Inequalities between gamma-polynomials of graph-associahedra

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Abstract

We prove a conjecture of Postnikov, Reiner and Williams by defining a partial order on the set of tree graphs with n vertices that induces inequalities between the γ-polynomials of their associated graph-associahedra. The partial order is given by relating trees that can be obtained from one another by operations called tree shifts. We also show that tree shifts lower the γ-polynomials of graphs that are not trees, as do the flossing moves of Babson and Reiner.

1 Introduction

For any building set $\mathcal{B}$ there is an associated simple polytope $P_{\mathcal{B}}$ called the *nestohedron* (see [Po] Section 7 and [PRW] Section 6). When $\mathcal{B} = \mathcal{B}(G)$ is the building set determined by a graph G, $P_{\mathcal{B}(G)}$ is the well-known graph-associahedron of G (see [BV], [Er], [PRW] Sections 7 and 12, and [Vol]). The numbers of faces of $P_{\mathcal{B}}$ of each dimension are conveniently encapsulated in its γ-polynomial $\gamma(\mathcal{B}) = \gamma(P_{\mathcal{B}})$ (see [PRW] Section 1 for the definition). Postnikov, Reiner and Williams conjectured the following monotonicity property of the γ-polynomials of the graph-associahedra of trees.

Conjecture 1. [PRW, Conjecture 14.1]. There exists a partial order $\leqslant$ on the set of (unlabelled, isomorphism classes of) trees with n vertices, with the following properties:

- $\mathrm{Path_n}$ is the unique $\leqslant$-minimal element,
- $K_{1,n-1}$ is the unique $\leqslant$-maximal element,
- $T \leqslant T'$ implies $\gamma(\mathcal{B}(T)) \leqslant \gamma(\mathcal{B}(T'))$.

Here $\mathrm{Path_n}$ denotes the graph that is a path with n vertices, and $K_{1,n-1}$ is the graph with n vertices with exactly one vertex of degree $n-1$ and $n-1$ vertices of degree 1.

This conjecture implies the following lower and upper bounds for the γ-polynomial of a tree T with n vertices

$$\gamma(\mathcal{B}(\mathrm{Path_n})) \leqslant \gamma(\mathcal{B}(T)) \leqslant \gamma(\mathcal{B}(K_{1,n-1})). \tag{1}$$

These upper and lower bound theorems have been proven by Buchstaber and Volodin [BV, Theorem 9.4]. Moreover, they show that the lower bound is attained only for $\mathrm{Path_n}$ and the upper bound is attained only for $K_{1,n-1}$. Their proof relies on some general results about γ-polynomials of flag nestohedra which were announced in [Vol] and whose proofs are included in [BV]; see Lemmas 10, 11, 14 and theorems 9, 12 and 13. Note that the methods of Buchstaber and Volodin require one to work with the more general class of flag nestohedra in order to deduce the results about graph-associahedra. In this paper we make use of these theorems to show that Conjecture 1 can be proven with the relation of tree shifts that we define.

We also use these theorems to show that flossing moves lower the γ-polynomial. Flossing moves were originally defined in [BR] Section 4.2 and it was suggested in [PRW] Section 14 that they might lower the γ-polynomial. Our definition of flossing move is more general than that in [BR] as it can be applied to any pair of leaves that floss a vertex, and it does not have to be applied to a tree graph.

Section 2 contains preliminary definitions and results relating to polytopes and building sets. Section 3 contains more specific results relating to the γ-polynomial that are needed for the main theorems in Sections 4 and 5. Section 4 introduces tree shifts and in Theorem 15 we show that they lower the γ-polynomial of the associated graph-associahedra. We then prove Conjecture 1, in Theorem 16. Section 5 introduces flossing moves and Theorem 17 shows that they lower the γ-polynomials.

2 Building sets and nestohedra

A *building set* $\mathcal{B}$ on a finite set S is a set of non empty subsets of S such that

- For any $I,\ J \in \mathcal{B}$ such that $I \cap J \neq \emptyset$, $I \cup J \in \mathcal{B}$.
- $\mathcal{B}$ contains the singletons $\{i\}$, for all $i \in S$.

$\mathcal{B}$ is *connected* if it contains S. For any building set $\mathcal{B}$, $\mathcal{B}_{max}$ denotes the set of maximal elements of $\mathcal{B}$ with respect to inclusion. The elements of $\mathcal{B}_{max}$ form a disjoint union of S, and if $\mathcal{B}$ is connected then $\mathcal{B}_{max} = \{S\}$. Building sets $\mathcal{B}_1$, $\mathcal{B}_2$ on S are *equivalent*, denoted $\mathcal{B}_1 \cong \mathcal{B}_2$, if there is a permutation $\sigma : S \to S$ that induces a one to one correspondence $\mathcal{B}_1 \to \mathcal{B}_2$.

Example 2. Let G be a graph with no loops or multiple edges, with n vertices labelled distinctly from $[n]$. Then the graphical building set $\mathcal{B}(G)$ is the set of subsets of $[n]$ such that the induced subgraph of G is connected. $\mathcal{B}(G)_{max}$ is the set of connected components of G.

Let $\mathcal{B}$ be a building set on S and $I \subseteq S$. The *restriction of* $\mathcal{B}$ *to* I is the building set

$$\mathcal{B}|_I := \{J \mid J \subseteq I, \text{ and } J \in \mathcal{B}\} \text{ on } I.$$

The *contraction of* $\mathcal{B}$ *by* I is the building set

$$\mathcal{B}/I := \{J - (J \cap I) \mid J \in \mathcal{B},\ J \not\subseteq I\} \text{ on } S - I.$$

Example 3. If G is a graph on $[n]$, and $I \in \mathcal{B}(G)$, then $\mathcal{B}(G)/I = \mathcal{B}(G')$ where G' is the graph on $[n]-I$ such that any two vertices $i,\ j \in [n]-I$ are adjacent if they are adjacent in G, or both i and j are adjacent to vertices in I in the full graph G.

Given a building set $\mathcal{B}$, a subset $N \subseteq \mathcal{B} \backslash \mathcal{B}_{max}$ is a *nested set* if it satisfies

- For any $I,\ J \in N$, either $I \subseteq J$, $J \subseteq I$, or $I \cap J = \emptyset$.
- For any collection of $k \geqslant 2$ disjoint subsets $J_1, \dots, J_k \in N$, the union $J_1 \cup \cdots \cup J_k \notin \mathcal{B}$.

The *nested set complex* $\Delta_{\mathcal{B}}$ is the simplicial complex on $\mathcal{B} - \mathcal{B}_{max}$ whose faces are the nested sets. We associate a polytope to a building set as follows. Let $e_1, \dots, e_n$ denote the endpoints of the coordinate vectors in $\mathbb{R}^n$. Given $I \subseteq [n]$, define the simplex $\Delta_I := ConvexHull(e_i \mid i \in I)$. Let $\mathcal{B}$ be a building set on $[n]$. The *nestohedron* $P_{\mathcal{B}}$ is a polytope given by the Minkowski sum of the simplices Δ_I for all $I \in B$

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I.$$

If $\mathcal{B}$ is a graphical building set $P_{\mathcal{B}}$ is known as the *graph-associahedron*. The nestohedron is related to the nested sets of any building set $\mathcal{B}$, as described in the following theorem.

Theorem 4. *[Po, Theorem 7.4] [FS, Theorem 3.14]. Let $\mathcal{B}$ be a building set on $[n]$. The nestohedron $P_{\mathcal{B}}$ is a simple polytope of dimension $n - |\mathcal{B}_{max}|$. The simplicial polytope polar dual to $P_{\mathcal{B}}$ has boundary complex isomorphic to $\Delta_{\mathcal{B}}$.*

For a simple d dimensional polytope P, the f-polynomial, h-polynomial and γ-polynomial are polynomials in $\mathbb{Z}[t]$ defined as follows:

$$f(P)(t) := f_0 + f_1 t + \cdots + f_d t^d,$$

where f_i is the number of i-dimensional faces of P. The h-polynomial is given by

$$h(P)(t+1) := f(P)(t),$$

and it is known to be positive and symmetric. Since it is symmetric, it can be written

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i},$$

for some $\gamma_i \in \mathbb{Z}$, and the γ-polynomial is given by

$$\gamma(P)(t) := \gamma_0 + \gamma_1 t + \cdots + \gamma_{\lfloor \frac{d}{2} \rfloor} t^{\lfloor \frac{d}{2} \rfloor}.$$

If a polytope P is combinatorially equivalent to $P_1 \times P_2 \times \cdots \times P_n$ where $P_1, \ldots, P_n$ are a set of polytopes, then by the definition of combinatorial equivalence we have that $f(P) = f(P_1)f(P_2)\cdots f(P_n)$, and consequently $\gamma(P) = \gamma(P_1)\gamma(P_2)\cdots\gamma(P_n)$. When $\mathcal{B}$ is a building set, we denote the γ-polynomial for $P_{\mathcal{B}}$ by $\gamma(\mathcal{B})$. If $\mathcal{B}$ and $\mathcal{B}'$ are building sets, the notation $\gamma(\mathcal{B}) \leqslant \gamma(\mathcal{B}')$ implies that for all i the coefficient of t^i in $\gamma(\mathcal{B})$ is less than or equal to the coefficient of t^i in $\gamma(\mathcal{B}')$.

A $d-1$-dimensional face of a d-dimensional polytope is called a *facet*. A simple polytope P is *flag* if any collection of pairwise intersecting facets has non empty intersection. A building set $\mathcal{B}$ is *flag* if $P_{\mathcal{B}}$ is flag. The conditions in Proposition 5 determine whether a building set is flag.

Proposition 5. *[PRW, Proposition 7.1]. For a building set $\mathcal{B}$, the following are equivalent:*

(1) $P_{\mathcal{B}}$ is flag.

(2) If $J_1, \ldots, J_m$, $m \geqslant 2$, are disjoint and $J_1 \cup \cdots \cup J_m \in \mathcal{B}$, then the sets can be reindexed so that for some k such that $1 \leqslant k \leqslant m-1$, $J_1 \cup \cdots \cup J_k \in \mathcal{B}$ and $J_{k+1} \cup \cdots \cup J_m \in \mathcal{B}$.

(3) If $N \subseteq \mathcal{B} \backslash \mathcal{B}_{max}$ such that

- *for any $I,\ J \in N$ either $I \subseteq J$, $J \subseteq I$ or $I \cap J = \emptyset$,*
- *for any $I,\ J \in N$ such that $I \cap J = \emptyset$, one has $I \cup J \notin \mathcal{B}$,*

then N is a nested set.

It follows from Proposition 5 that a graphical building set is flag. A *minimal flag building set* $\mathcal{D}$ on a set S is a connected building set on S that is flag, such that that no proper subset of its elements form a connected flag building set on S. Minimal flag building sets are described in detail in [PRW, Section 7.2]. They take the form of a binary tree, where the vertices biject to elements of $\mathcal{D}$, and the direct descendants of any non leaf vertex that represents an element $I \in \mathcal{D}$ are the two elements in $\mathcal{D}$ whose disjoint union is I. For any minimal flag building set $\mathcal{D}$, $\gamma(\mathcal{D}) = 1$ (see [PRW] Section 7.2).

Let $\mathcal{B}$ be a building set. A *binary decomposition* or *decomposition* of a non singleton element $I \in \mathcal{B}$ is a set $\mathcal{D} \subseteq \mathcal{B}$ that forms a minimal flag building set on I. Suppose that $I \in B$ has a binary decomposition $\mathcal{D}$. The two maximal elements $D_1,\ D_2 \in \mathcal{D} - \{I\}$ with respect to inclusion are the *maximal components* of I in $\mathcal{D}$. The following lemma gives another definition of when a building set is flag.

Lemma 6. *A building set $\mathcal{B}$ is flag if and only if every non singleton $I \in \mathcal{B}$ has a binary decomposition.*

Proof. The only if part follows immediately from [PRW, Proposition 7.3].

For the if part, suppose that $\mathcal{B}$ is a building set and every element has a binary decomposition. We show that $\mathcal{B}$ is flag by showing that part (3) of Proposition 5 holds. Suppose by contradiction that (3) does not hold so that there exists a set $\mathcal{S} = \{S_1, \dots, S_k\} \subset \mathcal{B}$, $k \geqslant 3$, such that $S_i \cap S_j = \emptyset$, $S_i \cup S_j \notin \mathcal{B}$ for all $i \neq j$, and $S_1 \cup \dots \cup S_k = I \in \mathcal{B}$. Fix a decomposition $\mathcal{D}$ of I. Now consider all one element sets of $\mathcal{D}$ (the set of all $\{i\}$ such that $i \in I$). They are each a subset of one element of $\mathcal{S}$. Suppose by induction that all elements in $\mathcal{D}$ that are sets with $\leqslant i$ elements are a subset of one element of $\mathcal{S}$. Then any $i+1$ element subset of $\mathcal{D}$ must also be contained in one element of $\mathcal{S}$. This is true since each $i+1$ element subset of $\mathcal{D}$ is the union of two elements of $\mathcal{D}$ each with less than $i+1$ elements. These two subsets must be contained in the same element of $\mathcal{S}$ since if they were contained in two distinct elements then their union would intersect two elements S_i and S_j of $\mathcal{S}$ which implies $S_i \cup S_j \in \mathcal{B}$. As the size of the elements of the decomposition increase, they are eventually equal to I, which implies that $k = 1$, a contradiction since $k \geqslant 3$. $\square$

Corollary 7. *A building set $\mathcal{B}$ is flag if and only if for every non singleton $I \in \mathcal{B}$, there exists two elements $D_1,\ D_2 \in \mathcal{B}$ such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = I$.*

Lemma 8. *Suppose $\mathcal{B}$ is a flag building set. If $I, J \in \mathcal{B}$ and $J \subsetneq I$, then there is a decomposition of I in $\mathcal{B}$ that contains J.*

Proof. Consider the set $\{J, \{i_1\}, \dots, \{i_k\}\}$ where $\{i_1, \dots, i_k\} = I - J$. This is a set of disjoint elements whose union is in $\mathcal{B}$. Therefore, by Proposition 5 part (2) we can reindex these sets until we obtain two disjoint sets each in $\mathcal{B}$ whose union is I. We can repeatedly perform this same procedure on the elements in $\{J, \{i_1\}, \{i_2\}, \dots, \{i_k\}\}$ that are subsets of each of the new sets obtained at each step. All of the new sets obtained with reindexing, together with a decomposition of J, and the element I are a decomposition of I that contains J. $\square$

3 Face shavings of flag building sets

The following Theorem is proven by Volodin [Vol].

Theorem 9. *[Vol, Lemma 6]. Let $\mathcal{B}$ and $\mathcal{B}'$ be connected flag building sets on $[n]$ such that $\mathcal{B} \subseteq \mathcal{B}'$. Then $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by successively adding elements so that at each step the set is a flag building set.*

Suppose that a connected flag building set $\mathcal{B}'$ on $[n]$ is obtained from a flag building set $\mathcal{B}$ on $[n]$ by adding an element I. Then I has a binary decomposition in $\mathcal{B}'$ with two maximal components D_1, D_2. This implies that $P_{\mathcal{B}'}$ can be obtained by shaving the codimension 2 face of $P_{\mathcal{B}}$ that corresponds to the nested set $\{D_1, D_2\}$.

Lemma 10. *Let $\mathcal{B}$ be a building set with nestohedron $P_{\mathcal{B}}$. Suppose that F_0 is a facet of $P_{\mathcal{B}}$ corresponding to a (non-maximal) building set element I. Then the face poset of F_0 is isomorphic to the poset of faces of $P_{\mathcal{B}|_I} \times P_{\mathcal{B}/I}$.*

Proof. The poset of faces of F_0 is the subposet of the faces of P, consisting of faces that are contained in F_0. Since the facet F_0 corresponds to the nested set $\{I\}$, the set of faces of P that are contained in F_0 correspond to nested sets that contain I. The complex of nested sets of $\mathcal{B}$ that contain I is isomorphic to $\Delta_{\mathcal{B}|_I} \times \Delta_{\mathcal{B}/I}$. The isomorphism is given by

$$(N_1, N_2) \in \Delta_{\mathcal{B}|_I} \times \Delta_{\mathcal{B}/I} \mapsto N_1 \cup N_2' \cup \{I\},$$

where $N_2' := \{D \mid D \in N_2 \text{ and } D \cup I \notin \mathcal{B}\} \cup \{D \cup I \mid D \in N_2,\ D \cup I \in \mathcal{B}\}$. It is not too hard to see that this is a map to nested sets that contain I, that preserves the inclusion relation, and that is injective and surjective. $\square$

[Vol, Proposition 5] states that if a polytope Q can be obtained from a simple n-dimensional polytope P by shaving a face G of dimension k to obtain a new facet F_0, then F_0 is combinatorially equivalent to $G \times \Delta^{n-k-1}$, where Δ^d denotes the d-dimensional simplex. If G is of dimension $n-2$ then F_0 is combinatorially equivalent to $G \times \Delta^1$, so that $\gamma(F_0) = \gamma(G)\gamma(\Delta^1) = \gamma(G)$. Hence, in the case that the polytopes are flag nestohedra, using Lemma 10, we can rewrite [Vol, Corollary 1] as:

Lemma 11. *[Vol, Corollary 1]. If $\mathcal{B}'$ is a flag building set on $[n]$ obtained from a flag building set $\mathcal{B}$ on $[n]$ by adding an element I then*

$$\begin{aligned} \gamma(\mathcal{B}') =& \gamma(\mathcal{B}) + t\gamma(\mathcal{B}'|_I)\gamma(\mathcal{B}'/I) \\ =& \gamma(\mathcal{B}) + t\gamma(\mathcal{B}|_I)\gamma(\mathcal{B}/I). \end{aligned}$$

Proof. The first identity is a direct consequence of the preceding discussion. From the definition of the contraction of a building set we have that $\mathcal{B}'/I = \mathcal{B}/I$ so that $\gamma(\mathcal{B}'/I) = \gamma(\mathcal{B}/I)$. Let D_1, D_2 be the maximal components of I in the decomposition of I in $\mathcal{B}'$. They are unique since $I \notin \mathcal{B}$. Using Lemma 14 below we have that $\mathcal{B}'|_I = \mathcal{D}[\mathcal{B}|_{D_1}, \mathcal{B}|_{D_2}]$ where $\mathcal{D}$ is the building set $\{\{1\}, \{2\}, [2]\}$. Hence

$$\gamma(\mathcal{B}'|_I) = \gamma(\mathcal{D})\gamma(\mathcal{B}|_{D_1})\gamma(\mathcal{B}|_{D_2}) = \gamma(\mathcal{D})\gamma(\mathcal{B}|_I) = \gamma(\mathcal{B}|_I).$$

$\square$

Note that if $\mathcal{B}$ is a flag building set on $[n]$ and $I \in \mathcal{B}$, then $\mathcal{B}/I$ and $\mathcal{B}|_I$ are flag building sets. This is obvious for $\mathcal{B}|_I$. For the claim about $\mathcal{B}/I$, we let $D \in \mathcal{B}/I$. Then if $D \in \mathcal{B}$ there exist two elements D_1, D_2 in $\mathcal{B}/I$ such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = I$. If $D \notin \mathcal{B}$ then $D \cup I \in \mathcal{B}$, and since $I \subseteq I \cup D$, by Lemma 8, I is in a decomposition $\mathcal{D}$ of $I \cup D$ and this implies there are two elements $D_1, D_2 \in \mathcal{D}$ such that $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = D \cup I$, and I is a proper subset of either D_1 or D_2. Let $\overline{D_i}$ denote the image of D_i in the contraction. Then $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\overline{D_1} \cup \overline{D_2} = D$.

Using Theorem 9 and Lemma 11 [Vol] shows the following two Theorems. Their proof uses the inductive hypothesis that both $\gamma(\mathcal{B}'|_I)$ and $\gamma(\mathcal{B}'/I)$ of Lemma 11 are such that $\gamma(\mathcal{B}'|_I) \geqslant 0$ and $\gamma(\mathcal{B}'/I) \geqslant 0$.

Theorem 12. *[Vol, Theorem 2]. For any flag nestohedron $P_{\mathcal{B}}$ we have*

$$\gamma(\mathcal{B}) \geqslant 0.$$

Theorem 13. *[Vol, Theorem 3] [BV, Theorem 1.1]. If $\mathcal{B}$ and $\mathcal{B}'$ are connected flag building sets on $[n]$ and $\mathcal{B} \subseteq \mathcal{B}'$, then $\gamma(\mathcal{B}) \leqslant \gamma(\mathcal{B}')$.*

The following construction is due to Erokhovets [Er]. Let $[i,j]$ denote the interval $\{i, i+1, \ldots, j\}$. Let $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ be connected building sets on $[n], [k_1], \ldots, [k_n]$ respectively, and let $\overline{[k_i]}$ denote the interval $[\sum_{j=1}^{i-1} k_j + 1, \sum_{j=1}^{i} k_j]$. Define the connected building set $\mathcal{B}[\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n]$ on $[k_1 + k_2 + \cdots + k_n]$, where $\mathcal{B}|_{\overline{[k_i]}}$ is equivalent to $\mathcal{B}_i$, and add the elements $\overline{[k_{i_1}]} \cup \overline{[k_{i_2}]} \cup \cdots \cup \overline{[k_{i_m}]}$ for every $\{i_1, i_2, \ldots, i_m\} \in \mathcal{B}$.

Lemma 14. *[Er]. Let $\mathcal{B}, \mathcal{B}_1, \ldots, \mathcal{B}_n$ be connected building sets on $[n], [k_1], \ldots, [k_n]$ respectively. Let $\mathcal{B}' = \mathcal{B}[\mathcal{B}_1, \ldots, \mathcal{B}_n]$. Then $P_{\mathcal{B}'}$ is combinatorially equivalent to $P_{\mathcal{B}} \times P_{\mathcal{B}_1} \times \cdots \times P_{\mathcal{B}_n}$.*

4 Tree shifts

Our goal of this section is to prove Theorem 15.

We will now introduce the tree shift operation mentioned in Theorem 15. We call a degree one vertex of an arbitrary graph a leaf (this is the standard name for a degree one vertex of a tree).

Let G be a connected graph with n vertices labelled 1 to n, with the following properties and extra data (for a vertex v we also denote the set $\{v\}$ by v):

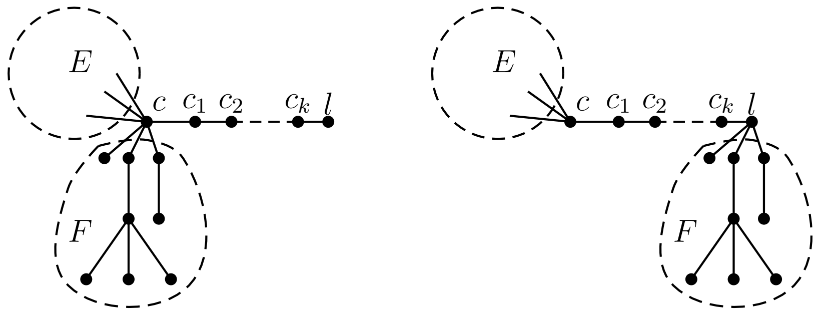
1. G has a leaf l and the nearest vertex to l of degree greater than 2 is labelled c. The vertices in the path from c to l are labelled $c, c_1, c_2, \ldots, c_k, l$.

2. There exists a set of vertices F of $G - \{c, c_1, \ldots, l\}$ such that $F \cup c$ is a subgraph of G that forms a tree, and such that there is no vertex of $G - (c \cup F)$ that is connected to a vertex in F.

3. $G - (F \cup \{c, c_1, c_2, \ldots, c_k, l\}) \neq \emptyset$, and is denoted E.

A *tree shift* is the following move applied to a graph with the properties described. Informally, we remove F and reattach F to l. More formally, we remove any edge (v, c) where $v \in F$, and replace it with the edge (v, l) (see Figure 1).

Figure 1: A graph G followed by the tree shift of G.



Theorem 15. *Let G be a connected graph, and let G' be a resulting tree shift of G. Then $\gamma(\mathcal{B}(G')) \leqslant \gamma(\mathcal{B}(G))$.*

Proof. We suppose that G has n vertices, and we label G as in the definition of a tree shift. We assume by induction that for any connected graph H with less than n vertices, if H' is a tree shift of H, then $\gamma(\mathcal{B}(H')) \leqslant \gamma(\mathcal{B}(H))$. When $n < 4$ no tree shift is possible so the result is vacuously true. Let v be a leaf of G (and G') contained in F. The set $\overline{\mathcal{B}} := \mathcal{B}(G - v) \cup \{\{v\}, [n]\}$ is a flag building set contained in $\mathcal{B}(G)$ and $\overline{\mathcal{B}'} = \mathcal{B}(G' - v) \cup \{\{v\}, [n]\}$ is a flag building set contained in $\mathcal{B}(G')$, hence, by Theorem 9 we can add elements to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ so that at each step the set obtained is a flag building set. Similarly, we can add elements to $\overline{\mathcal{B}'}$ to obtain $\mathcal{B}(G')$ so that at each step the set we obtain is a flag building set. By Lemma 11 and Theorem 12 each time an element is added to these flag building sets the γ-polynomial of the resulting building set increases. We will construct an injection

$$\mathcal{B}(G') - \overline{\mathcal{B}}' \to \mathcal{B}(G) - \overline{\mathcal{B}}$$

$$I' \mapsto I,$$

and show that the increase in the γ-polynomial when adding I' is less than or equal to the increase when adding I. This shows that

$$\gamma(\mathcal{B}(G')) - \gamma(\overline{\mathcal{B}'}) \leqslant \gamma(\mathcal{B}(G)) - \gamma(\overline{\mathcal{B}}). \tag{2}$$

By Lemma 14

$$\gamma(\overline{\mathcal{B}}) = \gamma(\mathcal{B}(G - v))$$

and

$$\gamma(\overline{\mathcal{B}'}) = \gamma(\mathcal{B}(G' - v)),$$

so that Equation 2 becomes

$$\gamma(\mathcal{B}(G')) - \gamma(\mathcal{B}(G' - v)) \leqslant \gamma(\mathcal{B}(G)) - \gamma(\mathcal{B}(G - v)).$$

By induction, since $G' - v$ is a tree shift of $G - v$, or is equal to $G - v$, we have

$$\gamma(\mathcal{B}(G' - v)) \leqslant \gamma(\mathcal{B}(G - v))$$

so that

$$\gamma(\mathcal{B}(G')) \leqslant \gamma(\mathcal{B}(G)).$$

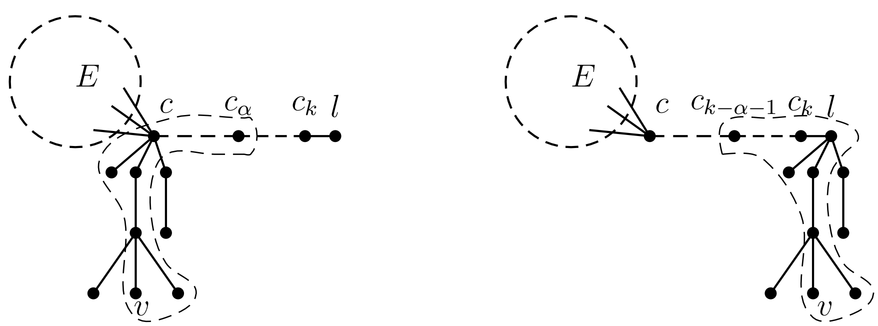
We will now construct the injection. Suppose that $I'_1, I'_2, \ldots, I'_k$ are the building set elements that are added to $\overline{\mathcal{B}}'$ to obtain $\mathcal{B}(G')$ (in order) and $I'_j \subseteq I'_i$. Then $j > i$, since $I'_j \cap (I'_i - \{v\}) \neq \emptyset$ and $I'_j \cup (I'_i - \{v\}) = I'_i$ which implies that when I'_j is in the building set I'_i must be too. Similarly, no subset of an element is added before it when we are adding sets to obtain $\mathcal{B}(G)$.

Let $\mathcal{B}'_m$ be the building set $\overline{\mathcal{B}}' \cup \{I'_1, I'_2, \ldots, I'_m\}$. By Lemma 11 we have that

$$\gamma(\mathcal{B}'_m) - \gamma(\mathcal{B}'_{m-1}) = t\gamma(\mathcal{B}'_{m-1}|_{I'_m})\gamma(\mathcal{B}'_{m-1}/I'_m).$$

Suppose that $I'_m \cap E = \emptyset$, so that $I'_m = D \cup \{l, c_k, \ldots, c_{k-\alpha+1}\}$ for some $D \subseteq F$ and let $I_m = D \cup \{c, c_1, \ldots, c_\alpha\}$, one of the elements that is added to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G)$. Note that we may have $c_{k-\alpha+1} = c$ and $c_\alpha = l$. Note also that I_m is not necessarily the mth element that is added to $\overline{\mathcal{B}}$ (see Figure 2).

Figure 2: The set I_m followed by the set I'_m.



We let $\mathcal{B}_m$ denote the building set obtained after adding the elements up to and including I_m to $\overline{\mathcal{B}}$. Let $\widetilde{\mathcal{B}}_{m-1}$ denote the building set $\mathcal{B}_m - \{I_m\}$ (note that $\widetilde{\mathcal{B}}_{m-1}$ is not necessarily equal to $\mathcal{B}_{m-1}$ since I_{m-1} is not necessarily added directly before I_m). Then by Lemma 11

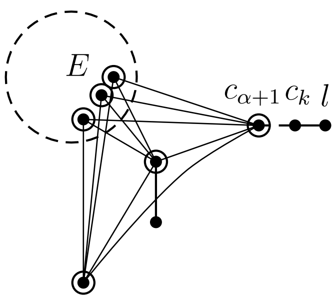
$$\gamma(\mathcal{B}_m) - \gamma(\widetilde{\mathcal{B}}_{m-1}) = t\gamma(\widetilde{\mathcal{B}}_{m-1}|_{I_m})\gamma(\widetilde{\mathcal{B}}_{m-1}/I_m).$$

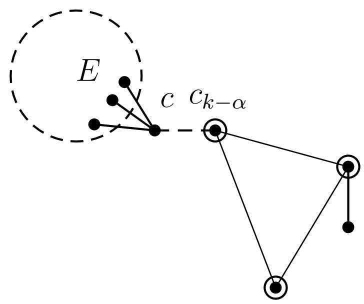
Since we do not add a subset of a set before adding the set, we have that

$$\widetilde{\mathcal{B}}_{m-1}|_{I_m} = \mathcal{B}(G)|_{I_m - \{v\}} \cup \{\{v\}\} \cong \mathcal{B}(G')|_{I'_m - \{v\}} \cup \{\{v\}\} = \mathcal{B}'_{m-1}|_{I'_m}.$$

We let K' denote the set of vertices in $G' - I'_m$ that are adjacent in G' to a vertex in I'_m, and we let K denote the set of vertices in $G - I_m$ that are adjacent in G to a vertex in I_m. Then $\mathcal{B}'_{m-1}/I'_m = \mathcal{B}(G')/I'_m$. This is true since we know that $\mathcal{B}'_{m-1}/I'_m \subseteq \mathcal{B}(G')/I'_m$ since $\mathcal{B}'_{m-1} \subseteq \mathcal{B}(G')$. To show that $\mathcal{B}'_{m-1}/I'_m \supseteq \mathcal{B}(G')/I'_m$, note that $\mathcal{B}(G')/I'_m = \mathcal{B}(\hat{G}')$ where $\hat{G}'$ is the graph $G' - I'_m$ with additional edges so that the restriction to K' is a complete graph. The elements of $\mathcal{B}(\hat{G}')$ that are the edges between elements in K' are in $\mathcal{B}'_{m-1}/I'_m$ because any two vertices in K' are linked by a path of vertices contained in $I'_m - v$. By a similar argument we have that $\widetilde{\mathcal{B}}_{m-1}/I_m = \mathcal{B}(G)/I_m$. Note that $\mathcal{B}(G)/I_m = \mathcal{B}(\hat{G})$ where $\hat{G}$ denotes the graph $G - I_m$ with additional edges so that the restriction to K is a complete graph, (see Figure 3).

Figure 3: The graph $\hat{G}$ for the contraction $\mathcal{B}_{m-1}/I_m = \mathcal{B}(\hat{G})$ followed by the graph $\hat{G}'$ for the contraction $\mathcal{B}'_{m-1}/I'_m = \mathcal{B}(\hat{G}')$. The vertices K and K' are drawn with an additional ring around them.

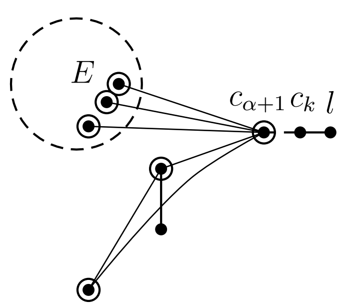




We also have that $\gamma(\mathcal{B}'_{m-1}/I'_m) \leqslant \gamma(\widetilde{\mathcal{B}}_{m-1}/I_m)$ because $\hat{G}'$ can be obtained from $\hat{G}$ by first removing edges (which lowers the γ-polynomial of the corresponding graphical building set by Theorem 13) and then performing a tree shift on a graph with fewer than n vertices (or doing no tree shift in the case that $c_\alpha = c_k$ or $c_\alpha = l$), which we assume lowers the γ-polynomial (see Figure 4). Hence

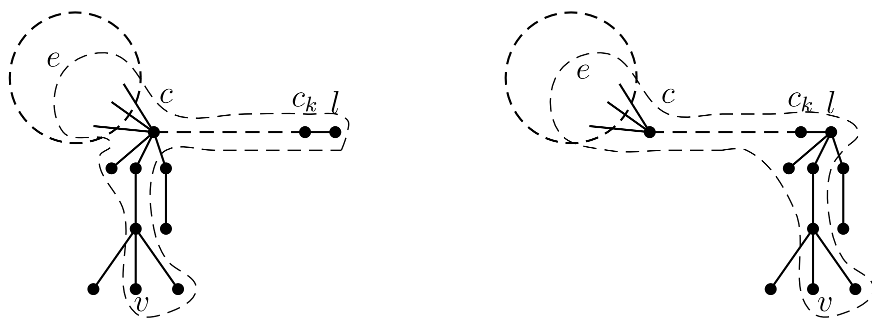
$$\begin{aligned}
\gamma(\mathcal{B}'_m) - \gamma(\mathcal{B}'_{m-1}) &= t\gamma(\mathcal{B}'_{m-1}|_{I'_m})\gamma(\mathcal{B}'_{m-1}/I'_m) \\
&\leqslant t\gamma(\widetilde{\mathcal{B}}_{m-1}|_{I_m})\gamma(\widetilde{\mathcal{B}}_{m-1}/I_m) \\
&= \gamma(\mathcal{B}_m) - \gamma(\widetilde{\mathcal{B}}_{m-1}).
\end{aligned}$$

Figure 4: The graph that is obtained after removing edges from $\hat{G}$ in Figure 3. The tree shift of this graph gives the graph $\hat{G}'$ of Figure 3.



Now suppose that $I'_m \cap E \neq \emptyset$, so that $\{c, c_1, \ldots, c_k, l\} \subseteq I'_m$. Let I_m denote I'_m, which is a set that is also added to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ (see Figure 5). Define $\mathcal{B}'_{m-1}$, $\widetilde{\mathcal{B}}_{m-1}$ as in the previous case.

Figure 5: The set I_m followed by the set I'_m.



Then we have that $\widetilde{\mathcal{B}}_{m-1}|_{I_m} = \mathcal{B}'_{m-1}|_{I'_m}$ and $\widetilde{\mathcal{B}}_{m-1}/I_m = \mathcal{B}'_{m-1}/I'_m$ which are both equal to $\mathcal{B}(G)/I_m$. This can be shown by arguments similar to those used in the case where $I'_m \cap E = \emptyset$. Hence in this case we also have

$$\begin{aligned}\gamma(\mathcal{B}'_m) - \gamma(\mathcal{B}'_{m-1}) &= t\gamma(\mathcal{B}'_{m-1}|_{I'_m})\gamma(\mathcal{B}'_{m-1}/I'_m) \\ &\leqslant t\gamma(\widetilde{\mathcal{B}}_{m-1}|_{I_m})\gamma(\widetilde{\mathcal{B}}_{m-1}/I_m) \\ &= \gamma(\mathcal{B}_m) - \gamma(\widetilde{\mathcal{B}}_{m-1}).\end{aligned}$$

Since for every element I'_m that is added to $\overline{\mathcal{B}'}$ to obtain $\mathcal{B}(G')$ there is a corresponding element I_m that is added to $\overline{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ that increases the γ-polynomial by at least as much as I'_m we have that

$$\gamma(\mathcal{B}(G')) - \gamma(\overline{\mathcal{B}'}) \leqslant \gamma(\mathcal{B}(G)) - \gamma(\overline{\mathcal{B}})$$

as desired. □

By applying Theorem 15 to the case where the graph is a tree we obtain the following Theorem, which is predicted by [PRW, Conjecture 14.1].

Theorem 16. *Let S be the set of all tree graphs on n nodes. Define the relation $T' \leqslant T$ if T' can be obtained by applying any number of tree shifts to T. Then $\leqslant$ defines a partial order on S with the following properties.*

- $\mathrm{Path_n}$ *is the unique $\leqslant$-minimum element.*
- $K_{1,n-1}$ *is the unique $\leqslant$-maximum element.*
- $T' \leqslant T$ *implies* $\gamma(\mathcal{B}(T')) \leqslant \gamma(\mathcal{B}(T))$.

Proof. This relation is a partial order on S, since given any $a, b \in S$ we have that if $a \leqslant b$ and $b \leqslant a$ then $a = b$ because any tree shift decreases the number of leaves by one.

$\mathrm{Path_n}$ is $\leqslant$-minimal since no tree has fewer leaves than $\mathrm{Path_n}$. Let T be a tree that is not $\mathrm{Path_n}$. We can apply a tree shift to T since if we travel along the path from any leaf inwards we must eventually meet a vertex of degree three or more. Hence T is not $\leqslant$-minimal, so that $\mathrm{Path_n}$ is the unique $\leqslant$-minimum element.

$K_{1,n-1}$ is $\leqslant$-maximal because no tree has more leaves than $K_{1,n-1}$. Suppose that T' is a tree that is not $K_{1,n-1}$. We can perform a *reverse shift*, which sends T' to a tree T such that we can apply a tree shift to T to obtain T'. T' must contain two adjacent vertices c and l, neither of which is a leaf. To obtain T, we attach the component of $T' - \{c, l\}$ that was attached to l in T, and attach it to c, so that the vertices that were attached to l are now attached to c. Hence T' is not $\leqslant$-maximal, so that $K_{1,n-1}$ is the unique $\leqslant$-maximum element.

By Theorem 15, if $T' \leqslant T$ then $\gamma(\mathcal{B}(T')) \leqslant \gamma(\mathcal{B}(T))$. □

Theorem 15 provides a new (arguably more explicit) proof of the bounds on the γ-polynomial of trees (Equation 1) than that provided in [BV, Theorem 9.4, (1)].

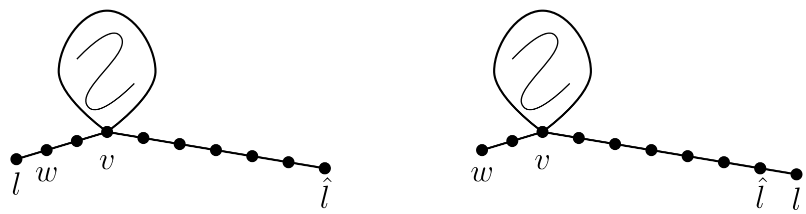
5 Flossing moves

Let G be a graph with n vertices labelled 1 to n. A pair of leaves $l, \hat{l}$ in G *floss* a vertex $v \in G$ if there is a unique path in G from l to $\hat{l}$ of minimal length, and v is the unique branched vertex (having degree $\geqslant 3$) on this path. [BR, Proposition 4.8] shows that for any tree graph T that is not $\mathrm{Path_n}$, there exists a triple of vertices $(l, \hat{l}, v)$ in which the vertices $l, \hat{l}$ floss the vertex v. When $l, \hat{l}$ floss a vertex v, relabel so that

$$\mathrm{dist}_G(l, v) \leqslant \mathrm{dist}_G(\hat{l}, v),$$

where $\mathrm{dist}_G(v_1, v_2)$ denotes the number of edges in a minimal path in G between vertices v_1 and v_2. Flossing moves are defined in [BR], and it was suggested in [PRW] that they might lower the γ-polynomial of the graph-associahedra. We show that this is true for flossing moves that are a generalisation of those given in [BR]. Let G be a graph with a triple of vertices $(l, \hat{l}, v)$ such that $l, \hat{l}$ are leaves that floss the vertex v (and $\mathrm{dist}_G(l, v) \leqslant \mathrm{dist}_G(\hat{l}, v)$). A *flossing move* on G is obtained by removing the edge (l, w) and adding an edge $(\hat{l}, l)$ where w is the nearest vertex (possibly v) to l. We let $r := \mathrm{dist}_G(l, v) + 1$ (the number of vertices in the chain from l to v), and $\hat{r} := \mathrm{dist}_G(\hat{l}, v) + 1$ (see Figure 6).

Figure 6: A graph G followed by a flossing move applied to G. In this example we have $r = 4$ and $\hat{r} = 7$. The loop represents G minus the path of vertices from l to $\hat{l}$ that contains v.



Theorem 17. *Let G be a connected graph, and let G' be the resulting flossing move of G. Then $\gamma(\mathcal{B}(G')) \leqslant \gamma(\mathcal{B}(G))$.*

Proof. We suppose that G has n vertices, and we label G by $l, \hat{l}, r, \hat{r}, v$ and w, as in the definition of flossing move. We assume by induction that for any graph with $< n$ vertices, that a flossing move lowers the γ-polynomial. When $n < 4$ no flossing move is possible so the result is vacuously true. $\mathcal{B}(G)$ is a flag building set on $[n]$, and the building set $\hat{\mathcal{B}}$ that is obtained from $\mathcal{B}(G)$ by removing all building set elements that contain $\{l, w\}$ apart from $[n]$ is also a flag building set on $[n]$. Hence by Theorem 9, $\mathcal{B}(G)$ can be obtained from $\hat{\mathcal{B}}$ by successively adding building set elements so that at each step the set is a flag building set. Similarly, $\mathcal{B}(G')$ can be obtained from $\hat{\mathcal{B}}$ by successively adding building set elements so that at each step the set is a flag building set. Similar to the arguments used in the proof of Theorem 15, we construct an injection

$$\mathcal{B}(G') - \hat{\mathcal{B}} \to \mathcal{B}(G) - \hat{\mathcal{B}}$$

$$I' \mapsto I.$$

We then show that the increase in the γ-polynomial when adding the element in $\mathcal{B}(G') - \hat{\mathcal{B}}$ is less than or equal to the increase when adding the corresponding element in $\mathcal{B}(G) - \hat{\mathcal{B}}$ which proves the Theorem.

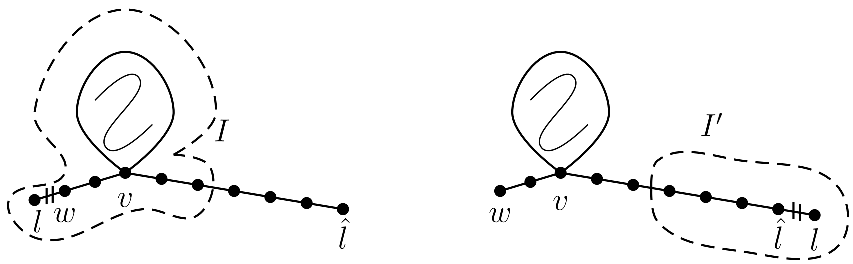
Let $I_1, I_2, \ldots, I_k$ be the building set elements of $\mathcal{B}(G') - \hat{\mathcal{B}}$. Suppose for some $i \neq j$ that $I_j \subseteq I_i$. Then $j > i$, since $I_j \cap (I_i - \{l\}) \neq \emptyset$ and $I_j \cup (I_i - \{l\}) = I_i$ which implies that when I_j is in the building set I_i must be too.

Let P be the set of vertices in the minimal path from l to $\hat{l}$. Let I' be an element that is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G')$. There are three cases for I' that we will consider.

- $|I'| \leqslant \hat{r}$,
- $|I'| \geqslant \hat{r} + 1$, and I' does not contain all of $G - P$,
- $|I'| \geqslant \hat{r} + 1$, and I' contains all of $G - P$.

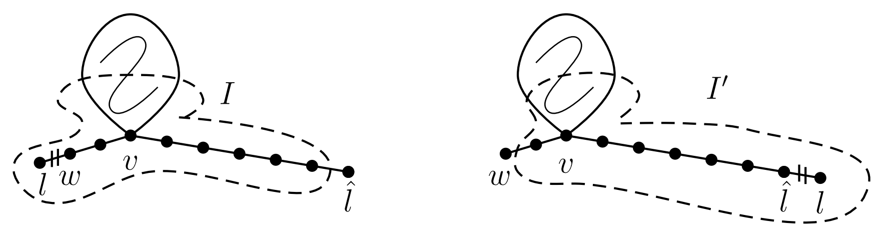
Suppose that $|I'| \leqslant \hat{r}$, and let I be the element of $\mathcal{B}(G') - \hat{\mathcal{B}}$ such that $|I \cap P| = r + \hat{r} - |I'|$, and I contains all of $G - P$. In each case we let $\mathcal{B}_1$ (respectively $\mathcal{B}_2$) denote the building sets we have before adding I (respectively I'). Then $\mathcal{B}_1|_I = \mathcal{B}_2/I' \cup \{\{l\}\}$, so that $\gamma(\mathcal{B}_1|_I) = \gamma(\mathcal{B}_2/I')$. Also, $\mathcal{B}_1/I \cup \{\{l\}\} = \mathcal{B}_2|_{I'}$, so that $\gamma(\mathcal{B}_1/I) = \gamma(\mathcal{B}_2|_{I'})$ (see Figure 7).

Figure 7: The graph G followed by G'. Keeping with the values of Figure 6, we have $|I'| = 5$ and $|I \cap P| = 6$.



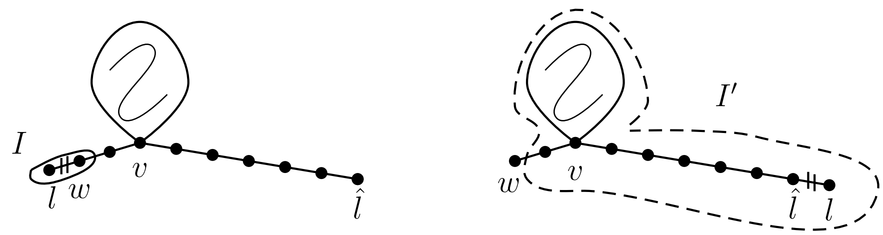
Suppose that $|I'| \geqslant \hat{r} + 1$, and suppose that I' does not contain all of $G - P$. Let I be the element of $\mathcal{B}(G) - \hat{\mathcal{B}}$ such that $|I \cap P| = |I' \cap P|$, and $I \cap (G - P) = I' \cap (G - P)$. Then we have that $\mathcal{B}_1/I \cong \mathcal{B}_2/I'$, and $\mathcal{B}_1|_I = \mathcal{B}(G_1) \cup \{\{l\}\}$, and $\mathcal{B}_2|_{I'} = \mathcal{B}(G_2) \cup \{\{l\}\}$ where G_2 is a graph obtained from a graph G_1 by a flossing move (or if $\mathrm{dist}_G(l, v) = 1$, $G_2 = G_1$). By induction on the number of vertices of the graphs involved we have that $\gamma(\mathcal{B}(G_2)) \leqslant \gamma(\mathcal{B}(G_1))$ so that $\gamma(\mathcal{B}_2|_{I'}) \leqslant \gamma(\mathcal{B}_1|_I)$ (see Figure 8).

Figure 8: The graph $\mathcal{B}_1$ followed by $\mathcal{B}_2$. We have $|I'| \geqslant 7$.



Suppose that $|I'| \geqslant \hat{r} + 1$ and I' contains all of $G - P$. Let I be the element of $\mathcal{B}(G) - \hat{\mathcal{B}}$ such that $|I| = r + \hat{r} - |I' \cap P|$. Then $\mathcal{B}_1/I \cup \{\{l\}\} = \mathcal{B}_2|_{I'}$, and $\mathcal{B}_1|_I = \mathcal{B}_2/I' \cup \{\{l\}\}$. Hence $\gamma(\mathcal{B}_1/I) = \gamma(\mathcal{B}_2|_{I'})$ and $\gamma(\mathcal{B}_1|_I) = \gamma(\mathcal{B}_2/I')$ (see Figure 9).

Figure 9: The graph $\mathcal{B}_1$ followed by $\mathcal{B}_2$. We have $|I| = 2$ and $|I' \cap P| = 9$.



Note that no element $I \in \mathcal{B}(G) - \hat{\mathcal{B}}$ is used more than once, since in the first case we have that $|I| \geqslant r$ and I contains all of $G - P$. In the second case we have that $|I| \geqslant \hat{r} + 1 > r$ and I does not contain all of $G - P$. In the third case we have that $|I| = r + \hat{r} - |I' \cap P| \leqslant r + \hat{r} - (\hat{r} + 1) = r - 1$.

By Lemma 11 the change in the γ-polynomial when adding I' is given by

$$\gamma(\mathcal{B}_2 \cup \{I'\}) - \gamma(\mathcal{B}_2) = t\gamma(\mathcal{B}_2/I')\gamma(\mathcal{B}_2|_{I'}),$$

and when adding I it is given by

$$\gamma(\mathcal{B}_1 \cup \{I\}) - \gamma(\mathcal{B}_1) = t\gamma(\mathcal{B}_1/I)\gamma(\mathcal{B}_1|_I).$$

Since for every element I' that is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G')$, there is an element I that is added to $\hat{\mathcal{B}}$ to obtain $\mathcal{B}(G)$ such that $\gamma(\mathcal{B}_2/I')\gamma(\mathcal{B}_2|_{I'}) \leqslant \gamma(\mathcal{B}_1/I)\gamma(\mathcal{B}_1|_I)$ we have that $\gamma(\mathcal{B}(G')) \leqslant \gamma(\mathcal{B}(G))$. $\square$

It is exactly when $\mathrm{dist}_G(l, v) = 1$ that a flossing move is a kind of tree shift. This is exactly when a flossing move reduces the number of leaves. If we partition the set S of all tree graphs with n vertices by their number of leaves, then tree shifts send graphs between the parts, whilst flossing moves such that $\mathrm{dist}_T(l, v) \neq 1$ give relations between graphs with the same number of leaves. This is illustrated in the following example for tree graphs with seven vertices.

Example 18. Arrows are drawn between pairs of graphs with the same number of leaves when one (at the head) can be obtained from the other (at the tail) by a flossing move. Arrows are drawn from a graph with $i + 1$ leaves to one with i leaves when the graph at the head can be obtained from the graph at the tail by a tree shift.

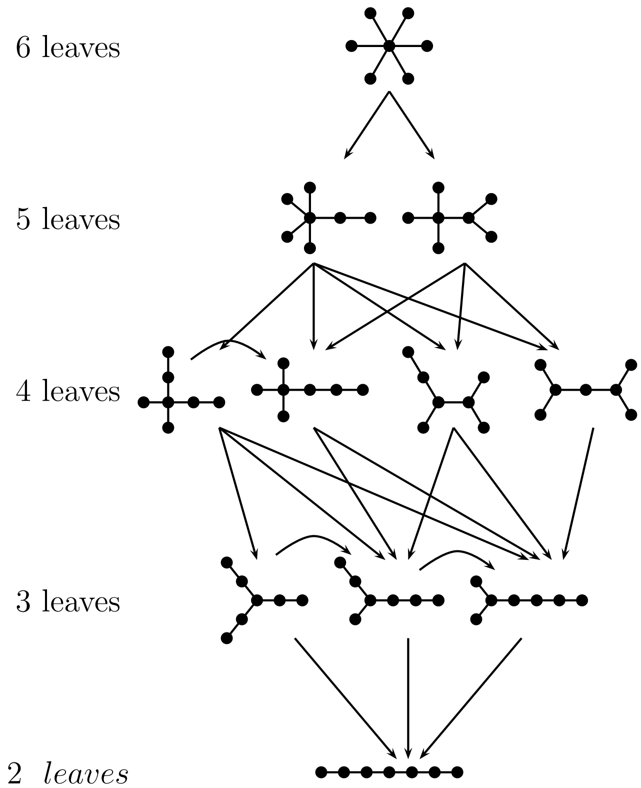


Figure 10: Tree graphs with 7 vertices and their tree shift and flossing move relations.

It is suggested in [PRW] that a move on a tree graph that increases the *Wiener index* [Wie] might approximately lower the γ-polynomial, although the only moves that we have found that increase the Wiener index and lower the γ-polynomial are tree shifts and flossing moves.

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