The 1/3-2/3 Conjecture for N-free ordered sets

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Abstract

A balanced pair in an ordered set $P = (V, \leqslant)$ is a pair (x, y) of elements of V such that the proportion of linear extensions of P that put x before y is in the real interval $[1/3, 2/3]$. We prove that every finite N-free ordered set which is not totally ordered has a balanced pair.

Keywords: Ordered set; Linear extension; N-free; Balanced pair; 1/3-2/3 Conjecture.

1 Introduction

Throughout, $P = (V, \leqslant)$ denotes a *finite ordered set*, that is, a finite set V and a binary relation $\leqslant$ on V which is reflexive, antisymmetric and transitive. A *linear extension* of $P = (V, \leqslant)$ is a linear ordering $\preceq$ of V which extends $\leqslant$, i.e. such that $x \preceq y$ whenever $x \leqslant y$.

Suppose an unknown linear extension L of P is to be determined using only comparisons between pairs of elements. At each step we ask a question of the form "is it true that $x \prec y$?". We will get the answer before we can ask another question. How many comparisons do we need to perform (in the worst case) in order to determine L completely? This is known as the problem of *comparison sorting*.

Suppose that at each step we can find a pair (x, y) of incomparable elements such that the proportion of linear extensions of P that put x before y, denoted $\mathbb{P}(x \prec y)$, equals $\frac{1}{2}$. Then we need at least $\log_2(e(P))$ comparisons where $e(P)$ denotes the number of linear extensions of P. This is not always possible as shown by the example (i) depicted in Figure 1. Indeed, in that example the only possible values for $\mathbb{P}(x \prec y)$ are 1/3 or 2/3.

Call a pair (x, y) of elements of V a *balanced pair* in $P = (V, \leqslant)$ if $1/3 \leqslant \mathbb{P}(x \prec y) \leqslant 2/3$. The 1/3-2/3 Conjecture states that every finite ordered set which is not totally ordered has a balanced pair. If true, the example (i) depicted in Figure 1 would show that the result is best possible. The 1/3-2/3 Conjecture first appeared in a paper of Kislitsyn [6]. It was also formulated independently by Fredman in about 1975 and again by Linial [7].

The 1/3-2/3 Conjecture is known to be true for ordered sets with a nontrivial automorphism [5], for ordered sets of width two [7], for semiorders [2], for bipartite ordered sets [10], for 5-thin posets [4], and for 6-thin posets [8]. See [3] for a survey.

In this paper we prove the 1/3-2/3 Conjecture for N-free ordered sets.

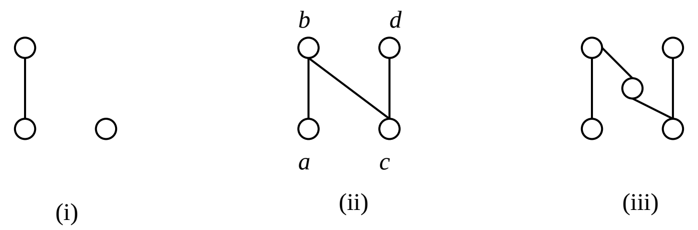

Figure 1:

Let $P = (V, \leqslant)$ be an ordered set. For $x, y \in V$ we say that y is an *upper cover* of x or that x is *a lower cover* of y if $x < y$ and there is no element $z \in V$ such that $x < z < y$. Also, we say that x and y are *comparable* if $x \leqslant y$ or $y \leqslant x$; otherwise we say that x and y are *incomparable*. A *chain* is a totally ordered set.

A 4-tuple (a, b, c, d) of distinct elements of V is an N in P if b is an upper cover of a and c, d is an upper cover of c and if these are the only comparabilities between the elements a, b, c, d (See Figure 1 (ii)). The ordered set P is *N-free* if it does not contain an N (the ordered set depicted in Figure 1 (iii) is N-free and the one depicted in Figure 1 (ii) is not).

Notice that every finite ordered set can be embedded into a *finite* N-free ordered set (see for example [9]). It was proved in [1] that the number of (unlabeled) N-free ordered sets is

$$2^{n \log_2(n) + o(n \log_2(n))}.$$

Our main result is this.

Theorem 1. *Every finite N-free ordered set which is not totally ordered has a balanced pair.*

The proof of Theorem 1 is similar to the proof of Theorem 2 of [7] stating that the 1/3-2/3 Conjecture is true for finite ordered sets of width two (these being the ordered sets covered by two chains).

2 Proof of Theorem 1

We start this section by stating some useful properties of N-free ordered sets.

Lemma 2. *Let $P = (V, \leqslant)$ be an N-free ordered set. If $x, y \in V$ have a common upper cover, then x and y have the same upper covers. Dually, if $x, y \in V$ have a common lower cover, then x and y have the same lower covers.*

Let $P = (V, \leqslant)$ be an ordered set. An element $m \in V$ is called *minimal* if for all $x \in V$ comparable to m we have $x \geqslant m$. We denote by $Min(P)$ the set of all minimal elements of P. We recall that the decomposition of P into *levels* is the sequence $P_0, \cdots, P_l, \cdots$ defined by induction by the formula

$$P_l := Min(P - \cup\{P_{l'} : l' < l\}).$$

In particular, $P_0 = Min(P)$.

Lemma 3. *Let $P = (V, \leqslant)$ be an N-free ordered set and let $P_0, \cdots, P_h$ be the sequence of its levels. Then for every $x \in V$, there exists $i \leqslant h$ such that all upper covers of x are in P_i.*

Proof. If x has at most one upper cover, then the conclusion of the lemma holds. So we may assume that x has at least two distinct upper covers x_1 and x_2 belonging to two distinct levels. Let $j < k$ be such that $x_1 \in P_j$ and $x_2 \in P_k$. Then x_2 has a lower cover $x_3 \in P_{k-1}$. We claim that $(x_3, x_2, x, x_1\}$ is an N in P contradicting our assumption that P is N-free. Indeed, since x_1 and x_2 are upper covers of x we infer that they must be incomparable. Moreover, x_1 and x_3 are incomparable because otherwise $x_1 < x_3 < x_2$ (notice that $x_3 < x_1$ is not possible since $j \leqslant k - 1$) which contradicts our assumption that x_2 is an upper cover of x. Similarly we have that x and x_3 are incomparable proving our claim. The proof of the lemma is now complete. □

Let $P = (V, \leqslant)$ be an ordered set. For $x \in V$ define $D(x) := \{y \in V : y < x\}$ and $U(x) := \{y \in V : x < y\}$.

Lemma 4. *Let P be an N-free ordered set and let $P_0, \cdots, P_h$ be the sequence of its levels. Let $0 \leqslant i \leqslant h$ be such that i is the largest with the property that P_i contains two distinct elements with the same set of lower covers. Then for every $x \in P_i$ we have that $U(x) \cup \{x\}$ is a chain.*

Proof. Let $x \in P_i$ be such that $U(x) \neq \emptyset$ and suppose that $U(x)$ is not a chain. There is then an element $y \in U(x) \cup \{x\}$ having at least two distinct upper covers, say y_1, y_2. From Lemma 3 we deduce that y_1 and y_2 are in the same level P_j with $i < j$. Because P is N-free it follows from Lemma 2 that y_1 and y_2 have the same set of lower covers. This contradicts our choice of i. □

We recall that an incomparable pair (x, y) of elements is *critical* if $U(y) \subseteq U(x)$ and $D(x) \subseteq D(y)$. The following lemma is true for ordered sets that are not necessarily N-free.

Lemma 5. *Suppose (x, y) is a critical pair in P and consider any linear extension of P in which $y < x$. Then the linear order obtained by swapping the positions of y and x is also a linear extension of P. Moreover, $\mathbb{P}(x \prec y) \geqslant \frac{1}{2}$.*

Proof. Let L be a linear extension that puts y before x and let z be such that $y \prec z \prec x$ in L. Then z is incomparable with both x and y since (x, y) is a critical pair of P. Therefore, the linear order L' obtained by swapping x and y is a linear extension of P. The map $L \mapsto L'$ from the set of linear extensions that put y before x into the set of linear extensions that put x before y is clearly one-to-one. Hence, $\mathbb{P}(y \prec x) \leqslant \mathbb{P}(x \prec y)$ and therefore $\mathbb{P}(x \prec y) \geqslant \frac{1}{2}$. □

We now prove Theorem 1.

Proof. Let $P = (V, \leqslant)$ be an N-free ordered set not totally ordered and $P_0, \cdots, P_h$ be the sequence of its levels. If P_0 is a singleton, say $P_0 = \{p_0\}$, then p_0 will be the minimum element in any linear extension of the ordered set. Therefore, nothing will change if p_0 is deleted from the ordered set. So we may assume without loss of generality that P_0 has at least two distinct elements. Notice that any two such elements have the same set of lower covers: the empty set. Now let $0 \leqslant i \leqslant h$ be such that i is the largest with the property that P_i contains two distinct elements with the same set of lower covers and let $a, b \in P_i$ be such elements. If $U(b) = U(a) = \emptyset$, then $\mathbb{P}(a \prec b) = \frac{1}{2}$ and we are done. Otherwise we may suppose without loss of generality that $U(b) \neq \emptyset$. From Lemma 4 we deduce that $U(b) \cup \{b\}$ is a chain, say $U(b) \cup \{b\}$ is the chain $b = b_1 < \cdots < b_n$. We prove the theorem by contradiction. We may assume without loss of generality that

$$\mathbb{P}(a \prec b_1) < \frac{1}{3}.$$

Indeed, if $U(a) \neq \emptyset$, then the situation is symmetric with respect to a and b and therefore such an assumption is possible. Otherwise, $U(a) = \emptyset$ and hence (b_1, a) is a critical pair (this is because $D(a) = D(b_1)$ by assumption) yielding $\mathbb{P}(b_1 \prec a) > \frac{2}{3}$ (Lemma 5) or equivalently $\mathbb{P}(a \prec b_1) < \frac{1}{3}$.

Define now the following quantities

$$\begin{aligned} q_1 &= \mathbb{P}(a \prec b_1), \\ q_j &= \mathbb{P}(b_{j-1} \prec a \prec b_j)(2 \leqslant j \leqslant n), \\ q_{n+1} &= \mathbb{P}(b_n \prec a). \end{aligned}$$

Lemma. The real numbers q_j $(1 \leqslant j \leqslant n+1)$ satisfy:

(i) $0 \leqslant q_{n+1} \leqslant \cdots \leqslant q_1 \leqslant \frac{1}{3}$,

(ii) $\sum_{j=1}^{n+1} q_j = 1$.

Proof. Since $q_1, \cdots, q_{n+1}$ is a probability distribution, all we have to show is that $q_{n+1} \leqslant \cdots \leqslant q_1$. To show this we exhibit a one-to-one mapping from the event that $b_j \prec a \prec b_{j+1}$ whose probability is q_{j+1} into the event that $b_{j-1} \prec a \prec b_j$ whose probability is q_j $(1 \leqslant j \leqslant n)$. Notice that in a linear extension for which $b_j \prec a \prec b_{j+1}$ every element z between b_j and a is incomparable to both b_j and a. Indeed, such an element z cannot be comparable to b_j because otherwise $b_j < z$ in P but the only element above b_j is

b_{j+1} which is above a in the linear extension. Now z cannot be comparable to a as well because otherwise $z < a$ in P and hence $z < b = b_1 < b_j$ (by assumption we have that $D(a) = D(b)$). The mapping from those linear extensions in which $b_j \prec a \prec b_{j+1}$ to those in which $b_{j-1} \prec a \prec b_j$ is obtained by swapping the positions of a and b_j. This mapping clearly is well defined and one-to-one. $\square$

Theorem 1 can be proved now: let r be defined by

$$\sum_{j=1}^{r-1} q_j \leqslant \frac{1}{2} < \sum_{j=1}^{r} q_j$$

Since $\sum_{j=1}^{r-1} q_j = \mathbb{P}(a \prec b_{r-1}) \leqslant \frac{1}{2}$, it follows that $\sum_{j=1}^{r-1} q_j < \frac{1}{3}$. Similarly $\sum_{j=1}^{r} q_j = \mathbb{P}(a \prec b_r)$ must be $> \frac{2}{3}$. Therefore $q_r > \frac{1}{3}$, but this contradicts $\frac{1}{3} > q_1 \geqslant q_r$. $\square$

References

[1] Bayoumi I. Bayoumi, M. El-Zahar and Soheir M. Khamis. Asymptotic enumeration of N-free partial orders. Order, 6:219-225, 1989.

[2] G. Brightwell. Semiorders and the $1/3 - 2/3$ conjecture. Order, 5:369–380, 1989.

[3] G. Brightwell. Balanced pairs in Partial orders. Discrete Mathematics, 201:25–52, 1999.

[4] G. Brightwell and C. D. Wright. The $1/3 - 2/3$ conjecture for 5-thin posets. SIAM. J. Discrete Mathematics, 5:467–474, 1992.

[5] B. Ganter, G. Hafner and W. Poguntke. On linear extensions of ordered sets with a symmetry. *Special issue: ordered sets (Oberwolfach, 1985).* Discrete Mathematics, 63:153–156, 1987.

[6] S. S. Kislitsyn. Finite partially ordered sets and their associated set of permutations. Matematicheskiye Zametki, 4:511–518, 1968.

[7] N. Linial. The information theoretic bound is good for merging. SIAM J. Comput., 13:795–801, 1984.

[8] Peczarski, Marcin. The gold partition conjecture for 6-thin posets. Order, 25:91-103, 2008.

[9] M. Pouzet and N. Zaguia. N-free extensions of posets. Note on a theorem of P. A. Grillet. Contrib. Discrete Mathematics, 1:80–87, 2006. (electronic).

[10] W. T. Trotter, W. G. Gehrlein, P. C. Fishburn. Balance theorems for height-2 posets. Order, 9:43–53, 1992.