that the tiling $[0,1)^d+\lambda$, $\lambda\in\Lambda$, contains a column. This inspired Ott-Heinrich Keller [11] to consider the problem of the existence of columns in arbitrary cube tilings. In [12] he conjectured that starting from dimension 7 there are cube tilings without columns. This has been confirmed in dimension 10 by Jeffrey Lagarias and Peter Shor [18] and in dimension 8 by John Mackey [22]. There are quite a few other papers stemming from Keller's problem [2, 3, 5, 6, 8, 13, 14, 19, 21, 24, 28].

A new stimulus came from Fuglede's conjecture [7]. Several papers appeared at almost the same time where the set determining a cube tiling is characterized as follows: $[0,1)^d+t$, $t\in T$, is a cube tiling of $\mathbb{R}^d$ if and only if the system of functions $\exp(2\pi\,\mathrm{i}\langle t,x\rangle)$, $t\in T$, is an orthonormal basis of $L^2([0,1]^d)$ ([9, 10, 15, 17]).

A reader who seeks an exposition concerning cube tilings is advised to consult [16, 26, 29].

Let us suppose that $[0,1)^d+t$, $t\in T$, is a cube tiling and that $x\in T$. The simplest unit cube tiling to which the cube $[0,1)^d+x$ belongs is $[0,1)^d+u$, $u\in x+\mathbb{Z}^d$. We discuss here in what way these two tilings are intertwined. To be more specific, we show that there are certain number-theoretic characteristics of the intersection $(x+\mathbb{Z}^d)\cap T$.

A *standard block* in $\mathbb{R}^d$ is a set of the form $X = X_1\times\cdots\times X_d$, where $X_i \in \{[0,1),\mathbb{R}\}$. The translate $X+x$ of a standard block $X\subseteq\mathbb{R}^d$ on $x\in\mathbb{R}^d$ is called a *block* in $\mathbb{R}^d$. A block which is a translate of $[0,1)^d$ is said to be a *unit cube*. If $F = F_1\times\cdots\times F_d$ is a block in $\mathbb{R}^d$, then the set $N_F = \{i\colon F_i = \mathbb{R}\}$ is called the *cosupport* of F. For future reference, we define the set $\mathbb{Z}^d_{N_F}$ by the equation

$$\mathbb{Z}^d_{N_F} = \{(k_1,\ldots,k_d)\in\mathbb{Z}^d\colon k_i = 0, \text{whenever } i\notin N_F\}.$$

If F is a block, then its position vector $v = v(F)\in\mathbb{R}^d$ is defined as follows

$$v_i = \begin{cases} x_i, & \text{if } F_i = [0,1)+x_i,\\ 0, & \text{if } F_i = \mathbb{R}.\end{cases}$$

Clearly, $F - v$ is a standard block.

A family $\mathscr{F}$ of disjoint blocks contained in $\mathbb{R}^d$ is a *block tiling* of $\mathbb{R}^d$ if $\bigcup\mathscr{F} = \mathbb{R}^d$. If $\mathscr{F}$ consists of unit cubes only, then we refer to $\mathscr{F}$ as a *cube tiling*.

Suppose that there is a unit cube J which belongs to a block tiling $\mathscr{F}$. Then the family of cubes $\{(J+k)\colon k\in\mathbb{Z}^d\}\cap\mathscr{F}$ is called a *simple component* of $\mathscr{F}$ (containing J).

Theorem 1 *Suppose that a unit cube J belongs to a block tiling $\mathscr{F}$. Let $\mathscr{C}$ be a simple component of $\mathscr{F}$ containing J. Then there is a block tiling $\mathscr{G}$ such that*

(1) $v(G)_i \in v(J)_i + \mathbb{Z}$, for $G\in\mathscr{G}$ and $i\notin N_G$;

(2) $J\in\mathscr{G}$ and the simple component of $\mathscr{G}$ containing J equals $\mathscr{C}$.

In particular, $\mathscr{C}$ consists of all unit cubes belonging to $\mathscr{G}$.

Proof. We may assume that $J = [0,1)^d$. Let us fix $i\in\{1,\ldots,d\}$ and $\alpha\in(0,1)$. Let $\mathscr{F}^\alpha = \{F\in\mathscr{F}\colon v(F)_i\in\alpha+\mathbb{Z}\}$. If $\mathscr{F}^\alpha$ is non-empty, then since $\mathscr{F}$ is a tiling, the family $\mathscr{S} := \{F\in\mathscr{F}^\alpha\colon v(F)_i = \alpha\}$ is non-empty. For $F\in\mathscr{S}$, let $\bar{F}$ be a block defined so that $\bar{F}_i = \mathbb{R}$ and $\bar{F}_j = F_j$ whenever $j\neq i$. Let $\mathscr{G}_\alpha = \{\bar{F}\colon F\in\mathscr{S}\}$. Again by the fact that $\mathscr{F}$ is a tiling, we deduce that $\bigcup\mathscr{F}^\alpha = \bigcup\mathscr{G}_\alpha$. If $\mathscr{F}^\alpha$ is empty, then we let $\mathscr{G}_\alpha$ to be empty. Observe now that the family $\mathscr{F}^{(i)} := (\mathscr{F}\setminus\bigcup_{\alpha\in(0,1)}\mathscr{F}^\alpha)\cup\bigcup_{\alpha\in(0,1)}\mathscr{G}_\alpha$ is a block tiling which has the following properties:

- ⋄ the simple components of $\mathscr{F}$ and $\mathscr{F}^{(i)}$ containing J coincide;
- ⋄ if $F\in\mathscr{F}^{(i)}$, then $v(F)_i\in\mathbb{Z}$.

Let us define inductively a sequence of block tilings $(\mathscr{F}^{[i]}\colon i = 1,\ldots,d)$:

$$\mathscr{F}^{[1]} = \mathscr{F}^{(1)},\ \ \mathscr{F}^{[2]} = (\mathscr{F}^{[1]})^{(2)},\ \ldots,\ \mathscr{F}^{[d]} = (\mathscr{F}^{[d-1]})^{(d)}.$$

It is clear that the simple components of $\mathscr{F}$ and $\mathscr{F}^{[d]}$ containing J coincide and $v(F) \in \mathbb{Z}^d$ whenever $F \in \mathscr{F}^{[d]}$. Therefore, it suffices to declare $\mathscr{G} = \mathscr{F}^{[d]}$. □

We say that a set $T \subset \mathbb{R}^d$ *determines* a cube tiling $\mathscr{F}$ of $\mathbb{R}^d$ if

$$T = v(\mathscr{F}) = \{v(F) \colon F \in \mathscr{F}\}.$$

Let T determine a cube tiling $\mathscr{F}$ and $t \in T$. Then the set $(t + \mathbb{Z}^d) \cap T$ consists of all position vectors of a certain simple component of $\mathscr{F}$. As an immediate consequence we have

THEOREM 2 *Given a set T determining a cube tiling of $\mathbb{R}^d$. Let $x \in T$ and let $D \subset x + \mathbb{Z}^d$ be a finite discrete box with sidelengths equal to m; that is, $D = D_1 \times \cdots \times D_d$ and $m = |D_1| = \cdots = |D_d|$. Then $|T \cap D|$ is divisible by m.*

Proof. Let $\mathscr{F}$ be the cube tiling determined by T. Let $\mathscr{C}$ be the simple component of $\mathscr{F}$ which contains $J = [0,1)^d + x$. Let $\mathscr{G}$ be the block tiling of $\mathbb{R}^d$ as described in Theorem 1. Since for each $G \in \mathscr{G}$ we have $G = [0,1)^d + v(G) + \mathbb{Z}^d_{N_G}$, it follows that the set

$$\mathscr{P} = \{(v(G) + \mathbb{Z}^d_{N_G}) \cap D \colon G \in \mathscr{G}\} \setminus \{\emptyset\}$$

is a partition of D. Observe that N_G is non-empty if $G \in \mathscr{G} \setminus \mathscr{C}$. Thus, $H_G := (v(G) + \mathbb{Z}^d_{N_G}) \cap D$ is empty or it is a discrete box such that $H_i = D_i$, for $i \in N_G$. Now, it follows that $|H_G|$ is divisible by m. Therefore, the number

$$|T \cap D| = m^d - \sum_{G \in \mathscr{G} \setminus \mathscr{C}} |H_G|.$$

is divisible by m. □

Let us underline that the sets $D_1 - x_1, \ldots, D_d - x_d$ are not necessarily intervals of consecutive integers.

Theorem 2 can be generalized to finite discrete boxes of arbitrary size. To this end we shall need a deep result of Coppersmith and Steinberger [4, Theorem 1] concerning the sum of the entries of a cyclotomic array.

Let $\mathbb{N}$ be the set of all positive integers. If $m \in \mathbb{N}$, then $[m]$ denotes, as usual, the initial segment $\{1, \ldots, m\}$. If $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, then $[\boldsymbol{m}] := [m_1] \times \cdots \times [m_d]$. A non-negative integer n is *representable* by $\boldsymbol{m}$ if there are non-negative integers $n_1, \ldots, n_d$ such that

$$n = n_1 m_1 + \cdots + n_d m_d.$$

In other words, the amount n can be changed using coins of denominations $m_1, \ldots, m_d$. As a consequence of this interpretation, the problem of representability is often called *the coin exchange problem* (see e.g. [1, 25]). A *line* in $[\boldsymbol{m}]$ is any set of the form

$$\{x_1\} \times \cdots \times \{x_{s-1}\} \times [m_s] \times \{x_{s+1}\} \times \cdots \times \{x_d\},$$

where $s \in [d]$, and $x_i \in [m_i]$. A subset Q of $[\boldsymbol{m}]$ is said to be *complementable by lines* if its complement $[\boldsymbol{m}] \setminus Q$ can be represented as a union of disjoint lines. A characteristic function $f\colon [\boldsymbol{m}] \to \{0,1\}$ is called a *fiber* if its support is a line. Following Steinberger [27], a mapping $A\colon [\boldsymbol{m}] \to \mathbb{Z}$ is said to be a *cyclotomic array* if it is a linear combination of fibers with integer coefficients. The result of Coppersmith and Steinberger reads that the sum $\sum_{\boldsymbol{x}\in[\boldsymbol{m}]} A(\boldsymbol{x})$ is representable by $\boldsymbol{m}$ if A is a nonnegative cyclotomic array. As an immediate consequence we have

PROPOSITION 3 *For each $\boldsymbol{m} \in \mathbb{N}^d$, if $Q \subseteq [\boldsymbol{m}]$ is complementable by lines, then $|Q|$ is representable by $\boldsymbol{m}$.*

THEOREM 4 *Given a set T determining a cube tiling of $\mathbb{R}^d$. Let $x \in T$ and let $D \subset x + \mathbb{Z}^d$ be a finite discrete box of size $m_1 \times \cdots \times m_d$. Then $|T \cap D|$ is representable by $\boldsymbol{m} = (m_1, \ldots, m_d)$.*

Proof. Since the proof is a slight modification of the proof of Theorem 2, the notation used there is preserved. Let us pick a system of bijections $\varphi_i\colon D_i \to [m_i]$, $i \in [d]$, and define a bijection $\varphi\colon D \to [\boldsymbol{m}]$ by the formula $\varphi = \varphi_1 \times \cdots \times \varphi_d$. Let $Q = \varphi(T \cap D)$. The set $[\boldsymbol{m}] \setminus Q$ is a disjoint union of the sets $\varphi(H_G)$, $G \in \mathscr{G} \setminus \mathscr{C}$. By the definition of φ and the fact that each H_G is a box with side D_i, for some i, it follows that each $\varphi(H_G)$ is a disjoint union of lines. Now, as Q satisfies the assumption of Proposition 3, $|Q|$ is representable by $\boldsymbol{m}$. □

COROLLARY 5 *Given a set T determining a cube tiling of $\mathbb{R}^d$. Let $x \in T$ and let B be a half-open box of size $m_1 \times \cdots \times m_d$, that is, there is $y \in \mathbb{R}^d$ such that $B = [y_1, y_1 + m_1) \times \cdots \times [y_d, y_d + m_d)$. Then $|T \cap (x + \mathbb{Z}^d) \cap B|$ is representable by $\boldsymbol{m} = (m_1, \ldots, m_d)$.*

Proof. Define $D := (x + \mathbb{Z}^d) \cap B$. As D is a discrete box of size $m_1 \times \cdots \times m_d$, Theorem 4 applies to reach the conclusion. □

COROLLARY 6 *Given a set T determining a cube tiling of $\mathbb{R}^d$. Let $x \in T$ and let B be a closed box of size $m_1 \times \cdots \times m_d$, that is, there is $y \in \mathbb{R}^d$ such that $B = [y_1, y_1 + m_1] \times \cdots \times [y_d, y_d + m_d]$. Then the cardinality of the set $Q := \{t \in T \cap (x + \mathbb{Z}^d)\colon (t + [0,1)^d) \cap B \neq \emptyset\}$ is representable by $(m_1 + 1, \ldots, m_d + 1)$.*

Proof. Define

$$D := (x + \mathbb{Z}^d) \cap ((y_1 - 1, y_1 + m_1] \times \cdots \times (y_d - 1, y_d + m_d])$$

D is a discrete box of size $(m_1 + 1) \times \cdots \times (m_d + 1)$. Moreover, $Q = T \cap D$. Again, Theorem 4 leads to the conclusion. □

Let $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and let T determine a cube tiling of $\mathbb{R}^d$. This tiling is said to be ***m**-periodic* if for every vector of the standard basis $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ one has

$$T + m_i e_i = T.$$

We define the *(flat) torus* $\mathbb{T}^d_{\boldsymbol{m}}$ to be the set $[0, m_1) \times \cdots \times [0, m_d)$ with addition $\bmod \boldsymbol{m}$:

$$x \oplus y := ((x_1 + y_1) \bmod m_1, \ldots, (x_d + y_d) \bmod m_d).$$

We can extend the notion of a cube so that it will apply to flat tori: Cubes in $\mathbb{T}^d_{\boldsymbol{m}}$ are the sets of the form $[0,1)^d \oplus t$, where $t \in \mathbb{T}^d_{\boldsymbol{m}}$. It is clear that we can speak about cube tilings of $\mathbb{T}^d_{\boldsymbol{m}}$ and that there is a canonical 'one-to-one' correspondence between these tilings and the $\boldsymbol{m}$-periodic tilings of $\mathbb{R}^d$. We say that $T \subset \mathbb{T}^d_{\boldsymbol{m}}$ determines a cube tiling of $\mathbb{T}^d_{\boldsymbol{m}}$ if $[0,1)^d \oplus t$, $t \in T$ is a tiling. Let $\mathbb{Z}^d_{\boldsymbol{m}} = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_d}$. By the analogy to cube tilings of $\mathbb{R}^d$, every set $T \cap (x \oplus \mathbb{Z}^d_{\boldsymbol{m}})$, where $x \in T$, determines a simple component of the tiling $[0,1)^d \oplus t$, $t \in T$. As a consequence of Corollary 5 we have

Theorem 7 *If T determines a cube tiling of a torus $\mathbb{T}^d_{\boldsymbol{m}}$ and S determines a simple component of this tiling, then $|S|$ is $\boldsymbol{m}$ representable.*

Our main result (Theorem 4) holds in a more general setting. We need some additional terminology in order to formulate such a generalization (compare [21]).

Let V be a non-empty set. A family $\mathscr{V} \subseteq 2^V \setminus \{\emptyset, V\}$ is *distinctive* if for every $A \in \mathscr{V}$, there is a unique partition $\mathscr{C}_A \subseteq \mathscr{V}$ of V such that $A \in \mathscr{C}_A$.

The family of all unit segments $[0,1) + x$, $x \in \mathbb{R}$, is distinctive while the family of unit squares $[0,1)^2 + x$, $x \in \mathbb{R}^2$, is not. On the other hand, the family of all translates of a regular hexagon in $\mathbb{R}^2$ with three consecutive sides removed is distinctive.

Let X be the Cartesian product of sets X_i, $i \in [d]$. A non-empty subset A of X is called a *box* (in X) if $A = A_1 \times \cdots \times A_d$ and $A_i \subseteq X_i$ for each $i \in [d]$.

Let $\mathscr{X}_i \subseteq 2^{X_i} \setminus \{\emptyset, X_i\}$, $i \in [d]$. We denote by $\mathscr{X} = \mathscr{X}_1 \otimes \cdots \otimes \mathscr{X}_d$ the family of all boxes $A \subseteq X = X_1 \times \cdots \times X_d$ such that $A_i \in \mathscr{X}_i$ for every $i \in [d]$. If each $\mathscr{X}_i$ is distinctive, then $\mathscr{X}$ is called a *free family of boxes* on X. Let us fix $A \in \mathscr{X}$. Then $\mathscr{C}_A = \mathscr{C}_{A_1} \otimes \cdots \otimes \mathscr{C}_{A_d} \subseteq \mathscr{X}$ is a partition of X. We refer to $\mathscr{C}_A$ as a *simple partition* of X (determined by A).

The announced generalization can be proved along the same lines as Theorem 4.

Theorem 8 *Let $\mathscr{X}$ be a free family of boxes on $X = X_1 \times \cdots \times X_d$. Let $\mathscr{K} \subseteq \mathscr{X}$ be a partition of X, and $A \in \mathscr{K}$. Let $\mathscr{D}$ be a finite discrete box of size $\boldsymbol{m} \in \mathbb{N}^d$ contained in the simple partition $\mathscr{C}_A$, that is, $\mathscr{D} = \mathscr{D}_1 \otimes \cdots \otimes \mathscr{D}_d$, $\mathscr{D}_i \subseteq \mathscr{C}_{A_i}$ and $|\mathscr{D}_i| = m_i$, for $i \in [d]$. Then $|\mathscr{D} \cap \mathscr{K}|$ is representable by $\boldsymbol{m}$.*

Acknowledgments. We thank Mihalis Kolountzakis who brought our attention to the connection between Proposition 3 that had been conjectured in a previous version of this note and a paper of Tsit Yuen Lam and Ka Hin Leung on vanishing sums of roots of unity [20] which eventually led us to the result of Coppersmith and Steinberger on cyclotomic arrays.

We also wish to express our thanks to an anonymous referee for valuable comments.

References

[1] M. Beck and S. Robins, *Computing the Continuous Discretely,* Springer, New York, 2007.

[2] K. Corrádi and S. Szabó, Cube tiling and covering a complete graph, *Discrete Math.* **85** (1990), 319–321.

[3] K. Corrádi and S. Szabó, A combinatorial approach for Keller's conjecture, *Period. Math. Hungar.* **21** (1990), 95–100.

[4] D. Coppersmith and J. Steinberger, On the Entry Sum of Cyclotomic Arrays, *INTEGERS* **6** (2006), # A26, pp 28.

[5] M. Dutour Sikirić and Y. Itoh, Combinatorial cube packings in the cube and the torus, *European J. Combin.* **31** (2010), 517–534.

[6] M. Dutour Sikirić, Y. Itoh and A. Poyarkov, Cube packings, second moment and holes, *European J. Combin.* **28** (2007), 715–725.

[7] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, *J. Funct. Anal.* **16** (1974), 101–121.

[8] G. Hajós, *Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter*, Math. Z. **47** (1941), 427–467.

[9] A. Iosevich and S. Pedersen, Spectral and tiling properties of the unit cube, *Inter. Math. Res. Notices* **16** (1998), 819–828.

[10] P. Jorgensen and S. Pedersen, Spectral pairs in Cartesian coordinates, *J. Fourier Anal. Appl.* **5** (1999), 285–302.

[11] O. H. Keller, *Über die lückenlose Erfülung des Raumes Würfeln*, J. Reine Angew. Math. **163** (1930), 231–248.

[12] O. H. Keller, *Ein Satz über die lückenlose Erfüllung des 5- und 6-dimensionalen Raumes mit Würfeln*, J. Reine Angew. Math. **177** (1937), 61–64.

[13] A. P. Kisielewicz, K. Przesławski, Polyboxes, cube tilings and rigidity, *Discrete Comput. Geom.* **40** (2008), 1–30.

[14] M. N. Kolountzakis, Lattice tilings by cubes: whole, notched and extended, *Electron. J. Combin.* **5** (1998), #R14, pp 11.

[15] M. N. Kolountzakis, Packing, tiling, orthogonality and completeness, *Bull. London Math. Soc.* **32** (2000), 589–599.

[16] M. N. Kolountzakis, The study of translational tiling with Fourier Analysis, *Fourier Analysis and Convexity*, 131–187, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004.

[17] J. C. Lagarias, J. A. Reeds and Y. Wang, Orthonormal bases of exponentials for the n-cube, *Duke. Math. J.* **103** (2000), 25–37.

[18] J. C. Lagarias and P. W. Shor, Keller's cube-tiling conjecture is false in high dimensions, *Bull. Amer. Math. Soc.* **27** (1992), 279–287.

[19] J. C. Lagarias and P. W. Shor, Cube tilings and nonlinear codes, *Discrete Comput. Geom.* **11** (1994), 359–391.

[20] T.Y. Lam, K.H. Leung, On vanishing sums of roots of unity, *J. Algebra* **224** no. 1 (2000), 91–109.

[21] M. Łysakowska and K. Przesławski, Keller's conjecture on the existence of columns in cube tilings of $\mathbb{R}^n$, *Advances in Geometry* **12** (2012), 329–352.

[22] J. Mackey, A cube tiling of dimension eight with no facesharing, *Discrete Comput. Geom.* **28** (2002), 275–279.

[23] H. Minkowski, *Diophantische Approximationen*, Teubner, Leipzig, 1907.

[24] O. Perron, Über lückenlose Ausfüllung des n-dimensionalen Raumes durch kongruente Würfel, *Math. Z.* **46** (1940), 1–26.

[25] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, New York, 2005.

[26] S. K. Stein and S. Szabó, *Algebra and Tiling: Homomorphisms in the Service of Geometry*, American Mathematical Association, Washington, 1994.

[27] J. Steinberger, Minimal vanishing sums of roots of unity with large coefficients, *Proc. London Math. Soc.* **97** (2008), 689–717.

[28] S. Szabó, Cube tilings as contributions of algebra to geometry, *Beiträge Algebra Geom.* **34** (1993), 63–75.

[29] Ch. Zong, *The Cube: A Window to Convex and Discrete Geometry*, Cambridge Tracts in Mathematics (168), 2006.