The Existence of Near Generalized Balanced Tournament Designs

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Abstract

In this paper, we complete the existence of near generalized balanced tournament designs (NGBTDs) with block size 3 and 5. As an application, we obtain new classes of optimal constant composition codes.

Keywords: near generalized balanced tournament design; constant composition codes; optimal; existence

1 Introduction

Let X be a set of v *points*, and $\mathcal{A}$ a collection of subsets (called *blocks*) of X. A (v, k, λ) *balanced incomplete block design* (BIBD), or a (v, k, λ)-BIBD, is a pair $(X, \mathcal{A})$ such that any pair of distinct points of X occurs in precisely λ blocks. A $(km + 1, k, k - 1)$-BIBD $(X, \mathcal{A})$ is called a *near generalized balanced tournament design* (NGBTD), or an NGBTD(k, m) in short, if its blocks can be arranged into an $m \times (km + 1)$ array in such a way that

(1) the blocks in each column form a partial parallel class which partition $X \setminus \{x\}$ for some point $x \in X$;

(2) each point of X is contained in precisely k cells of each row.

By the definition, any NGBTD can be identified with its corresponding block array defined above.

NGBTDs are a generalization of odd balanced tournament designs (OBTDs). Particularly, an NGBTD$(2, m)$ is an OBTD(m) whose existence was completed in [5]. Lamken [4] almost finished the existence of NGBTD$(3, m)$ with four possible exceptions. Recently, Shan [8] presented a nearly complete solution for the existence of NGBTD(k, m)'s for

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$k = 4$ and 5 with four possible exceptions. We collect the existence of NGBTD(k, m)'s for $2 \leqslant k \leqslant 5$ as follows.

Theorem 1. ([4, 5, 8])
(1) *There exists an NGBTD*$(2, m)$ *for any positive integer* m*;*
(2) *There exists an NGBTD*$(3, m)$ *for any positive integer* m *and* $m \notin \{3, 38, 39, 118\}$*;*
(3) *There exists an NGBTD*$(4, m)$ *for any positive integer* m*;*
(4) *There exists an NGBTD*$(5, m)$ *for any positive integer* m *and* $m \notin \{15, 32, 40, 45\}$.

A *group divisible design* of block size k and index λ, or a (k, λ)-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a finite set of (*points*), $\mathcal{G}$ is a partition of X into subsets (called *groups*), and $\mathcal{B}$ is a set of subsets of size k (called *blocks*) of X, such that every pair of points from distinct groups occurs in exactly λ blocks, and any pair of points from the same group occur in no block. The type of the GDD is defined to be the multiset $T = \{|G| : G \in \mathcal{G}\}$, which is usually denoted by an "exponential" notation: a type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ means u_i occurrences of g_i for $1 \leqslant i \leqslant s$.

A set of blocks of a GDD $(X, \mathcal{G}, \mathcal{B})$ is called a *partial α-parallel class* over $X \backslash S$ if each point of $X \backslash S$ occurs in exactly α blocks, while any point of S occurs in no block. If $S = \emptyset$, it is called an *α-parallel class* over X. Whenever $\alpha = 1$, we simply say a (partial) parallel class, instead of a (partial) α-parallel class. A GDD is called *resolvable* if its blocks can be partitioned into parallel classes.

We need a special type of GDDs, called *frame generalized doubly resolvable packings* (FGDRPs), which was first introduced in [11] to construct optimal constant composition codes.

Let $(X, \mathcal{G}, \mathcal{B})$ be a $(k, k-1)$-GDD of type g^u where $g = kh$. Suppose that the group set $\mathcal{G} = \{G_1, G_2, \cdots, G_u\}$. Define

$$R_i = \{(i-1)h + j : j = 1, \cdots, h\}$$
$$C_i = \{(i-1)kh + j : j = 1, \cdots, kh\}$$

for $1 \leqslant i \leqslant u$. The blocks of the GDD can be arranged into an $hu \times gu$ array F which satisfies the following properties:
(1) Each cell of F is either empty or contains a block of $\mathcal{B}$;
(2) Let F_t be the subarray indexed by the elements of R_t and C_t. Then F_t is empty for $t = 1, 2, \cdots, u$, i.e., the main diagonal of F consists of u empty subarrays of size $h \times kh$;
(3) For any $x \in R_i$ $(1 \leqslant i \leqslant u)$, the blocks in row x form a partial k-parallel class over $X \backslash G_i$;
(4) For any $y \in C_j$ $(1 \leqslant j \leqslant u)$, the blocks in column y form a partial parallel class over $X \backslash G_j$.

Then we refer to this GDD as an FGDRP(k, g^u). Actually, FGDRPs can be defined in a more general way in which all the groups are not necessarily the same size. Nevertheless, we use the definition here for our purpose. The interest reader may refer to [10, 11] for more details on FGDRPs.

In this note, we completely establish the existence of NGBTD(k, m)'s for $k = 3$ and 5 by removing all the remaining cases in Theorem 1. We also present an application of NGBTDs to optimal constant composition codes.

2 The Existence of NGBTDs with Block Size 3

In this section, we give a complete solution to the existence of NGBTDs with block size 3.

Theorem 2. [8, 9] *Suppose that there exists an FGDRP*(k, g^u) *where* $g = kh$*. Let* w *be any nonnegative integer. If there exists an NGBTD*$(k, h+w)$ *which contains an NGBTD*(k, w) *as a subarray, then an NGBTD*$(k, hu + w)$ *exists.*

Theorem 3. [9, 10] *Let* g *and* u *be positive integers with* $g \equiv 0 \pmod 3$ *and* $u \geqslant 5$*. Then an FGDRP*$(3, g^u)$ *exists except possibly for* $(g, u) \in \{(6, 15), (9, 18), (9, 28), (9, 34), (30, 15)\}$.

Theorem 4. *For any integer* $m \geqslant 1$ *and* $m \neq 3$*, there exists an NGBTD*$(3, m)$*. There does not exist an NGBTD*$(3, 3)$*.*

Proof : By Theorem 3, there exists an FGDRP$(3, 3^m)$ for any $m \geqslant 5$. An NGBTD$(3, 1)$ exists trivially with all the subsets containing any three points as blocks. Then we apply Theorem 2 with $k = 3, h = 1, u = m, w = 0$ to obtain an NGBTD$(3, m)$ for any $m \geqslant 5$.

We know that there does not exist an NGBTD$(3, 3)$ definitely by an exhaustive search with the aid of a computer.

Combining with Theorem 1, we complete the proof. □

Remark: Here we have fixed the four possible exceptions in [4] and complete the existence of NGBTD$(3, m)$'s. Moreover, Theorem 4 actually provides an alternative proof for the existence of NGBTD$(3, m)$'s.

3 The Existence of NGBTDs with Block Size 5

This section serves to complete the existence of NGBTDs with block size 5, by removing the four outstanding cases in Theorem 1.

Lemma 5. *There exists an NGBTD*$(5, 15)$*.*

Proof : We construct the desired NGBTD on the Abelian group $X = \mathbb{Z}_{19} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Here we write the element (a, b, c) of X as $\underline{a}bc$ for brevity. The following blocks form a partial parallel class which partitions $X \backslash \{\underline{18}11\}$.

$\{\underline{0}01, \underline{0}00, \underline{0}10, \underline{0}11, \underline{1}00\}$	$\{\underline{16}01, \underline{3}01, \underline{12}10, \underline{9}10, \underline{14}01\}$	$\{\underline{14}11, \underline{13}11, \underline{7}01, \underline{17}01, \underline{2}00\}$
$\{\underline{15}01, \underline{12}00, \underline{18}01, \underline{3}11, \underline{11}01\}$	$\{\underline{14}00, \underline{9}11, \underline{13}00, \underline{12}11, \underline{8}10\}$	$\{\underline{13}01, \underline{15}00, \underline{4}00, \underline{11}11, \underline{2}10\}$
$\{\underline{1}01, \underline{3}00, \underline{10}11, \underline{11}00, \underline{17}11\}$	$\{\underline{1}11, \underline{4}01, \underline{6}11, \underline{9}01, \underline{14}10\}$	$\{\underline{3}10, \underline{16}11, \underline{7}11, \underline{10}00, \underline{5}01\}$
$\{\underline{5}00, \underline{6}10, \underline{12}01, \underline{2}11, \underline{17}00\}$	$\{\underline{4}11, \underline{7}00, \underline{1}10, \underline{16}10, \underline{8}11\}$	$\{\underline{5}10, \underline{10}10, \underline{6}00, \underline{7}10, \underline{11}10\}$
$\{\underline{6}01, \underline{4}10, \underline{15}10, \underline{8}00, \underline{17}10\}$	$\{\underline{18}00, \underline{16}00, \underline{9}00, \underline{8}01, \underline{2}01\}$	$\{\underline{18}10, \underline{15}11, \underline{13}10, \underline{10}01, \underline{5}11\}$

It is easy to check that the desired design is produced by the action of the group X on above partial parallel class. □

Before we move to the next construction, we need the well-known starter-adder method, which is widely used to produce some designs with orthogonal properties (see, for example, [1, 2, 5, 7, 8, 13]).

Let G be an additively Abelian group of order u. Let $g = kt$. A *starter* S for an FGDRP(k, g^u) over $\mathbb{Z}_g \times G$ with groups $G_x = \mathbb{Z}_g \times \{x\}$ $(x \in G)$ consists of t sets of k-tuples (base blocks), $S_1, S_2, \cdots, S_t$, which satisfies the following properties:

(1) For any i $(1 \leqslant i \leqslant t)$, S_i contains exactly $u-1$ base blocks B_{ij}, $j = 1, 2, \cdots, u-1$;

(2) $S = \bigcup_{i=1}^{t} S_i$ forms a partition of $\mathbb{Z}_g \times (G\backslash\{0\})$ and every element of $\mathbb{Z}_g \times (G\backslash\{0\})$ occurs exactly $k-1$ times in the difference list of S.

A corresponding *adder* $A(S)$ for the starter S consists t permutations (not necessarily distinct) of $G\backslash\{0\}$

$$A(S_i) = (a_{i1}, a_{i2}, \cdots, a_{i(u-1)}), 1 \leqslant i \leqslant t$$

such that for any i $(1 \leqslant i \leqslant t)$, $\bigcup_{j=1}^{u-1}(B_{ij} + (0, a_{ij}))$ contains exactly k elements (not necessarily distinct) from each group G_x for $x \in G\backslash\{0\}$ and no element of G_0.

Theorem 6. [10] *If there exists a starter-adder pair* $(S, A(S))$ *for an FGDRP*(k, g^u) *over* $\mathbb{Z}_g \times G$ *defined above, then there exists an FGDRP*(k, g^u)*.*

Lemma 7. *There exists an FGDRP*$(5, 25^9)$*.*

Proof : We take $\mathbb{Z}_{25} \times GF(9)$ as the point set and $\{\mathbb{Z}_{25} \times \{x\} \mid x \in GF(9)\}$ as the groups. Let $GF(9)^* = GF(9)\backslash\{0\}$. Suppose that ω is a primitive element of $GF(9)$ satisfying $\omega^2 = \omega + 1$ and C_0^2 is the square residue of $GF(9)^*$. We display the starter S and the corresponding adder $A(S)$ in Table 1. Then we apply Theorem 6 to produce the desired design. □

Table 1: The starters and corresponding adders for an $FGDRP(25^9)$

	S	$A(S)$	
S_1 :	$\{(0,\omega^1), (8,\omega^5), (8,\omega^2), (13,\omega^3), (14,\omega^7)\}\cdot(1,h)$	$(0,1)\cdot(1,h)$	$h \in C_0^2$
	$\{(9,\omega^0), (10,\omega^7), (21,\omega^1), (23,\omega^2), (15,\omega^6)\}\cdot(1,h)$	$(0,\omega)\cdot(1,h)$	
S_2 :	$\{(2,\omega^6), (11,\omega^5), (12,\omega^2), (14,\omega^0), (15,\omega^1)\}\cdot(1,h)$	$(0,1)\cdot(1,h)$	
	$\{(5,\omega^4), (6,\omega^6), (17,\omega^3), (20,\omega^7), (24,\omega^1)\}\cdot(1,h)$	$(0,\omega)\cdot(1,h)$	
S_3 :	$\{(10,\omega^0), (18,\omega^1), (18,\omega^6), (19,\omega^3), (23,\omega^7)\}\cdot(1,h)$	$(0,1)\cdot(1,h)$	
	$\{(2,\omega^1), (4,\omega^3), (4,\omega^6), (11,\omega^2), (13,\omega^4)\}\cdot(1,h)$	$(0,\omega)\cdot(1,h)$	
S_4 :	$\{(3,\omega^6), (7,\omega^1), (19,\omega^0), (22,\omega^2), (22,\omega^7)\}\cdot(1,h)$	$(0,1)\cdot(1,h)$	
	$\{(1,\omega^2), (6,\omega^1), (9,\omega^7), (16,\omega^0), (24,\omega^6)\}\cdot(1,h)$	$(0,\omega)\cdot(1,h)$	
S_5 :	$\{(1,\omega^1), (12,\omega^5), (16,\omega^3), (17,\omega^6), (20,\omega^0)\}\cdot(1,h)$	$(0,1)\cdot(1,h)$	
	$\{(0,\omega^0), (3,\omega^1), (5,\omega^3), (7,\omega^2), (21,\omega^4)\}\cdot(1,h)$	$(0,\omega)\cdot(1,h)$	

The second type of starter-adder construction is called intransitive starter-adder method, which involves infinite points. The reader may refer to [2, 5, 7, 8, 10, 13] for more details.

Let $GF(u-1)$ be the Galois field with $u-1$ elements and $X = \mathbb{Z}_g \times (GF(u-1) \bigcup \{\infty\})$ where $g = kt$. Let $G = \mathbb{Z}_g \times GF(u-1)$. An *intransitive starter* S for an FGDRP(k, g^u) over X with groups $\{G_x = \mathbb{Z}_g \times \{x\} \mid x \in GF(u-1) \bigcup \{\infty\}\}$ is defined as a triple (S, R, C) satisfying the following properties.

(1) S consists of t sets of k-tuples (base blocks), $S_1, S_2, \cdots, S_t$. For any i $(1 \leqslant i \leqslant t)$, S_i contains precisely $u-2$ base blocks, B_{ij} $(1 \leqslant i \leqslant u-2)$, in which there exist exactly k base blocks containing one infinite point each from G_∞;

(2) R consists of t base blocks over G, denoted by $R_1, R_2 \cdots, R_t$, in which each contains no infinite points;

(3) C consists of t base blocks over G, denoted by $C_1, C_2 \cdots, C_t$, in which each contains no infinite points;

(4) $S \bigcup R$ forms a partition of $X \backslash G_0$;

(5) the difference list from the base blocks of $S \bigcup R \bigcup C$ contains every element of $G \backslash G_0$ precisely $k-1$ times, and no element in G_0.

A corresponding *adder* $A(S)$ for S consists of t permutations on $GF(u-1) \backslash \{0\}$,

$$A(S_i) = (a_{i1}, a_{i2}, \cdots, a_{i(u-2)}) \ (1 \leqslant i \leqslant t)$$

For any i $(1 \leqslant i \leqslant t)$, the multiset $\bigcup_{j=1}^{u-2}(B_{ij} + (0, a_{ij})) \bigcup \{C_i\}$ contains exactly k elements (not necessarily distinct) from each group G_x for $x \in GF(u-1) \backslash \{0\}$ and no element of G_0.

Theorem 8. [10] *If there exists an intransitive starter (S, R, C) over X defined above and a corresponding adder $A(S)$, then there exists an FGDRP(k, g^u).*

Lemma 9. *There exists an FGDRP$(5, 20^8)$.*

Proof : Let $X = \mathbb{Z}_{20} \times (\mathbb{Z}_7 \cup \{\infty\})$ be the point set and $\{\mathbb{Z}_{20} \times \{x\} \mid x \in \mathbb{Z}_7 \bigcup \{\infty\}\}$ the group set. Here we apply Theorem 8 with $k = 5, t = 4$. In the following table, we present S_i, R_i, C_i and $A(S_i)$ for $i = 1$ and 2.

$A(S_1)$	S_1	$A(S_2)$	S_2
(0,6)	{(3,3), (19,6), (0,2), (17,5), (2,4)}	(0,6)	{(4,5), (17,4), (9,2), (12,3), (15,6)}
(0,5)	{–, (7,5), (14,3), (13,1), (4,6)}	(0,5)	{–, (15,5), (2,6), (11,3), (19,4)}
(0,4)	{–, (2,5), (0,4), (9,6), (1,2)}	(0,4)	{–, (4,4), (6,6), (12,1), (13,2)}
(0,3)	{–, (8,1), (3,6), (19,2), (10,5)}	(0,3)	{–, (18,3), (11,1), (12,2), (7,6)}
(0,2)	{–, (16,2), (16,4), (1,6), (8,3)}	(0,2)	{–, (3,2), (5,4), (9,3), (16,6)}
(0,1)	{–, (5,1), (5,5), (10,4), (14,2)}	(0,1)	{–, (6,3), (6,5), (14,1), (18,2)}
C_1 :	{(4,2), (14,4), (14,3), (0,6), (0,5)}	C_2 :	{(6,3), (5,1), (8,6), (8,2), (7,5)}
R_1 :	{(7,4), (11,5), (1,3), (17,1), (0,6)}	R_2 :	{(10,6), (18,1), (15,4), (13,3), (8,2)}

For $i = 3$ and 4, let $S_i = S_{i-2} \cdot (1, -1)$, $R_i = R_{i-2} \cdot (1, -1)$, $C_i = C_{i-2} \cdot (1, -1)$ and $A(S_i) = A(S_{i-2}) \cdot (1, -1)$. Then it is easy to check that $(\cup_{i=1}^4 S_i, \cup_{i=1}^4 R_i, \cup_{i=1}^4 C_i)$ is an intransitive starter and $A(S) = \cup_{i=1}^4 A(S_i)$ is the corresponding adder for an FGDRP$(5, 20^8)$. Here the twenty points from $\mathbb{Z}_{20} \times \{\infty\}$ can be distributed to the five blocks of size four in each S_i for $1 \leqslant i \leqslant 4$ in an arbitrary way. So we use the symbol "–" to denote any point from $\mathbb{Z}_{20} \times \{\infty\}$. □

Lemma 10. *There exists an FGDRP*$(5, 20^{10})$.

Proof : Here we take $\mathbb{Z}_{20} \times (GF(9) \cup \{\infty\})$ as the point set and $\{\mathbb{Z}_{20} \times \{x\} \mid x \in GF(9) \cup \{\infty\}\}$ as the group set. Suppose that ω is a primitive element of $GF(9)$ satisfying $\omega^2 = \omega + 1$. We apply Theorem 8 with $k = 5, t = 4$. First we display S_1, R_1, C_1 and $A(S_1)$ in the following table.

$A(S_1)$	S_1
(0,1)	{(16,ω^0), (7,ω^6), (14,ω^2), (10,ω^7), (1,ω^1)}
(0,ω)	{(3,ω^0), (18,ω^1), (18,ω^2), (6,ω^3), (4,ω^7)}
(0,ω^2)	{(6,ω^0), (14,ω^1), (2,ω^2), (3,ω^3), (1,ω^4)}
(0,ω^3)	{−, (4,ω^0), (13,ω^1), (0,ω^4), (9,ω^5)}
(0,ω^4)	{−, (8,ω^4), (15,ω^2), (8,ω^5), (11,ω^6)}
(0,ω^5)	{−, (12,ω^5), (16,ω^3), (15,ω^7), (17,ω^6)}
(0,ω^6)	{−, (0,ω^7), (9,ω^6), (19,ω^4), (19,ω^5)}
(0,ω^7)	{−, (2,ω^1), (10,ω^2), (12,ω^0), (17,ω^5)}
C_1:	{(15,ω^7), (2,ω^3), (14,ω^0), (5,ω^5), (11,ω^4)}
R_1:	{(11,ω^5), (5,ω^2), (5,ω^7), (13,ω^6), (7,ω^1)}

Then, for $1 \leqslant i \leqslant 4$, let $S_i = S_1 \cdot (1, \omega^{2(i-1)})$, $R_i = R_1 \cdot (1, \omega^{2(i-1)})$, $C_i = C_1 \cdot (1, \omega^{2(i-1)})$ and $A(S_i) = A(S_1) \cdot (1, \omega^{2(i-1)})$. It is an easy matter to verify that $(\cup_{i=1}^4 S_i, \cup_{i=1}^4 R_i, \cup_{i=1}^4 C_i)$ is the required intransitive starter and $A(S) = \cup_{i=1}^4 A(S_i)$ is the corresponding adder. Similarly with Lemma 9, the twenty points from $\mathbb{Z}_{20} \times \{\infty\}$ can be distributed to the five blocks of size four in each S_i for $1 \leqslant i \leqslant 4$ in an arbitrary way. So we still use the symbol "−" to denote any point from $\mathbb{Z}_{20} \times \{\infty\}$. □

Now we are in a position to complete the existence of NGBTDs with block size five.

Theorem 11. *For any positive integer* m*, there exists an NGBTD*$(5, m)$.

Proof : By Theorem 1, we need only to show an NGBTD$(5, m)$ exists for each $m \in \{15, 32, 40, 45\}$.

An NGBTD$(5, 15)$ is given in Lemma 5. For $m \in \{32, 40, 45\}$, we apply Theorem 2 with $k = 5, w = 0$ and other suitable parameters displayed in the following table to obtain the desired NGBTD$(5, m)$.

m	g	h	u	the source of an FGDRP(k, g^u)
32	20	4	8	Lemma 9
40	20	4	10	Lemma 10
45	25	5	9	Lemma 7

4 Applications to Constant Composition Codes

Let $Q = \{a_t : 0 \leqslant t \leqslant m - 1\}$ be an arbitrary alphabet set with m elements. A code $C \subseteq Q^n$ over Q with size M and minimum distance d is referred to as a *constant composition code* (CCC), or an $(n, M, d, [w_0, w_1, \cdots, w_{m-1}])_m$-CCC, if each codeword has precisely w_i occurrences of a_i for any i $(0 \leqslant i \leqslant m-1)$. Here the definition implies $n = \sum_{0 \leqslant i \leqslant m-1} w_i$.

Since the constant composition $[w_0, w_1, \cdots, w_{m-1}]$ is essentially an unordered multiset, we usually write it in an exponential notation: a constant composition $[a_1^{u_1} a_2^{u_2} \cdots a_s^{u_s}]$ indicates u_i occurrences of a_i for $1 \leqslant i \leqslant s$ for brevity. We denote the maximum size M of an $(n, M, d, [w_0, w_1, \cdots, w_{m-1}])_m$-CCC by $A_m(n, d, [w_0, w_1, \cdots, w_{m_1}])$. A CCC with this size is called *optimal*. The following upper bound was established by Luo et al. [6].

Theorem 12. [6] *If* $nd - n^2 + (w_0^2 + w_1^2 + \cdots + w_{m-1}^2) > 0$*, then*

$$A_m(n, d, [w_0, w_1, \cdots, w_{m_1}]) \leqslant \frac{nd}{nd - n^2 + (w_0^2 + w_1^2 + \cdots + w_{m-1}^2)}.$$

The study of optimal CCCs has attracted extensive attention due to their numerous applications (see, for example, [3, 6] and the references therein). Particularly, Ding and Yin [3] presented a combinatorial characterization of constant composition codes and established an equivalent relationship between CCCs and a class of designs called generalized doubly resolvable packings (GDRPs) which are defined below.

Let X be a set of v elements (called points) and $\mathcal{A}$ be a collection of subsets (called blocks) of X. Then the pair $(X, \mathcal{A})$ is called a λ-*packing* of order v, if every pair of distinct points of X occurs in at most λ blocks. Furthermore, it is termed a *generalized doubly resolvable packing* (GDRP), if the blocks of $\mathcal{A}$ can be arranged into an $m \times n$ array $\mathcal{R}$ which satisfies the following properties:

(1) Each cell of $\mathcal{R}$ is either empty or contains one block;

(2) For $0 \leqslant i \leqslant m-1$, the blocks in row i of $\mathcal{R}$ form a w_i-parallel class, that is, every point occurs in exactly w_i blocks;

(3) The blocks in every column of $\mathcal{R}$ form a parallel class, that is, every point occurs in exactly one block.

We denote such a GDRP by a GDRP$(m \times n, \lambda; v)$. The multiset $T = \{w_0, w_1, \cdots, w_{m-1}\}$ is called the type of the GDRP. For more details, the interested reader may refer to [3, 11, 12].

Theorem 13. [3, 12] *The existence of a GDRP*$(m \times n, \lambda; v)$ *of type* $\{w_0, w_1, \cdots, w_{m-1}\}$ *is equivalent to an* $(n, M, d, [w_0, w_1, \cdots, w_{m-1}])_m$*-CCC, where* $M = v$ *and* $d = n - \lambda$.

Theorem 14. *If there exists an NGBTD*(k, m)*, then there exists an optimal* $(km+1, km+1, k(m-1)+2, [1^1 k^m])_{m+1}$*-CCC.*

Proof : It is readily checked that an NGBTD(k, m) is a GDRP$(m \times (km+1), k-1; km+1)$ of type $\{1, k, \cdots, k\}$. By Theorem 13, we have an $(n, M, d, [w_0, w_1, \cdots, w_{m-1}])_m$-CCC with $n = M = km+1, d = k(m-1)+2, w_0 = 1, w_1 = w_2 = \cdots = w_{m-1} = k$. In addition, by Theorem 12, we have

$$\begin{aligned}
&A_m(km+1, k(m-1)+2, [1^1k^m]) \\
&\leqslant \frac{(km+1)(k(m-1)+2)}{(km+1)(k(m-1)+2)-(km+1)^2+(1^2+k^2+\cdots+k^2)} \\
&= \frac{(km+1)(k(m-1)+2)}{(km+1)(1-k)+(1+mk^2)} \\
&= km+1
\end{aligned}$$

Hence the obtained CCC is optimal. Then the proof is complete. □

Theorem 15. *Let m, k be integers satisfying $m \geqslant 1$, $2 \leqslant k \leqslant 5$ and $(k, m) \notin (3, 3)$. Then there exists an optimal $(km+1, km+1, k(m-1)+2, [1^1k^m])_{m+1}$-CCC.*

Proof : By Theorem 1, 4 and 11, there exists an NGBTD(k, m) for any integers m and k where $m \geqslant 1$, $2 \leqslant k \leqslant 5$ and $(k, m) \notin (3, 3)$. Then the conclusion follows from Theorem 14. □

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